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THE

CAMBRIDGE AND DUBLIN

MATHEMATICAL JOURNAL.

ON CERTAIN GENERAL PROPERTIES OF HOMOGENEOUS FUNCTIONS.

By J. J. Sylvester, M.A., F.R.S.

Let χ denote the operation

$$x_1 \cdot \frac{d}{da_1} + x_2 \cdot \frac{d}{da_2} + \dots + x_n \cdot \frac{d}{da_n},$$

and Δ the operation

$$a_1 \cdot \frac{d}{dx_1} + a_2 \cdot \frac{d}{dx_2} + \dots + a_n \cdot \frac{d}{dx_n}:$$

and now suppose that ω, a homogeneous function of ι dimensions of $a_1 a_2 \dots a_n$, and not of any of the quantities $x_1 x_2 \dots x_n$ is subjected to the successive operations indicated by $\Delta^s . \chi^r$.

We have $\Delta^s . \chi^r . \omega = \Delta^{s-1} . \Delta . \chi^r . \omega$,

$$\Delta \chi^r . \omega = \left(a_1 \frac{d}{dx_1} + a_2 \cdot \frac{d}{dx_2} + \dots + a_n \cdot \frac{d}{dx_n}\right)$$

$$\times \left(x_1 \cdot \frac{d}{da_1} + x_2 \cdot \frac{d}{da_2} + \dots + x_n \cdot \frac{d}{da_n}\right)^r . \omega$$

$$= r\left(a_1 \cdot \frac{d}{da_1} + a_2 \cdot \frac{d}{da_2} + \dots + a_n \cdot \frac{d}{da}\right) \chi^{r-1} . \omega$$

$$= r\,(\iota - r + 1)\,\chi^{r-1} . \omega,$$

for $\chi^{r-1} . \omega$ is of $(r-1)$ dimensions, lower than ω (which is of ι dimensions) in $a_1 a_2 \dots a_n$.

Hence $\Delta^s . \chi^r . \omega = r\,(\iota - r + 1)\,\Delta^{s-1} . \chi^{r-1} . \omega$

$$= \&c. = \{r.(r-1) \dots (r-s+1)\}$$

$$\times \{(\iota - r + 1)(\iota - r + 2) \dots (\iota - r + s)\}\, \chi^{r-s} . \omega \dots (1).$$

Now in the expression $\chi^r . \omega (a_1 a_2 , \ldots . a_n)$, suppose that we write

$$x_1 = u_1 + a_1 . \epsilon,$$
$$x_2 = u_2 + a_2 . \epsilon,$$
$$\ldots\ldots\ldots\ldots$$
$$x_n = u_n + a_n . \epsilon,$$

we have, by Taylor's theorem,

$$\chi^r . \omega = U^r . \omega + A (U^r . \omega) \epsilon + A^2 . U^r . \omega \frac{\epsilon^2}{1.2} + \ldots .$$
$$+ A_r . U^r . \omega . \frac{\epsilon^r}{1.2.3. \ldots . r},$$

where $U^r . \omega$ denotes what $\chi^r . \omega$ becomes, on substituting u's for x's, and A now represents

$$u_1 \frac{d}{da_1} + u_2 . \frac{d}{da_2} + \ldots . + u_n . \frac{d}{da_n} .$$

This expansion stops spontaneously at the $(r + 1)$th term, because $\chi^r . \omega$ is only of r dimensions in $x_1 x_2 , \ldots . x_n$.

Applying now theorem (1), we obtain

$$\chi^r . \omega = U^r . \omega + r . (\iota - r + 1) U^{r-1} . \omega \epsilon + r . \frac{r-1}{2} . \{(\iota - r + 1)(\iota - r + 2)\}$$
$$. U^{r-2} . \omega . \epsilon^2 + \ldots + \{(\iota - r + 1)(\iota - r + 2) \ldots \iota\} \omega \epsilon^r \ldots . (2).$$

In using this theorem in the course of the ensuing pages, it will be found convenient to assign to ϵ a specific value, and I shall suppose it equal to $\frac{x_n}{a_n}$; this gives

$$u_1 = x_1 - \frac{a_1}{a_n} . x_n ,$$
$$u_2 = x_2 - \frac{a_2}{a_n} . x_n ,$$
$$\ldots\ldots\ldots\ldots$$
$$u_n = x_n - \frac{a_n}{a_n} . x_n$$
$$= 0.$$

And inasmuch as the U symbol now contains $a_1 a_2 \ldots . a_n$, so that $U . U^r$ no longer equals U^{r+1}, I shall write U_r for U^r. Theorem (2) will thus assume the form

$$\chi^r . \omega = U_r . \omega + r . (\iota - r + 1) U_{r-1} . \omega . \frac{x_n}{a_n} + r . \frac{r-1}{2} \{(\iota - r + 1)(\iota - r + 2)$$
$$U_{r-2} . \omega \left(\frac{x_n}{a_n}\right)^2 \Big\} + \ldots . + \{(\iota - r + 1) \ldots . \iota\} \omega \left(\frac{x_n}{a_n}\right)^r \ldots . (3),$$

where U_r for all values of r denotes what

$$\left(x_1 \cdot \frac{d}{da_1} + x_2 \cdot \frac{d}{da_2} + \ldots\ldots + x_{n-1} \cdot \frac{d}{da_{n-1}}\right)^r \cdot \omega$$

becomes, on substituting $u_1 u_2 \ldots\ldots u_{n-1}$ for $x_1 x_2 \ldots\ldots x_{n-1}$, *after* the processes of derivation have been completed: this it is essential to observe, because $u_1 u_2 \ldots\ldots\ldots u_{n-1}$ now involve $a_1 a_2 \ldots\ldots a_{n-1}, a_n$. The term $x_n \cdot \dfrac{d}{da_n}$ is omitted from the symbol of linear derivation, because in the substitutions x_n will be replaced by zero.

As an example of this last theorem, take $\omega = a^3 + b^3 + c^3 + kabc$;

then $\chi\,\omega = 3a^2x + 3b^2y + 3c^2z + kbcx + kcay + kabz,$

$$\chi^2\omega = 6ax^2 + 6by^2 + 6cz^2 + 2kcxy + 2kayz + 2kbzx,$$

$$\chi^3\omega = 6x^3 + 6y^3 + 6z^3 + 6kxyz.$$

$$U_1\omega = 3a^2\left(x - \frac{az}{c}\right) + 3b^2\left(y - \frac{bz}{c}\right) + kbc\left(x - \frac{az}{c}\right) + kca\left(y - \frac{bz}{c}\right),$$

$$U_2\omega = 6a\left(x - \frac{az}{c}\right)^2 + 6b\left(y - \frac{bz}{c}\right)^2 + 2kc\left(x - a \cdot \frac{z}{c}\right)\left(3 - \frac{bz}{c}\right),$$

$$U_3\omega = 6\left(x - \frac{az}{c}\right)^3 + 6\left(y - \frac{bz}{c}\right)^3,$$

and it will be found that the equations given by theorem (3) are satisfied, viz.

$$\chi\omega = U\omega + 3\,\frac{z}{c}\,.\,\omega,$$

$$\chi^2\omega = U_2\omega + 2.2\,\frac{z}{c}\,U\omega + 2.3\,\frac{z^2}{c^2}\,\omega,$$

$$\chi^3\omega = U_3\omega + 3\,.\,\frac{z}{c}\,U_2\omega + 3.1.2\,\frac{z^2}{c^2}\,U(\omega),$$

$$+ 1.2.3\,\frac{z^3}{c^3}\,\omega.$$

Probably, as this theorem is of rather a novel character, the annexed sketch of a somewhat different course of demonstration may be not unacceptable to my readers.

We have $\chi(\omega) = \left(x_1 \dfrac{d}{da_1} + x_2 \dfrac{d}{da_2} + \ldots\ldots + x_n \cdot \dfrac{d}{da_n}\right)\omega$;

and by the well-known law for homogeneous functions,

$$\iota(\omega) = \left(a_1 \frac{d}{da_1} + a_2 \cdot \frac{d}{da_2} + \dots + a_n \cdot \frac{d}{da_n}\right)\omega.$$

Hence

$$\left(\chi - \iota \frac{x_n}{a_n}\right)\omega = \left(u_1 \cdot \frac{d}{da_1} + u_2 \cdot \frac{d}{da_2} + \dots + u_{n-1} \frac{d}{da_{n-1}}\right)\omega,$$
$$= U(\omega).$$

Hence

$$\chi(\omega) = \left(U + \iota \cdot \frac{x_n}{a_n}\right)\omega,$$

$$\chi^2 \cdot \omega = \left\{U + (\iota - 1)\frac{x_n}{a_n}\right\}\left(U + \iota \frac{x_n}{a_n}\right)\omega,$$

$$\chi^3 \omega = \left\{U + (\iota - 2)\frac{x_n}{a_n}\right\}\left\{U + (\iota - 1)\frac{x_n}{a_n}\right\}\left(U + \iota \frac{x_n}{a_n}\right)\omega,$$

&c. = &c.

But in performing the process indicated by the several factors it must be carefully borne in mind that $U . U_r$ is not $= U_{r+1}$; this would be the case were it not for the terms $-\frac{a_1}{a_n} . x_n, -\frac{a_2}{a_n} . x_n$, &c., which enter into $u_1 u_2 \dots u_{n-1}$. But on account of these terms, we have

$$U . U_r . \omega = \left(u_1 \frac{d}{da_1} + u_2 \cdot \frac{d}{da_2} + \dots + u_{n-1} \cdot \frac{d}{da_{n-1}}\right)$$
$$\left(u_1 \frac{d}{da_1} + u_2 \cdot \frac{d}{da_2} + \dots + u_{n-1} \cdot \frac{d}{da_{n-1}}\right)^r .$$
$$= U_{r+1} . \omega - r \frac{x_n}{a_n} . \left\{u_1 \frac{d}{da_1} + u_2 \cdot \frac{d}{da_2} + \dots + u_{n-1} \cdot \frac{d}{da_{n-1}}\right\}^{r-1} \omega,$$

for $$\frac{d}{da_1} . u_1 = \frac{d}{da_2} . u_2 = \dots = \frac{d}{da_{n-1}} . u_{n-1} = -\frac{x_n}{a_n}.$$

Hence $$U . U_r . \omega = U_{r+1} . \omega - r . \frac{x_n}{a_n} . U_r . \omega.$$

Let $\frac{x_n}{a_n}$ be called ε; we find

$$\chi = U + \iota\varepsilon,$$
$$\chi^2 = \{U + (\iota - 1)\varepsilon\}(U + \iota\varepsilon)$$
$$= U . U + (2\iota - 1)\varepsilon U + (\iota - 1)\iota\varepsilon^2$$
$$= U_2 + 2(\iota - 1)\varepsilon U + (\iota - 1)\iota\varepsilon^2;$$

$$\chi' = \{U + (\iota - 2)\,\epsilon\}\,\chi^2,$$
$$= U.U_2 + 2(\iota - 1)\,\epsilon\,.\,U.U + (\iota - 1)\,\iota\,.\,\epsilon^2 U$$
$$+ (\iota - 2)\,\epsilon U_2 + 2(\iota - 2)(\iota - 1)\,\epsilon^2 U + (\iota - 2)(\iota - 1)\,\iota\,.\,\epsilon^3$$
$$= U_3 + 3(\iota - 2)\,\epsilon U_2 + 3(\iota - 2)(\iota - 1)\,\epsilon^2.U + (\iota - 2)(\iota - 1)\,\iota\,.\,\epsilon^3.$$

The same process being continued will lead to results identical with those previously obtained and expressed in theorem (3).

The expansion of χ', treated according to this second method, appears to require the solution of the partial equation in differences

$$a_{r+1,\,s+1} = a_{r,\,s+1} + (\iota - 2r)\,a_{r,s},$$

$a_{s,1}$ being given as unity for $s = 1$ and as zero for all other values of s.

It is probable however that the solution of this equation might be evaded by some artifice peculiar to the particular case to be dealt with. I do not propose to dwell upon this inquiry, which would be foreign to the object of my present research. It may however not be out of place to make the passing remark, that the equations expressing χ' in terms of powers of U admit easily of being reverted, as indeed may the more general form

$$\chi_r = \epsilon_r\,.\,u_{r-1} + \frac{1}{1.2}\,\epsilon_r\epsilon_{r-1}\,.\,u_{r-2} + \&c.,$$

which becomes the equation of formula (3), on making

$$\epsilon_r = r\,(\iota + 1 - r)\,\frac{x_n}{a_n}\quad \chi_r = \chi\,.\,\omega,\ \text{and}\ u_r = u_r\,.\,\omega\,;$$

for let
$$u_r = \epsilon_1\epsilon_2\,\ldots\,\epsilon_r\,.\,v_r,$$
$$\chi_r = \epsilon_1\epsilon_2\,\ldots\,\epsilon_r\,.\,y_r,$$

then
$$y_r = v_r + v_{r-1} + \frac{v_{r-2}}{1.2} + \frac{v_{r-3}}{1.2.3} + \&c.;$$

whence
$$v_r = \epsilon^{-\epsilon^{-\frac{d}{dr}}}\,.\,y_r,$$
$$= y_r - y_{r-1} + \frac{y_{r-2}}{1.2} - \frac{y_{r-3}}{1.2.3},\ \&c.:$$

and therefore $u_r = x_r - r\epsilon_r\,.\,u_{r-1} + r\,.\,\dfrac{r-1}{2}\,\epsilon_r\,.\,\epsilon_{r-1}\,.\,u_{r-2}$, &c.

Thus we obtain, from equation (3),

$$U^r\,.\,\omega = \chi^r\omega - r\,(r+1)\,\chi^{r-1}\,.\,\omega\,\frac{x_n}{a_n} + \&c.$$

As a first application of theorem (3), I shall proceed to shew how Joachimsthal's equation to the surface drawn from a given point (a, β, γ, δ) through the intersection of two surfaces $\phi(x, y, z, t) = 0$, $\theta(x, y, z, t) = 0$, may be expressed under the *explicit* form of the equation to a cone.

The equation in question is obtained by eliminating λ between

$$\phi \cdot \lambda^m + \chi \cdot \phi\lambda^{m-1} + \frac{1}{1.2}\chi^2 \cdot \phi \cdot \lambda^{m-2} + \&c. = 0,$$

$$\theta\lambda^n + \chi\theta \cdot \lambda^{n-1} + \frac{1}{1.2}\chi^2\theta \cdot \lambda^{n-2} + \frac{1}{1.2.3}\chi^3\theta \cdot \lambda^{n-3} + \&c. = 0,$$

where

$$\phi = \phi(a,\beta,\gamma,\delta)\, \theta = \theta(a,\beta,\gamma,\delta)\, \chi = x \cdot \frac{d}{da} + y \cdot \frac{d}{d\beta} + z \cdot \frac{d}{d\gamma} + t\frac{d}{d\delta}.$$

By theorem (3), these two equations, on writing $\frac{x_n}{a_n} = \epsilon$, becomes

$$\phi \cdot \lambda^m + \{U(\phi) + m\phi\epsilon\}\lambda^{m-1}$$
$$+ \{U^2\phi + 2(m-1)U\phi\epsilon + (m-1)m\phi \cdot \epsilon^2\}\frac{\lambda^{m-2}}{1.2} + \&c. = 0,$$

$$\theta\lambda^n + \{U(\theta) + n\theta.\epsilon)\lambda^{n-1} + (U^2\theta + \&c.)\frac{\lambda^{n-2}}{1.2} + \{U^3\theta + 3(n-2)U^2\theta\epsilon$$
$$+ 3.(n-2)(n-1)U\theta\epsilon^2 + (n-2)(n-1)n.\epsilon^3\}\frac{\lambda^{n-3}}{1.2.3} + \&c.$$

Now on writing $\lambda = \mu - \epsilon$, these equations take the forms

$$\phi \cdot \mu^m + U(\phi)\mu^{m-1} + .\ U^2\phi\frac{\mu^{m-2}}{1.2} + \&c. = 0,$$

$$\theta \cdot \mu^n + U(\theta) \cdot \mu^{n-1} + U^2\theta \cdot \frac{\mu^{n-2}}{1.2} + \&c. = 0,$$

as is easily seen by substituting back $\lambda + \epsilon$ in place of μ. Consequently ϵ no longer appears in the coefficients of the terms of the equations between which the elimination is to be performed, and the resultant will accordingly come out as a function only of ϕ, $U(\phi)$ $U^2(\phi)$, &c. *i.e.* of a, β, γ, δ, and of

$$x - \frac{a}{\delta}t,\quad y - \frac{\beta}{\delta}t,\quad z - \frac{\gamma}{\delta}t,$$

shewing that the equation in x, y, z, t, is of the form of that to a cone, as we know *à priori* it ought to be. Precisely a similar method may be applied to the elucidation of the corresponding theorem for a system of rays drawn from a

given point through the locus of the intersection of two curves.

Before entering upon some further and more interesting applications of theorem (3), it will be convenient to explain a nomenclature which has been employed by me on another occasion, and which is almost indispensable in inquiries of the nature we are now engaged upon. Homogeneous functions may be characterized by their degree, by the number of letters which enter into them, and lastly, by the lowest number of linear functions of the letters which may be introduced in place of the letters to represent such functions. Any such linear function I designate as an order, and am now able to discriminate between the number of letters and the number of orders which enter into a given function. The latter number, *generally* speaking, is the same as the former; it can never exceed it, but *may* be any number of units less than it.

I need scarcely observe that a pair of points becoming coincident, a conic becoming a pair of lines, a conoid becoming a cone, and so forth, for the higher realms of space, will be expressed by the homogeneous function of the second order which characterizes such loci,* losing one order, *i.e.* having an order less than the number of letters entering therein. Calling such characteristic $\phi\,(x, y, z \ldots t)$, it is well known that the condition of such loss of an order is that the determinant

$$\left.\begin{array}{llll} \dfrac{d^2\phi}{dx^2} & \dfrac{d^2\phi}{dx\,dy} & \ldots & \dfrac{d^2\phi}{dx\,dt} \\ \dfrac{d^2\phi}{dy\,dx} & \dfrac{d^2\phi}{dy^2} & \ldots & \dfrac{d^2\phi}{dy\,dt} \\ \dfrac{d^2\phi}{dt\,dx} & \dfrac{d^2\phi}{dt\,dy} & \ldots & \dfrac{d^2\phi}{dt^2} \end{array}\right\} = 0.$$

A conoid becoming a pair of planes, a cone becoming a pair of coincident lines, a pair of points becoming indeterminate, will, in like manner, be denoted by their characteristic, losing two orders, and so forth, for the higher degrees of degradation. In like manner, in general, a homogeneous function of three letters of any degree losing an order, typifies that the locus to which it is the characteristic will break up into a system of right lines.

* If $U = 0$ is the equation to any locus, U may be said to *char* ... same, or to be its characteristic.

function of a, β, γ — the equations $\omega = 0$, χ(

$$\dots + \dots \frac{d}{d\gamma} + \dots\dots + t\frac{d}{d\delta}.$$

of the variables x, y, z — of the above equations one order less than the nu[illegible] of one letter will be atte[illegible]

$$\dots \frac{x_n}{a_n}\omega = 0,$$

$$\dots \frac{x_n}{a_n} U\omega + 2\left(\frac{x_n}{a_n}\right)^2\omega = 0\ ;$$

$$\omega = 0.$$

$$\dots\omega) = 0,$$

$$\dots\omega) = 0\ ;$$

contain one order less than the the resultant of the eliminatio[n] two orders less than the number consequently, whichever of the letters between $\chi(\omega) = 0$ and $\chi^2\omega = 0$, resultant equation will contain one of letters remaining.

that the tangent line to a conic points, the tangent plane to a lines, and so forth, for the higher instance, if we take $\omega(x, y, z, t) = 0$, and a, β, γ, δ, the coordinates to shall have $\omega(a, \beta, \gamma, \delta) = 0$,

$$\left(\dots \frac{d}{d\gamma} + t\frac{d}{d\delta}\right)\omega, \text{ i.e. } \chi\omega = 0,$$

$$\dots t)\ \text{i.e. } \chi^2\omega = 0\ ;$$

the coordinates of any point in the by the tangent plane.

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equation of a hyperlocus of the second degree at any

Consequently, by what has been shewn above, on eliminating any one of the four letters x, y, z, t, the resultant function of three letters will contain only two orders, and will thus represent a pair of lines, real or imaginary, intersecting one another at $\alpha, \beta, \gamma, \delta$.

The fact which has just been demonstrated (that the resultant of $\chi\omega = 0$, $\chi'\omega = 0$, loses an order if $\omega = 0$,) indicates that on expressing one of the quantities $x, y, z \ldots . t$ in terms of the others, by means of the first equation, and then substituting this value in the second, the determinant of the equation so obtained must be zero.

Now by virtue of a theorem which was given by me in a note to my paper in the preceding No. of this *Journal*, this determinant will be equal to the squared reciprocal of the coefficient in the equation $\chi(\omega) = 0$ of the letter eliminated multiplied by the determinant in respect to $x, y, z \ldots t, \lambda$ of

$$\chi^2(\omega) + \chi\omega \,.\, \lambda.$$

This latter determinant is therefore zero; but this determinant is the resultant of the equations

$$\left.\begin{array}{l}
\dfrac{d}{dx}\left(x\dfrac{d}{da} + y\dfrac{d}{db} + \&c.\right)^2 \omega + \dfrac{d}{dx}\left(x\dfrac{d}{da} + y\dfrac{d}{db} + \ldots\right)\omega = 0, \\
\dfrac{d}{dy}\left(x\dfrac{d}{da} + y\dfrac{d}{db} + \&c.\right)^2 \omega + \dfrac{d}{dy}\left(x\dfrac{d}{da} + y\dfrac{d}{db} + \ldots\right)\omega = 0, \\
\quad \&c. \qquad\qquad \&c. \qquad\qquad \&c. \qquad\qquad \&c. \\
\qquad\qquad \chi\omega = 0, \text{ i.e. } \left(x\dfrac{d}{da} + y\dfrac{d}{db} + \ldots .\right)\omega = 0,
\end{array}\right\}.$$

Thus we obtain the singular law, that the symmetrical eterminant

$$\begin{array}{cccc}
\dfrac{d}{da}\cdot\dfrac{d}{da}\,\omega, & \dfrac{d}{da}\cdot\dfrac{d}{db}\,\omega, \ldots . & \dfrac{d}{da}\cdot\dfrac{d}{dl}\,\omega, & \dfrac{d}{da}\,\omega, \\
\dfrac{d}{db}\cdot\dfrac{d}{da}\cdot\omega, & \dfrac{d}{db}\cdot\dfrac{d}{db}\cdot\omega, \ldots . & \dfrac{d}{db}\cdot\dfrac{d}{dl}\,\omega, & \dfrac{d}{dc}\cdot\omega, \\
\dfrac{d}{dc}\cdot\dfrac{d}{da}\cdot\omega, & \dfrac{d}{dc}\cdot\dfrac{d}{db}\cdot\omega, \ldots . & \dfrac{d}{dc}\cdot\dfrac{d}{dl}\cdot\omega, & \dfrac{d}{dc}\cdot\omega, \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\
\dfrac{d}{dl}\cdot\dfrac{d}{da}\cdot\omega, & \dfrac{d}{dl}\cdot\dfrac{d}{db}\cdot\omega, \ldots . & \dfrac{d}{dl}\cdot\dfrac{d}{dl}\cdot\omega, & \dfrac{d}{dl}\cdot\omega, \\
\dfrac{d}{da}\cdot\omega, & \dfrac{d}{db}\cdot\omega, \ldots\ldots\ldots & \dfrac{d}{dl}\cdot\omega, & 0,
\end{array}$$

ero when ω is *zero*.

This is easily shewn independently by means of a remarkable and I believe novel theorem, relative to homogeneous functions.

If ω be any homogeneous function of ι dimensions of $a, b, c \dots l$, we have (by Euler's theorem already repeatedly applied), remembering that $\frac{d\omega}{da}, \frac{d\omega}{db} \dots \frac{d\omega}{dl}$ are all homogeneous,

$$-\iota\omega + \left(a\frac{d}{da} + b\frac{d}{db} + \dots + l\frac{d}{dc}\right)\omega = 0,$$

$$-(\iota - 1)\frac{d\omega}{da} + \left(a\frac{d}{da}\cdot\frac{d}{da} + b\cdot\frac{d}{da}\cdot\frac{d}{db} + \dots + l\frac{d}{da}\cdot\frac{d}{dc}\right)\omega = 0,$$

$$-(\iota - 1)\frac{d\omega}{db} + \left(a\frac{d}{db}\cdot\frac{d}{da} + \dots + l.\frac{d}{db}\cdot\frac{d}{dc}\right)\omega = 0,$$

&c. &c. &c.

$$-(\iota - 1)\frac{d\omega}{dc} + \left(l\frac{d}{dc}\cdot\frac{d}{da} + \dots + l\cdot\frac{d}{dl}\cdot\frac{d}{dl}\right)\omega = 0.$$

Between these equations we may eliminate *all the letters*, $a, b, c \dots l$, and we obtain the equation

$$\left.\begin{array}{llll}
\frac{d}{da}\cdot\frac{d}{da}\omega, & \frac{d}{da}\cdot\frac{d}{db}\omega, \dots & \frac{d}{da}\cdot\frac{d}{dl}\omega, & \frac{d}{da}\omega, \\
\frac{d}{db}\cdot\frac{d}{da}\omega, & \frac{d}{db}\cdot\frac{d}{db}\omega, \dots & \frac{d}{db}\cdot\frac{d}{dl}\omega, & \frac{d}{db}\omega, \\
\frac{d}{dc}\cdot\frac{d}{da}\omega, & \frac{d}{dc}\cdot\frac{d}{db}\omega, \dots & \frac{d}{dc}\cdot\frac{d}{dl}\omega, & \frac{d}{dc}\omega, \\
\dots & \dots & \dots & \dots \\
\frac{d}{dl}\cdot\frac{d}{da}\omega, & \frac{d}{dl}\cdot\frac{d}{db}\omega, \dots & \frac{d}{dl}\cdot\frac{d}{dl}\omega, & \frac{d}{dl}\omega, \\
\frac{d}{da}\omega, & \frac{d}{db}\omega, & \frac{d}{dl}\omega, & \frac{\iota}{\iota-1}\omega,
\end{array}\right\} = 0.$$

As a corollary to this theorem, we see that if $\omega = 0$, the determinant obtained in the previous investigation becomes zero, agreeing with what has been already shewn; in fact the last-named determinant is always equal to

$$\frac{\iota - \iota}{\iota}\omega \times \left\{\begin{array}{lll}
\frac{d}{da}\cdot\frac{d}{da}\omega, & \frac{d}{da}\cdot\frac{d}{db}\omega \dots & \frac{d}{da}\cdot\frac{d}{dl}\omega, \\
\dots & \dots & \dots \\
\dots & \dots & \dots \\
\frac{d}{dl}\cdot\frac{d}{da}\omega, & \frac{d\iota}{dl}\cdot\frac{d}{db}\cdot\omega \dots & \frac{d}{dl}\cdot\frac{d}{dl}\omega,
\end{array}\right\}.$$

This remarkable theorem, which I have communicated to friends nearly a twelvemonth back, is here, I believe, published for the first time.*

Suppose next that $\omega(x, y, z)$ is the characteristic of a line of any degree, to which a tangent is drawn at the point α, β, γ, using U in a manner correspondent to its previous signification to denote

$$\left(x - \frac{\alpha}{\gamma} z\right) \frac{d}{d\alpha} + \left(y - \frac{\beta}{\gamma} z\right) \frac{d}{d\beta},$$

and understanding by ω, $\omega(\alpha, \beta, \gamma)$, we have for determining the point of intersection, $\omega = 0$, $\chi(\omega) = 0$, $\chi^n(\omega) = 0$; and consequently, by aid of our theorem (3), we shall obtain

$$\omega = 0,$$

$$U(\omega) = 0,$$

$$U_n \cdot \omega + n \cdot U_{n-1} \cdot \omega + n \cdot \frac{n-1}{2} U_{n-2} \cdot \omega + \ldots + n \cdot \frac{n-1}{2} U_2 \cdot \omega = 0.$$

By means of the two latter equations, we obtain

$$\left(x - \frac{\alpha z}{\gamma}\right)^2 \cdot F\left(x - \frac{\alpha z}{\gamma}\right) = 0,$$

$$\left(y - \frac{\beta z}{\gamma}\right)^2 \cdot G\left(y - \frac{\beta z}{\gamma}\right) = 0,$$

where F and G are each of only $(n-2)$ dimensions, and serve to determine the intersections of the tangent with the curve, extraneous to the two coincident ones at the point of contact.

Again, suppose that ω is a function of any degree of any number of letters α, β, γ, &c., and that we have given $\omega = 0$, $\chi\omega = 0$, $\chi^2\omega = 0$, ... $\chi^m\omega = 0$; it is evident from our fundamental theorem that these equations may be replaced by

$$\omega = 0,\ U_1(\omega) = 0,\ U_2(\omega) = 0, \ldots, U_m(\omega) = 0;$$

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* Thus let z be a homogeneous function in x and y of ι dimensions, and let

$$\frac{dz}{dx}, \frac{dz}{dy}, \frac{d^2z}{dx^2}, \frac{d^2z}{dxdy}, \frac{d^2z}{dy^2},$$

be called p, q, r, s, t, we shall find

$$\left.\begin{matrix} r, & s, & p, \\ s, & t, & q, \\ p, & q, & \text{[illegible]} \end{matrix}\right\} = 0,$$

[illegible] $+ tp^2$

[illegible]

and consequently that the expulsion of $(m-1)$ letters, by a[illegible] of the last (m) of the given equations, will be attended by th[illegible] disappearance of m orders, or, in other words, the resulta[illegible] will be minus an order, I mean, will have one order le[illegible] than the number of letters remaining in it.

In applying to space conceptions the preceding theore[illegible] it will be convenient to use a general nomenclature for ge[illegible]metrical species of various dimensions.

Thus we may call a line a monotheme, a surface a dithem[illegible] the species beyond a tritheme, and so on, *ad infinitum.*

A system of points according to the same system of nome[illegible]clature would be called a kenotheme.

An n-theme has for its characteristic a homogeneo[illegible] function of $(n+2)$ letters.

Again, it will be convenient to give a general name to a[illegible] themes expressed by equations of the first degree. Righ[illegible] lines and planes agree in conveying an idea of levelness an[illegible] uniformity; they may both be said to be homalous. I sha[illegible] therefore employ the word homaloid to signify in gener[illegible] any theme of the first degree.

Now let $\omega(x, y, z \ldots\ldots t)$ be the characteristic to a[illegible] n-theme of the nth degree.

The number of letters $x, y, z \ldots\ldots t$ is $(n+2)$.

As usual, let ω represent $\omega(\alpha, \beta, \gamma \ldots\ldots \delta)$, and suppose

$$\omega = 0, \ \chi(\omega) = 0, \ \chi_2(\omega) = 0 \ \ldots\ldots \ \chi_n(\omega) = 0,$$

and consequently

$$U_1(\omega) = 0, \ \ U_2(\omega) = 0 \ \ldots\ldots \ U_n(\omega) = 0.$$

On eliminating $(n-1)$ letters between the n last equations, the resulting function will be of three letters but of only two orders, and of the $1.2.3 \ldots\ldots n$ degree. Hence we see tha[illegible] at every point of an n-theme of the nth degree, and lying in the tangent homaloid thereto, $1.2.3 \ldots\ldots n$ right lines may be drawn coinciding throughout with the n-theme.

Thus one right line can be drawn on each point of a lin[illegible] of the first order lying on the line; two right lines at each point of a surface of the second order lying on the surface; six right lines at each point of a hyperlocus of the thir[illegible] degree, and so forth.

It is obvious that a surface may be treated as the homa[illegible]loidal section of a tritheme, just as a plane curve may be regarded as a section of a surface. I shall proceed to shew upon this view, how we may obtain a theorem given by Mr. Salmon for surfaces of the third degree of a particular character from the law just laid do[illegible] cording to which

degree, *two* right lines can be drawn lying wholly upon th surface.*

The last geometrical application of the theorem (3) which shall make, refers to the equations employed by Mr. Salmo in No. XXI. (new series) of this *Journal*, to obtain the locus the points on any surface at which tangent lines can be draw passing through four consecutive points. I may remar in passing that these equations may be obtained by rathe simpler considerations than Mr. Salmon has employed to do, and without any reference to Joachimsthal's theorem for if we take ξ, η, ζ, θ, as the coordinates of any point one of the tangent lines above described, and if we take th first polar to the surface with this point as origin, three of the four original points will be found in such polar co secutive but distinct; and consequently in the second pola referred to the same origin, two will continue consecutiv but distinct, and consequently one will remain over in th third polar.

Hence writing the equation to the surface $\omega(x, y, z, t) =$ and using D to denote $\xi\dfrac{d}{dx} + \eta\dfrac{d}{dy} + \zeta\dfrac{d}{dz} + \theta\dfrac{d}{dt}$, we sh evidently have

$$\omega = 0 \ldots\ldots\ldots\ldots\ldots(1),$$

$$D(\omega) = 0 \ldots\ldots\ldots\ldots\ldots(2),$$

$$D^2(\omega) = 0 \ldots\ldots\ldots\ldots\ldots(3),$$

$$D^3(\omega) = 0 \ldots\ldots\ldots\ldots\ldots(4),$$

as obtained by Mr. Salmon. And the same kind of reasoning precisely applies to the theory of points of inflexion in curves three consecutive points in right line order in this case cor responding to four such in the case above considered.

If now we make
$$\xi - \frac{x}{t}\theta = u,$$

$$\eta - \frac{y}{t}\theta = v,$$

$$\zeta - \frac{z}{t}\theta = w,$$

the equations (2), (3), (4), by our theorem may be expressed

* If we have an indeterminate system of algebraical equations consisting of one quadratic and another n^c function of three variables, this may be completely resolved by considering the first as an equation to a surface of the second degree, finding at any point ther[...] the two lines which lie upon the surface, and determining their re[...] intersections with

$\iota_1\ \iota_2 \ldots\ldots \iota_n$ without elevation of degree; by a linear solution being understood a solution under the form

$$\begin{aligned} a_1 &= \alpha_1 + \lambda\beta_1, \\ a_2 &= \alpha_2 + \lambda\beta_2, \\ &\ldots\ldots\ldots\ldots \\ a_r &= \alpha_r + \lambda\beta_r, \end{aligned}$$

where λ is left indeterminate.

Let us suppose that $\alpha_1, \alpha_2 \ldots \alpha_r$, substituted respectively for $a_1, a_2 \ldots a_r$, satisfy the given system of equations. The determination of these values without elevation of degree will, from what has been said before, depend upon the linear solution of a system of equations differing from the given system by the omission of any one of them at pleasure.

Now make $D = \alpha_1 \dfrac{d}{da_1} + \alpha_2 . \dfrac{d}{da_2} + \ldots + \alpha_r . \dfrac{d}{da_r}$,

and then write

$$\left.\begin{aligned} D\phi_1 = 0, \quad D^2\phi_1 &= 0 \ldots D^{\iota_1} . \phi_1 = 0 \\ D\phi_2 = 0, \quad D^2\phi_2 &= 0 \ldots D^{\iota_2} . \phi_2 = 0 \\ \ldots\ldots\ldots\ldots&\ldots\ldots\ldots\ldots \\ D\phi_r = 0, \quad D^2\phi_r &= 0 \ldots D^{\iota_n} . \phi_n = 0 \end{aligned}\right\} (\theta).$$

The values of $a_1\ a_2 \ldots\ldots a_r$ derived from this system, say $(a_1)\ (a_2) \ldots\ldots (a)_r$,

give $\quad a_1 = \alpha_1 + \lambda(a)_1\ a_2 = \alpha_2 + \lambda(a)_2 \ldots a_r = \quad + \lambda(a)_r$,

a solution under the required form where λ is left indeterminate.

The solution of this new system without elevation of degree depends on the *linear solution* of all but one of them; this excepted one may be taken the one whose dimensions ι_r are the highest or as high as any of the quantities $\iota_1\ \iota_2 \ldots \iota_r$.

Consequently, if we use the symbol $(k_1\ k_2 \ldots k_r)$ to denote the number of letters required for the linear solution (without elevation of degree) of k_1 equations of the first degree, k_2 of the second, k_3 of the third...., k_r of the r^{th}, it would at first sight appear from the preceding reduction that we must have

$$(k_1 k_2 \ldots k_r) = \{K_1 K_2 \ldots K_{r-1} K'_r\};$$

where

$$\begin{aligned} K_1 &= k_1 + k_2 + \ldots + k_{r-1} + k_r, \\ K_2 &= k_2 + \ldots + k_{r-1} + k_r, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ K_{r-1} &= k_{r-1} + k_r, \end{aligned}$$

But now steps in our theorem (3), and shews that the system (θ) may be superseded by another, in which the variables, instead of being $a_1\, a_2 \ldots a_n$, will be

$$a_1 - \frac{a_1}{a_n}\, a_n, \quad a_2 - \frac{a_2}{a_n}\,.\, a_n \ldots a_{n-1} - \frac{a_{n-1}}{a_n}\,.\, a_n\,;$$

consequently the number of really independent variables is only $(n-1)$; we must therefore have

$$(k_1\, k_2 \ldots k_r) = 1 + \{K_1\, K_2 \ldots K_r'\}.$$

Since the introduction of a new simple equation is equivalent to the requirement of one more disposable letter, we may write the above more symmetrically under the form

$$(k_1\, k_2 \ldots k_r) = ('K_1\, K_2 \ldots K_{r-1} K_r'),$$

where

$$'K_1 = 1 + k_1 + k_2 + \ldots + k_r,$$
$$K_r' = \qquad\qquad k_r - 1.$$

By means of this formula of reduction $k_1\, k_2 \ldots k_r$ may be finally brought down to the form (L), and the value of (L) being the number of letters required for the linear solution of a system of L linear equations is evidently $L + 2$.

Thus, to determine the number of letters required for the linear solution of a single quadratic, we write

$$(0,\, 1) = (2) = 4.$$

For two quadratics, we write

$$(0,\, 2) = (3,\, 1) = (5) = 7\,;$$

for a quadratic and a cubic,

$$(0,\, 1,\, 1) = (3,\, 2) = (6,\, 1) = (8) = 10\,;$$

for two cubics,

$$(0,\, 0,\, 2) = (3,\, 2,\, 1) = (7,\, 3) = (11,\, 2) = (14,\, 1)$$
$$= (16) = 18.$$

These results coincide with those obtained by Sir William Hamilton in his Report on Mr. Gerard's Transformation of the Equation of the Fifth Degree in the *Transactions* of the British Association. I have much more to say on the subject of the linear solution of a system of indeterminate equations, and am, I believe, able to present the subject in a more general light than has hitherto been done; but my observations on this matter must be deferred until a subsequent communication.

26, *Lincoln's Inn Fields*,
August 26, 1850.

ON THE INTERSECTIONS OF TWO CONICS.

(In completion of a former Paper in the Journal.)

By J. J. Sylvester, M.A., F.R.S.

Let the two conics be written

$$U = ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy = 0,$$

$$V = \alpha x^2 + \beta y^2 + \gamma z^2 + 2\alpha' yz + 2\beta' zx + 2\gamma' xy = 0,$$

and make

$$U + \lambda V = Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy.$$

In my paper in the last number of the *Journal*, I shewe that the case of intersection of the two conics in two point was distinguishable from all other cases by the equatio $\square(U + \lambda V) = 0$ having two imaginary roots. When a the roots are real, the curves either intersect in four poin or not at all.

On the former supposition,

$$-C'^2 + AB, \quad -A'^2 + BC, \quad -B'^2 + CA,$$

which are quadratic functions of λ, will be negative for a three values of λ. On the contrary supposition, one valu of λ will make all these three quantities negative, but th other two values with each make them all three positive.

Hence we obtain a symmetrical criterion (which I strangel omitted to state in my former paper) by forming the quanti $A'^2 + B'^2 + C'^2 - AB - AC - BC$.

A cubic equation

$$Ly^3 + My^2 + Ny + P = 0$$

may be then constructed, of which the three values of th above function corresponding to three values of λ will b the roots.

The condition for *real* intersection is that $L, M, N,$. should be all of the same sign. The conics being suppose real, L and P are necessarily in both cases of the same sig The intersection is therefore satisfied if either $L, M, N,$ o M, N, P be of the same sign, and is consequently equivale to the condition that $\frac{M}{L}$ and $\frac{N}{L}$ shall be both positiv or $\frac{N}{P}$ and $\frac{M}{P}$ both positive. It does not appear to be possib in the nature of the question to find a criterion for disti guishing between the two cases, dependent on the sign one single function of the coefficients.

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* The solution is thus ... completely symmetrical, as was desired.

The case of double contact, abstraction being made of binary intersection, is a sort of intermediary state between intersection in four points and non-intersection; and accordingly, as shewn in my former paper for this case, the two equal values of λ will make the three quantities

$$AB - C'^2, \quad BC - A'^2, \quad CA - B'^2$$

all real; so that two of the values of y corresponding to the equal values of λ are zero, and the criterion becomes nugatory as it ought to do.

Again, when the two conics do not intersect, I distinguished two cases according as they lie each without, or one within the other, *i.e.* according as they have four common tangents or none.

But, as Mr. Cayley has well remarked to me, a similar distinction exists when the conics intersect in four points; in that case also they may have four common tangents or not any: when they intersect in two points they have necessarily two and only two common tangents There is no difficulty in separating these four cases.

Let the conics be written

$$(U) = \xi^2 + \eta^2 - \zeta^2,$$
$$(V) = A\xi^2 + B\eta^2 - C\zeta^2,$$

(U) and (V) being what U and V become when the coordinates are changed from x, y, z, to ξ, η, ζ.

A, B, C are the three values of λ in the equation

$$\square(V - \lambda U) = 0.$$

If the curves intersect $A - C$, $B - C$ must have different signs, *i.e.* C must be an intermediary quantity between A and B.

Again, the tangential equations to the conics expressed by the correlative system of coordinates will be

$$\xi_1^2 + \eta_1^2 - \zeta_1^2 = 0,$$
$$\frac{\xi_1^2}{A} + \frac{\eta_1^2}{B} - \frac{\zeta_1^2}{C} = 0;$$

and that these may have four real systems of roots,

$$\frac{1}{A} - \frac{1}{C}, \quad \frac{1}{C} - \frac{1}{B}$$

must have the same sign; and consequently, as $A - C$ and $C - B$ are supposed to have the same sign, A and B, and therefore all three A, B, C, have the same sign. We have therefore the following rule:

Let the equation in λ, viz. $\square(U+\lambda V)=0$ be called $\theta=0$, and the equation in y, above given, $\omega=0$. By an equation being congruent or incongruent, understand that its roots have all the same sign or not all the same sign.

Then ω congruent, θ congruent, implies that the intersections and common tangents are both real; ω congruent, θ incongruent, implies that the intersections are real, but the common tangents imaginary; ω incongruent, θ congruent, implies that the intersections and common tangents are both imaginary; ω incongruent, θ congruent, implies that the intersections are imaginary, but the common tangents real.

In like manner, as the cases of contact of lines are limiting cases to those which relate to the relative configurations of their points of intersection, so the cases of contact of surfaces are limiting cases in which the characters which usually separate the different forms of their curve of intersection exist blended and indistinguishable. The first step therefore to the study of the particular species of the curve of the fourth degree,* in which two surfaces of the second degree intersect, is to obtain the analytical and geometrical characters of their various species of contact. Accordingly I have made an enumeration of these different species, no less than 12 in number, many of them highly curious and I believe unsuspected, which the reader may consult in the *Philosophical Magazine* for February, 1851.

By the aid of these landmarks, I have little doubt, should time and leisure permit, of mapping out a natural arrangement of the principal distinctions of form between that class at least of lines in space of the fourth order which admit of being considered the complete intersection of two surfaces.

26, *Lincoln's Inn Fields,*
7th Dec. 1850.

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* I have found that the 16 points of spherical flexure in this curve are the four sets of four points in which it meets the four faces of the pyramid whose summits are the vertices of the four cones of the second degree in which the curve is completely contained, which 16 points reduce to 4 when the two surfaces have an ordinary contact, and to 1 when they have a cuspidal contact: of course in the case of contact the pyramid above described in a manner folds up and vanishes, as there are no longer 4 distinct vertices. I have found also that when the factors of $\square(U+\lambda V)$, (U and V being the characteristics of the two surfaces) are all unreal, the points of flexure are all unreal. When two factors are real and two imaginary, two of the faces of the pyramid (viz. its two real faces) will each contain one (and only one) pair of real points of flexure, and the other two planes none; and lastly, when the factors of $\square(U+\lambda V)$ are all real, then either all the points of flexure are imaginary, or else all the eight contained in a certain two of the pyramidal faces are real: and these two cases admit of being

ON CERTAIN "LOCI" CONNECTED WITH THE GEODESIC LINES OF SURFACES OF THE SECOND ORDER.

By John Y. Rutledge.

The present article, in addition to several theorems believed to be new, contains the demonstration of a very general theorem, which I indicated in a former number of the *Journal* (vol. v. p. 87). In the paper referred to, a particular case of the theorem under consideration was discussed, and the results proved to be neither uninteresting nor unimportant. Whatever, therefore, be the intrinsic interest and importance which the subject may be thought to possess, they will, I hope, be considerably increased by the results obtained in the following investigation.

When at the point of intersection (x, y, z) of three confocal surfaces, whose semi-major axes are (ρ, μ, ν), the three normals are drawn, and from this point a right line is drawn common tangent to any two confocal surfaces, whose semi-major axes are (α, α'), then the values of the cosines of the angles $(\iota\ \iota_,\ \iota_{,,})$, made by the common tangent chord with the normals, are (vol. v. p. 80),

$$\cos\iota = \frac{\xi.\sqrt{(\rho^2-\alpha^2)}.\sqrt{(\rho^2-\alpha'^2)}}{\rho.\sqrt{(\rho^2-b^2)}.\sqrt{(\rho^2-c^2)}},$$

$$\cos\iota_, = \frac{\eta.\sqrt{(\mu^2-\alpha^2)}.\sqrt{(\mu^2-\alpha'^2)}}{\mu.\sqrt{(\mu^2-b^2)}.\sqrt{(\mu^2-c^2)}},$$

$$\cos\iota_{,,} = \frac{\zeta.\sqrt{(\nu^2-\alpha^2)}.\sqrt{(\nu^2-\alpha'^2)}}{\nu.\sqrt{(\nu^2-b^2)}.\sqrt{(\nu^2-c^2)}};$$

where ξ, η, ζ are the coordinates of the common centre of the system of confocal surfaces referred to the three normals as axes of coordinates. Attending to the known values of ξ, η, ζ, it is easy to see that the following equation is true:

$$\rho^2\cos^2\iota + \mu^2\cos^2\iota_, + \nu^2\cos^2\iota_{,,}$$
$$= \rho^2 + \mu^2 + \nu^2 - \alpha^2 - \alpha'^2 \ldots\ldots\ldots\ldots (1).$$

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distinguished by a method analogous in its general features to that whereby I have shewn in the text above how to distinguish between the cases of 4 real and 4 imaginary points of intersection of two conics. Where the two surfaces have an ordinary contact, the curve of intersection, it is well known, has a double point; and where the surfaces have a higher contact, the curve has a cusp. Thus in the fact of the 16 flexures reducing to 4 and to 1 in these respective cases, we see a beautiful analogy to what takes place with the 9 flexures of a plane curve of the third degree, which [reduce] to 3 and 1, according as the curve has a double point or a cusp.

confocal surfaces μ and ν let the projected right line form the angles (ι', ι''), and it is easy to see that

$$\cos\iota' = \frac{\cos\iota_{\prime}}{\sin\iota}, \quad \cos\iota'' = \frac{\cos\iota_{\prime\prime}}{\sin\iota}.$$

From equation (1) we can now without any difficulty derive

$$(\rho^2 + \mu^2 \sin^2\iota' + \nu^2 \sin^2\iota'') \sin^2\iota + (\mu^2 + \nu^2) \cos^2\iota = a^2 + a'^2.$$

If we now write $\quad \mu^2 \sin^2\iota' + \nu^2 \sin^2\iota'' = \gamma^2,$

it is a well-known theorem that γ represents the semi-major axis of the particular confocal surface of the second order touched by the projected right line; since it is evidently tangent at the point (x, y, z) or (ρ, μ, ν) to a geodesic line upon the surface of the ellipsoid. We consequently have the interesting equation

$$(\rho^2 + \gamma^2) \sin^2\iota + (\mu^2 + \nu^2) \cos^2\iota = a^2 + a'^2 \ldots\ldots (7).$$

At every point of the ellipsoid the semi-major axes of the confocal surfaces, analogous to β and γ, are connected by the equations (3) and (7), from which we can easily deduce

$$(\rho^2 + \gamma^2 - \mu^2 - \nu^2) \sin^2\iota = \rho^2 - \beta^2 \ldots\ldots\ldots (8).$$

The confocal surfaces β and γ are, in general, variable from point to point of the ellipsoid: in two particular cases, however, as may be easily inferred from what has been already established, one surface becomes fixed while the other continues variable; viz. if we restrict the point (ρ, μ, ν) to a sphero-conic, β becomes fixed while γ varies; on the other hand, the point being restricted to a geodesic line, γ becomes fixed, while the surface β continues variable. In general, from any external point (x, y, z) four right lines can be drawn common tangents to any two confocal surfaces a and a': for the sake of perspicuity we will denote the right lines in their order by $\lambda, \lambda', \lambda'', \lambda'''$. In this arrangement the right lines λ and λ'' are opposite to each other, as are also the right lines λ' and λ''', while the right lines λ, λ' and λ'', λ''' are respectively adjacent to each other. Through the four right lines taken two by two, six planes can be drawn, which two by two intersect in the three normals to the three confocal surfaces (ρ, μ, ν), which intersect in the point (x, y, z). For instance we suppose the planes $\lambda\lambda''$ and $\lambda'\lambda'''$ to intersect in the normal to the ellipsoid ρ, the planes $\lambda\lambda'$ and $\lambda''\lambda'''$ to intersect in the normal to the hyperboloid μ, and the planes $\lambda\lambda'''$ and $\lambda'\lambda''$ to intersect in the normal to the hyperboloid ν. The equation (7) determines (γ), the semi-major axis of the

confocal surface which touches either of the planes $\lambda\lambda''$ or $\lambda'\lambda'''$. If we denote by γ' and γ'' the semi-major axes of the confocal surfaces, which touch the planes $\lambda\lambda$ or $\lambda''\lambda'$ and $\lambda\lambda$ or $\lambda'\lambda''$—since the reasoning in the case of the ellipsoid is in every respect applicable to the two hyperboloids—we shall have the following equations to determine γ' and γ'',

$$\left.\begin{array}{l}(\mu^2 + \gamma'^2) \sin^2\iota_, + (\rho^2 + \nu^2)\cos^2\iota_, = a^2 + a'^2 \\ (\nu^2 + \gamma''^2)\sin^2\iota_{,,} + (\rho^2 + \mu^2)\cos^2\iota_{,,} = a^2 + a'^2\end{array}\right\} \ldots\ldots(9).$$

From these equations, combined with the equation (7), we can deduce an expression connecting the semi-major axes of the confocal surfaces γ, γ', γ'', viz.

$$\gamma^2 \sin^2\iota + \gamma'^2 \sin^2\iota_, + \gamma''^2 \sin^2\iota_{,,} = a^2 + a'^2 \ldots .(10);$$

the analytical statement of a remarkable theorem, as may be easily inferred from its similarity to the equation (2), which has been demonstrated at the commencement of the present article. Hence we may also obtain

$$(\rho^2 - \gamma^2)\sin^2\iota + (\mu^2 - \gamma'^2)\sin^2\iota_, + (\nu^2 - \gamma''^2)\sin^2\iota_{,,} = 0 \ldots(11).$$

Through the point (x, y, z), four planes can be drawn normal each to one of the four right lines λ, λ', λ'', λ''', and which touch one and the same confocal surface, whose semi-major axis (β suppose) is always given by the equation

$$\rho^2 + \mu^2 + \nu^2 = a^2 + a'^2 + \beta^2.$$

This is obvious from the consideration of the reasoning by which the equation was originally obtained.

We now perceive that there are in general *nine* confocal surfaces of the second order, related in a very striking manner, and whose semi-major axes are connected by the equations (2), (3), (7), (9) and (11). From the preceding expressions many interesting and important particular theorems may be obtained, but which it is needless to enumerate at present. In the equation (7) substituting for $\cos\iota$ and $\sin\iota$, their values; also for $\mu^2 + \nu^2$ its value obtained from the relation

$$\rho^2 + \mu^2 + \nu^2 = b^2 + c^2 + R^2,$$

where R is the radius-vector drawn from the common centre of the confocal system to the point of intersection of the three confocal surfaces ρ, μ, and ν. Then after some reductions we shall obtain the following equation,

$$[illegible]^2 - \gamma^2 = \rho^2 - a^2 + \rho^2 - a'^2 - \frac{\xi^2 . (\rho^2 - R^2 + a^2 - b^2 + a'^2 - c^2)}{(A^2 - \xi^2)} \ldots(12).$$

Indicating by (A^2) the expression

$$\frac{\rho^2 . (\rho^2 - b^2)(\rho^2 - c^2)}{(\rho^2 - a^2)(\rho^2 - a'^2)}.$$

Hence, attending to the equation (3), we obtain a relation between the semi-major axes of the confocal surfaces β and γ, viz.

$$\rho^2 - \gamma^2 = \rho^2 - a^2 + \rho^2 - a'^2 - \frac{\xi^2 (\rho^2 - \beta^2)}{(A^2 - \xi^2)} \ldots (13).$$

This is the general equation, of which a particular case has been already published, vol. v. p. 87. Similarly, if we write

$$B^2 = \frac{\mu^2 (\mu^2 - b^2)(\mu^2 - c^2)}{(\mu^2 - a^2)(\mu^2 - a'^2)}, \quad C^2 = \frac{\nu^2 . (\nu^2 - b^2)(\nu^2 - c^2)}{(\nu^2 - a^2)(\nu^2 - a'^2)},$$

we can obtain equations connecting the semi-major axes of the confocal surfaces β, γ' and γ'', viz.

$$\mu^2 - \gamma'^2 = (\mu^2 - a^2 + \mu^2 - a'^2) - \frac{\eta^2 (\mu^2 - \beta^2)}{(B^2 - \eta^2)} \ldots (14),$$

and $$\nu^2 - \gamma''^2 = \nu^2 - a^2 + \nu^2 - a'^2 - \frac{\xi^2 (\nu^2 - \beta^2)}{(C^2 - \xi^2)} \ldots (15).$$

At the point of intersection of the confocal surfaces γ, γ', and γ'', let a plane be erected normal to a chord drawn from that point common tangent to the two confocal surfaces whose semi-major axes are a and a'. Then if β' denote the semi-major axis of the particular confocal surface which touches the plane just mentioned, from the equations (13), (14), and (15) we can find the interesting relation

$$\frac{\xi^2 (\rho^2 - \beta^2)}{(A^2 - \xi^2)} + \frac{\eta^2 (\mu^2 - \beta^2)}{(B^2 - \eta^2)} + \frac{\zeta^2 (\nu^2 - \beta^2)}{(C^2 - \zeta^2)}$$
$$= \beta^2 - a^2 + \beta'^2 - a'^2 \ldots\ldots\ldots (16).$$

In all that follows we shall, for the sake of brevity, restrict our investigations to the case of the ellipsoid (ρ), since the results thus obtained manifestly hold, with but slight modification, for the hyperboloids μ and ν. Through any point (x, y, z) of the ellipsoid, an infinity of geodesic lines may be supposed to pass, since in the normal drawn to the ellipsoid at this point an infinite number of osculating planes may be conceived to intersect. Now if from the point (x, y, z) four common tangent chords be drawn to the confocal surfaces, whose semi-major axes are a and a', they will determine two osculating planes which intersect the surface of the ellipsoid in the elements of two geodesic lines whose tangents touch

the same confocal surface. Let γ represent the semi-major axis of this latter surface, and it can be completely determined by the equation (12). Since, however,

$$R^2 = x^2 + y^2 + z^2, \quad \frac{1}{\xi^2} = \frac{x^2}{\rho^4} + \frac{y^2}{(\rho^2 - b^2)^2} + \frac{z^2}{(\rho^2 - c^2)^2},$$

we can determine γ in terms of the coordinates x, y, and z; hence we may obtain some important results. Suppose then that we are given any two determinate confocal surfaces a and a', and we select any point (x, y, z) upon a given confocal ellipsoid ρ. Through this point, in general, two geodesic lines pass, whose osculating planes contain each a pair of chords drawn from this point common tangents to the confocal surfaces a and a'. The tangents to the geodesic lines at the point (x, y, z) touch the same confocal surface, whose intersection with the given ellipsoid (ρ) forms the line of curvature which the geodesic lines themselves touch. If γ be the semi-major axis of this surface, by the equation (12) it can be determined in terms of the coordinates (x, y, z). If we next conceive a and a' to vary, it is manifest that we can determine all the geodesic lines which intersect in any point (x, y, z) or (ρ, μ, ν); while at the same time it is evident that a and a' vary from the focal curves to the hyperboloids μ and ν, which determine upon the surface of the ellipsoid the lines of curvature intersecting in the point in question. Many curious questions here remain to be discussed, which arise as we pass from osculating plane to osculating plane, according as we suppose that *one* alone of the surfaces a and a' varies, or we introduce the consideration of a simultaneous variation of both. From point to point of the ellipsoid (ρ), we in general obtain a different γ, since the equation (12) determines, not the γ of *any* geodesic line passing through the point, but only the γ of that particular geodesic line whose osculating plane at that point contains a pair of chords common tangents to the confocal surfaces a and a'. If, however, we conceive γ constant, we can readily determine upon all the geodesic lines which touch the *same* line of curvature, formed by the intersection of the confocal surfaces γ and ρ, a locus at every point of which the osculating planes to the respective geodesic lines contain pairs of common tangent chords to the fixed confocal surfaces a and a'. The equation of this locus will evidently be found by substituting in terms of x, y, and z the values of R and ξ in the equation (12), and finding the intersection of the surface of the second *order so determined* with the given

ellipsoid ρ. Let

$$M^2 = A^2(\gamma^2 + \rho^2 - a^2 - a'^2),$$
$$N^2 = \gamma^2 + \rho^2 - b^2 + \rho^2 - c^2,$$

and we shall have for the equation of the preceding locus,

$$x^2 . \left(\frac{M^2}{\rho^4} + \frac{\rho^2 - N^2}{\rho^2}\right) + y^2 . \left\{\frac{M^2}{(\rho^2 - b^2)^2} + \frac{\rho^2 - b^2 - N^2}{\rho^2 - b^2}\right\}$$
$$+ z^2 . \left\{\frac{M^2}{(\rho^2 - c^2)^2} + \frac{\rho^2 - c^2 - N^2}{\rho^2 - c^2}\right\} = 0 \; \ldots . \; (17).$$

We have consequently demonstrated the following theorem. *The locus of the points upon the surface of an ellipsoid at which the osculating planes touch a fixed confocal surface, and contain each a pair of chords common tangents to any two given confocal surfaces, is a curve of the second degree.*

The equation of the preceding curve projected upon the principal plane (yz), if we write

$$F^2 = \rho^2 . \{(\gamma^2 - a^2)(\gamma^2 - a'^2) - (\gamma^2 - b^2)(\gamma^2 - c^2)\},$$
$$G^2 = - b^2c^2(\gamma^2 - a^2 - a'^2) + a^2a'^2(\gamma^2 - b^2 - c^2),$$

may be found to be

$$b^2y^2 . \frac{(\rho^2 - c^2)(\gamma^2 - c^2) - (a^2 - c^2)(a'^2 - c^2)}{\rho^2 - b^2}$$
$$+ c^2z^2 . \frac{(\rho^2 - b^2)(\gamma^2 - b^2) - (a^2 - b^2)(a'^2 - b^2)}{\rho^2 - c^2}$$
$$= F^2 + G^2 \ldots\ldots\ldots\ldots\ldots\ldots (18).$$

Along the curve *locus*, determined by (17), let tangent planes be drawn to the ellipsoid, and perpendiculars upon them let fall from the centre; the perpendiculars will describe a cone of the second degree. Let us write

$$P^2 = \rho^4(\gamma^2 - a^2)(\gamma^2 - a'^2) - \rho^4(\gamma^2 - b^2)(\gamma^2 - c^2),$$
$$Q^2 = a^2a'^2(\rho^2 - b^2)(\rho^2 - c^2) - \rho^2(\rho^2 - \gamma^2)(a^2a'^2 - b^2c^2);$$

and the equation of the preceding cone may be found to be

$$(x^2 + y^2 + z^2)(P^2 + Q^2)$$
$$= (\rho^2 - a^2)(\rho^2 - a'^2) . \{b^2c^2x^2 + \gamma^2(b^2y^2 + c^2z^2)\} \ldots (19).$$

We have now obtained the most general solution possible of the theorem under consideration, the further investigation of which we must for the present omit. Before we conclude, however, it is as- ... -esting particular

ses which arise from the bifocal chords of the ellipsoid. If e equation of the ellipsoid be

$$\frac{x^2}{\rho^2} + \frac{y^2}{\rho^2 - b^2} + \frac{z^2}{\rho^2 - c^2} = 1,$$

d a or a' become b or c, the corresponding confocal surface ill become, in the first case, the focal hyperbola

$$\frac{x^2}{b^2} + \frac{z^2}{b^2 - c^2} = 1 \ldots\ldots\ldots\ldots\ldots (a),$$

the second, the focal ellipse

$$\frac{x^2}{c^2} + \frac{y^2}{c^2 - b^2} = 1 \ldots\ldots\ldots\ldots\ldots (a').$$

Similarly let a or a' become zero, and the corresponding nfocal surface will have then degenerated into its imagiry focal curve, whose equation is

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = -1 \ldots\ldots\ldots\ldots\ldots (a'').$$

Let us first suppose that $a = b$ and $a' = c$, the equation (17) ill will then become

$$\frac{b^2 y^2}{(\gamma^2 - b^2)(\rho^2 - b^2)^2} + \frac{c^2 z^2}{(\gamma^2 - c^2)(\rho^2 - c^2)^2} = 0 \ldots\ldots (20),$$

hich is evidently identical with the equation of the proected curve (18), indicating, while γ varies between b and c, he equation of two right lines in the principal plane (yz). ence it is easy to perceive that, in the present instance, e locus (17) becomes two plane curves formed by the interction with the ellipsoid of two planes which pass through e right lines (20) and intersect in the major axis of the rface. We consequently obtain the following theorem. *he locus of the points upon the surface of an ellipsoid, at hich the osculating planes contain pairs of bifocal chords d touch one and the same confocal surface, is a plane curve.* et γ be the semi-major axis of the confocal surface touched y the osculating planes, and writing

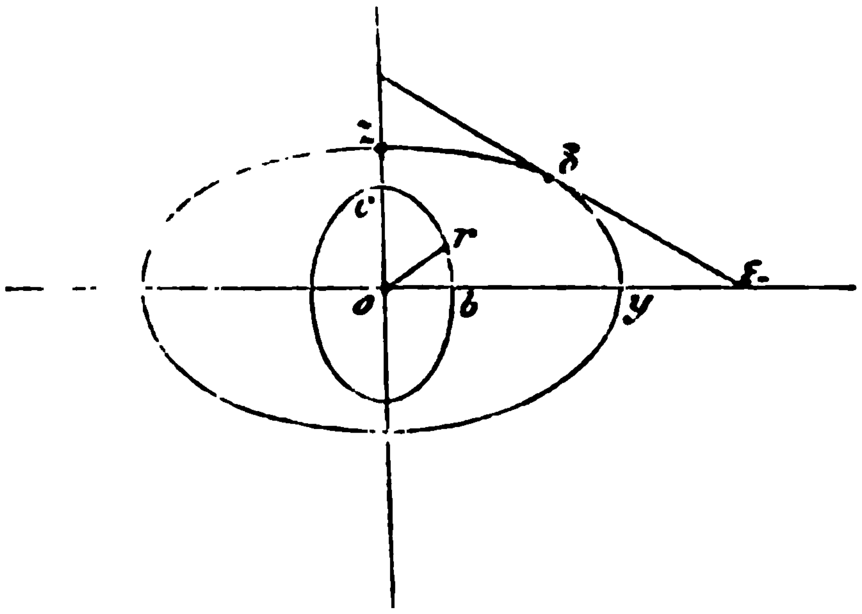
$$\tan^2\sigma = \frac{y^2}{z^2} \cdot \frac{(\rho^2 - c^2)^2}{(\rho^2 - b^2)^2},$$

e shall find from equation (20)

$$\frac{1}{\gamma^2} = \frac{\cos^2\sigma}{b^2} + \frac{\sin^2\sigma}{c^2}.$$

If, therefore, in the plane (yz) we construct an ellipse, conjugate of the imaginary focal curve a'' in the same principal plane, γ may be obtained by the following construction.

Let (oyz) represent the elliptic section in the plane (yz) and (obc) the constructed ellipse, conjugate of the imaginary focal curve in the same plane. Let us conceive δ to be the point in which the plane of the preceding locus intersects the principal ellipse, and at this point let a tangent $(\delta\epsilon)$ be drawn making with the axis of y the angle σ (suppose); then if a radius-vector (or) be drawn making with the axis b an angle equal to σ, the required semi-major axis γ will



be given by (or). It is manifest that γ varies between the limits b and c, and consequently the required confocal surface, whose equation is

$$\frac{x^2}{\gamma^2} + \frac{y^2}{\gamma^2 - b^2} + \frac{z^2}{\gamma^2 - c^2} = 1,$$

is always a hyperboloid of one sheet. This surface becomes identical with the focal hyperbola when γ is equal to b; when γ is equal to c, the surface becomes identical with the focal ellipse. All this is obviously as it ought to be, since, as may be easily seen, from *every* point of either focal curve may be drawn two bifocal chords, contained, as the case may be, by the principal plane corresponding to either the focal hyperbola or focal ellipse.

Let us next conceive that $a = b$ and $a' = o$, and without much difficulty we shall find that the equation (17) becomes

$$\frac{x^2}{\rho^4} c^2(\gamma^2 - b^2) + \frac{y^2}{(\rho^2 - b^2)^2} \gamma^2 (c^2 - b^2) = 0 \ldots\ldots(21);$$

which, if γ be less than b, [illegible] two planes intersecting in the least axis of [illegible]

If we now write
$$\tan^2\sigma = \frac{y^2}{x^2} \cdot \frac{\rho^4}{(\rho^2 - b^2)^2},$$
we shall find for the determination of γ,
$$\frac{\cos^2\sigma}{\gamma^2} = \frac{1}{b^2} - \frac{\sin^2\sigma}{c^2}.$$

This is the polar equation of a curve of the fourth degree whose radius-vector γ makes with the axis of y an angle equal to σ. The curve evidently intersects the axis of y in the origin of coordinates, and in two points whose distances from the origin are each equal to the semi-major axis of the focal hyperbola. It follows, therefore, that γ varies, in the present instance, between the limits zero and b; and that the confocal surface, of which γ is the semi-major axis, consequently is a hyperboloid of two sheets. The preceding curve of the fourth degree having been constructed in the principal plane (xy), it is easy to see that γ may be obtained by a geometrical construction similar to that indicated in the case already discussed, where a was supposed equal to b and a' equal to c. The equation (18), which in the general case represents the projection on the principal plane (yz) of the locus curve (17), becomes in the particular case under consideration,
$$\frac{z^2}{\rho^2 - c^2} + \frac{y^2}{\rho^2 - b^2} \cdot \frac{b^2}{c^2} \cdot \frac{\gamma^2(\rho^2 - c^2) - c^2(\rho^2 - b^2)}{(\gamma^2 - b^2)(\rho^2 - b^2)} = 1;$$
the equation of an ellipse, one of whose principal axes is equal to the least axis of the ellipsoid. The remaining case of the bifocal chords will be obtained by conceiving a equal c and a' equal zero, which will give for the determination of γ a curve of the fourth degree in the principal plane (xz). If we suppose σ to be the angle which any radius-vector makes with the axis of z, we shall have as the polar equation of the curve
$$\frac{\cos^2\sigma}{\gamma^2} = \frac{1}{c^2} - \frac{\sin^2\sigma}{b^2}.$$

It is easy to see that this curve intersects the axis of z in the origin of coordinates, and in two points whose distances from the origin are each equal to the semi-major axis of the focal ellipse. When σ becomes equal to the arc whose sign is $\pm\frac{b}{c}$, we perceive that γ becomes infinite, while at the same time it is manifest that γ can have no value equal to b. The properties of this curve we may *possibly* consider with greater

accuracy on some future occasion. Its equation in rectangular coordinates may readily be proved to be

$$z^2 \cdot \left\{\frac{z^2}{c^2} + \left(\frac{2}{c^2} - \frac{1}{b^2}\right) x^2 - 1\right\} = \left(\frac{1}{b^2} - \frac{1}{c^2}\right) x^4.$$

When γ varies between the limits zero and b, the confocal surface, of which γ is the semi-major axis, is a hyperboloid of *two* sheets; when it varies between the limits b and c, the confocal surface will be a hyperboloid of *one* sheet; and when γ becomes greater than c, the confocal surface will of course be an ellipsoid. The projection of the locus curve upon the principal plane (yz) is in the present instance

$$\frac{y^2}{\rho^2 - b^2} + \frac{z^2}{\rho^2 - c^2} \cdot \frac{c^2}{b^2} \cdot \frac{\gamma^2(\rho^2 - b^2) - b^2(\rho^2 - c^2)}{(\gamma^2 - c^2)(\rho^2 - c^2)} = 1,$$

the equation of a curve of the second degree, one of whose principal axes is equal to the mean axis of the ellipsoid. In all other respects conclusions may obviously be obtained analogous to those found in the instance last discussed, viz. where a was supposed equal to b, and a' equal zero. It is, therefore, needless at present to consider the subject farther in detail.

July 24th, 1850.

TWO ARITHMETICAL THEOREMS.

By Henry Wilbraham, M.A., Fellow of Trinity College, Cambridge.

It is well known that if the sum of the digits of any number be divisible by 9, the number itself will be also divisible by 9; and that if the sum of the digits in the even places be subtracted from the sum of those in the odd places, then if this difference be divisible by 11 the number itself will be also divisible by 11: but it has, I believe, escaped general observation that these two theorems respectively are only the most simple applications of two more general theorems. The first of these more general theorems is as follows:—

Let m be any number not divisible by 2 or 5, and suppose $\frac{1}{m}$ when reduced to a circulating decimal to have a recurring period of p digits; and let N be any other number, in which the number of digits is greater than p: if N be marked off into periods of p digits each (beginning at the units), and

these several periods, each considered as a number consisting of p digits, be added together, and the sum be n; then, if n be divisible by m, N also will be divisible thereby; and if n be not so divisible, the remainder, on dividing n by m, will be the same as that found on dividing N by m; or in other words, n will be congruous to N with respect to the modulus m.

The proof of this theorem is very obvious; for it is easily seen that, if $\frac{1}{m}$ is a circulating decimal which recurs in p digits, m is a submultiple of $10^p - 1$, and that $10^p - 1$ is the least number of the form $10^n - 1$ which is divisible by m; hence $10^p = km + 1$, (k being some whole number). If now the several periods into which N is divided be (beginning at the units) $a_1, a_2, a_3, \ldots$

$$\begin{aligned} N &= a_1 + 10^p a_2 + 10^{2p} a_3 + \ldots \\ &= a_1 + (km + 1)\, a_2 + (km + 1)^2\, a_3 + \ldots \\ &= a_1 + a_2 + a_3 + \ldots + Km \\ &= n + Km, \end{aligned}$$

which proves the truth of the rule.

It is evident that we may if we please mark off N into periods consisting of any multiple of p instead of p digits.

As an instance let $m = 271$, and $N = 2990105702178$; $\frac{1}{271} = .\dot{0}036\dot{9}$, and so has a recurring period of five digits, therefore $p = 5$; we must therefore mark off the periods in N thus, 299 | 01057 | 02178, and add together 2178, 1057, and 299, the sum of which is 3534; therefore $n = 3534$. On dividing 3534 by 271, we find the remainder to be 11; and this we should find to be the remainder also on dividing N by 271.

Cor. If the number of digits in N be a multiple of p, and N be a multiple of m, then if any number of digits be taken off from the left-hand side of the number N, and put on at the right-hand side of it, every one of the numbers so formed will be a multiple of m. If the number of digits in N be not a multiple of p, it may be made to be so, without altering the value of N, by adding a sufficient number of 0's at the left-hand side.

Thus 97643 is a multiple of 37; $\frac{1}{37}$ has a recurring period of 3 digits; therefore, if we write 097643 for 97643, the number of digits in it will be a multiple of the number in the recurring period: we shall find that each of the numbers 309764, 430976, 643097, 764309 *is a multiple* of 37.

For, as 37 *is prime to 2 and 5, 9764300* will be divisible

by 37, if and only if 97643 is so divisible; and applyi
to 9764300 the rule in the preceding theorem, we see th
it is divisible by 37, if and only if 764309 is so divisib
Therefore, if one of the numbers 97643, 309764, 43097
643097, 764309 is divisible by 37, all of them must be so.

The other general theorem which I mentioned, and
which the common rule respecting the divisibility of a nu
ber by 11 is a particular case, is as follows:—

If m be a prime number, and p be an even number,
may be divided into periods of $\frac{1}{2}p$ digits each; and of the
periods, if the 1st, 3rd, 5th, &c. be added together, and al
the 2nd, 4th, 6th, &c., and the latter sum subtracted fro
the former, and the difference be called n; then, as in t
former theorem, n will yield the same remainder as N
division by m. If n be negative and not divisible by
the proper remainder is the difference between n and t
multiple of m next *above* it numerically.

For as m is a measure of $10^p - 1$, being a prime numb
it must be a measure either of $10^{\frac{1}{2}p} - 1$ or $10^{\frac{1}{2}p} + 1$, which a
the factors of $10^p - 1$. But it cannot be a measure
$10^{\frac{1}{2}p} - 1$, for as we have seen $10^p - 1$ is the least number
the form $10^x - 1$ which is divisible by m; so that m mu
be a measure of $10^{\frac{1}{2}p} + 1$. Hence $10^{\frac{1}{2}p} = km - 1$; and
the several periods beginning at the units be b_1, b_2, b_3, . . .

$$N = b_1 + 10^{\frac{1}{2}p} b_2 + 10^p b_3 + \dots$$

$$= b_1 + (km - 1) b_2 + (km - 1)^2 b_3 + \dots$$

$$= b_1 - b_2 + b_3 - \dots + Km$$

$$= n + Km,$$

which proves the truth of the theorem.

It is not in this last case absolutely necessary that m shoul
be a prime number, but it may be a composite number pr
vided that it be prime to 2 and 5, and that (m_1, m_2, . . . bein
its prime factors) none of the fractions $\frac{1}{m_1}$, $\frac{1}{m_2}$, ... wh
reduced to a circulating decimal have its recurring peri
consisting of $\frac{1}{2}p$ or any submultiple of $\frac{1}{2}p$ figures.

As an example let $m = 13$; therefore $p = 6$; and le
$N = 4137219$. Mark off N into periods of 3 digits eac
4 | 137 | 219; add together 219 and 4, and therefrom su
tract 137; the result is 86, which on division by 13 giv
emainder of 8, which is the same as that given by dividi
by 13.

It will be observed that every case to which the second theorem is applicable may also be treated under the first theorem, but not *vice versâ*. Thus if any number be marked off into periods of two digits, and these periods be all added together, the result, on being divided by 11, will yield the same remainder as the original number.

11, *Lincoln's Inn Fields,*
June 21*st*, 1850.

APPLICATION OF COMBINATIONS TO THE EXPLANATION OF ARBOGAST'S METHOD.

By Professor De Morgan.

In the No. for November 1846, I gave some remarks tending to connect Arbogast's results with more modern notions of operation, and to simplify their deduction. The present paper is intended to point out a simple, and I think I may say fundamental, mode of arriving at the law of derivation from elementary combinations.

An easy rule has been long given for forming the combinations of m things out of n, when *no* repetition is allowed. If as easy a rule had been investigated for the case of combinations in which repetition *is* allowed, the mode of expanding

$$(a + bx + cx^2 + \dots)^n$$

would have followed very simply, and from it the expansion of any function of a polynomial. As it has happened, this latter rule was discovered imbedded in an application, and must be separated and brought back to its proper place.

Let there be *seven* letters a, b, c, d, e, f, g, of which it is required to write down all the combinations of *six*, with and without repetition. Let the letters of a combination always be written in the order of the alphabet. The first we should naturally think of is *aaaaaa*, the last *gggggg*: required a mode of proceeding from the first to the last, through every possible combination, without either repetition or omission of any one combination.

Every combination, as *bbdfff*, consists of parcels, each parcel containing one or more of the same letter. Every combination may be brought back to *aaaaaa*, through others, in a manner which never gives any choice of steps, by the following simple rule: Change the *first* letter of the *last* parcel (or its only letter, *if there be but one*) into the letter

It follows that the equation

$$F(a+x) = Fa + \frac{1}{1}\,\frac{x}{f(a+x)}\,(F'afa) + \ldots . \quad (2)$$

must reduce itself to an identity when the two sides are expanded in powers of x; or writing for shortness F, f instead of Fa, fa, and δ for $\dfrac{d}{da}$, we must have

$$\frac{1}{[r]^r}\,\delta^r F = S\left\{\frac{1}{[p]^p}\,\delta^{p-1}(\delta F\,.\,f^p)\,.\,\frac{1}{[r-p]^{r-1}}\,\delta^{r-p}f^{-p}\right\} \ldots . (3),$$

(where p extends from 0 to r). Or what comes to the same,

$$\frac{1}{[r]^r}\,\delta^r F = S\left\{\frac{1}{p\,[p-s]^{p-s}\,[r-p]^{r-p}\,[s-1]^{s-1}}\,\delta^{p-s}f^p.\delta^{r-p}f^{-p}.\delta^s F\right\} \ldots\ldots\ldots\ldots (4),$$

where s extends from 0 to $(r-p)$. The terms on the two sides which involve $\delta^r F$ are immediately seen to be equal; the coefficients of the remaining terms $\delta^s F$ on the second side must vanish, or we must have

$$S\left\{\frac{1}{p\,[p-s]^{p-s}\,[r-p]^{r-p}}\,(\delta^{p-s}f^p)\,(\delta^{r-p}f^{-p})\right\} = 0 \ldots . (5),$$

(s being less than r). Or in a somewhat more convenient form, writing p, q and k for $p-s$, $r-p$ and $r-s$,

$$S\left\{\frac{1}{(p+s)\,[p]^p.[s]^s}\,(\delta^p f^{p+s})\,(\delta^q f^{-p-s}\right\} = 0 \ldots . (6),$$

where s is constant and p and q vary subject to $p+q=k$, k being a given constant different from zero (in the case where $k=0$, the series reduces itself to the single term $\dfrac{1}{s}$). The direct proof of this theorem will be given presently.

II.

The following symbolical form of Lagrange's theorem was given by me in the *Mathematical Journal*, vol. III. p. 283.

If $$x = a + hfx \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

then $$Fx = \left(\frac{d}{da}\right)^{h\frac{d}{dh}-1} F'ae^{hfa}.$$

Suppose $fx = \phi(b + k\psi x)$, or $x = a + h\phi(b + k\psi x)$, then

$$Fx = \left(\frac{d}{da}\right)^{h\frac{d}{dh}-1} F'ae^{h\phi(b+k\psi a)}.$$

is obvious that we must have the sum of all products every *permutation* of n letters out of the set $a, b, c, \ldots\ldots$, Consequently, $a^p b^q c^r d^s \ldots\ldots$ enters with the coefficient

$$\frac{[p+q+r+s+\ldots\ldots] \text{ or } [n]}{[p]\,[q]\,[r]\,[s]\ldots\ldots},$$

where $[0]$ means 1, and $[p]$ means $1.1.2.3.\ldots\ldots p$. If the last arcel be, say d^s, the change of the last term introduces $d^{s-1}e$: he denominator of the coefficient was $\ldots\ldots [s]$, and ought to ecome $\ldots\ldots [s-1]\,[1]$; that is, multiplication by s ought to ccompany the change of d^s into $d^{s-1}e$. If the two last arcels, being consecutive in letters, be $c^r d^s$, the change the last but one introduces $c^{r-1}d^{s+1}$: the denominator of the efficient was ... $[r]\,[s]$, and ought to become $[r-1]\,[s+1]$; at is, multiplication by r and division by $s+1$ ought to ccompany the change of $c^r d^s$ into $c^{r-1}d^{s+1}$. And thus the le for the formation of the derivatives of a^n is completely ablished.

The number of descents by which we pass from a^n to $b^q c^r d^s \ldots\ldots$ is $0.p + 1.q + 2.r + 3.s + \ldots\ldots$; whence, in ming $(a + bx + cx^2 + dx^3 + \ldots\ldots)^n$, the term in $a^p b^q c^r d^s \ldots\ldots$ part of the coefficient of $x^{q+2r+3s+\ldots}$. If each degree of scent be made to occupy one line, each line gives the efficient of one power of x. Thus, by the instance chosen ove, we see that

$$\begin{aligned}(a + bx + cx^2 + dx^3)^3 &= a^3 + 3a^2bx + (3a^2c + 3ab^2)\,x^2 \\ &+ (3a^2d + 6abc + b^3)\,x^3 + (6abd + 3ac^2 + 3b^2c)\,x^4 \\ &+ (6acd + 3b^2d + 3bc^2)\,x^5 + (3ad^2 + 6bcd + c^3)\,x^6 \\ &+ (3bd^2 + 3c^2d)\,x^7 + 3cd^2x^8 + d^3x^9.\end{aligned}$$

Hence Arbogast's law of formation of any integer power of polynomial may be made a part of the most elementary gebra.

NOTES ON LAGRANGE'S THEOREM.

By Arthur Cayley.

I.

If in the ordinary form of Lagrange's theorem we write + x) for x, it becomes

$$x = hf(a + x),$$

$$F(a + x) = Fa + \frac{h}{1}\,F'afa + \&c.\ \ldots\ldots\ldots(1).$$

where $m + n + \&c. = s$, and as before Fa, fa, $\frac{d}{da}$ have b
replaced by F, f, δ. By comparing the coefficients of δ^m

$$\frac{1}{[t]^t}\,\frac{s-t}{s}\,\delta^t.f^s = \Sigma\left\{\frac{1}{[k]^n\,[p]^p\ldots}\,(\delta^n f^{s-t})\,(\delta^p f^n.)\ldots.\right\}\ldots.(11$$

where $n + p + \ldots. = t$, the last of the series $n, p, \ldots.$ al
vanishing. The formula (10) deduced, as above mentio
from Taylor's theorem and the subsequent formula (11)
an independent demonstration of it, not I believe mater
different from that which will presently be given, ar
be found in a memoir by M. Collins (in the second vol
of the *Petersburgh Transactions*), who appears to have m
very extensive researches in the theory of developmen
connected with the combinatorial analysis.

III.

To demonstrate the formula (6), consider, in the first p
the expression

$$S\,\frac{\phi p}{[p]^p\,[q]^q}\,\{(\delta^p f^{p+s})\,(\delta^q f^{-p-s-\theta})\},$$

where $p + q = k$. Since

$$\frac{1}{[p]^p\,[q]^q} = \frac{1}{k}\left(\frac{1}{[p-1]^{p-1}\,[q]^q} + \frac{1}{[p]^p\,[q-1]^{q-1}}\right).$$

This is immediately transformed into

$$\frac{1}{k}\,S\phi p\,\left\{\frac{p+s}{[p-1]^{p-1}\,[q]^q}\,(\delta^{p-1}f^{p+s-1}\,\delta f)\,(\delta^q f^{-p-s-\theta})\right.$$

$$-\frac{(p+s+\theta)}{[p]^p\,[q-1]^{q-1}}\,(\delta^p f^{p+s})\,(\delta^{q-1}.\,f^{-p-s-\theta-1}$$

$$=\frac{1}{k}\,S\,\frac{1}{[p]^p\,[q]^q}\,\{\phi\,(p+1)\,(p+s+1)\,(\delta^p.\,f^{p+s}\delta f)\,(\delta^q f^{-p-}$$

$$-\phi p\,.\,(p+s+\theta)\,(\delta^p f^{p+s})\,(\delta^q.\,f^{-p-s-\theta-1}$$

in which last expression $p + q = (\rho - 1)$. Of this,
separating the factor δf, the general term is

$$\frac{1}{k}\,\frac{1}{[a]^a}\,\delta^{a+1}f.S\left\{\frac{1}{[p-a]^{p-a}\,[q]^q}\,\phi(p+1)(p+s+1)\,(\delta^{p-a}f^{p+s})\,(\delta^q f^{-}\right.$$

$$-\frac{1}{[p]^p\,[q-a]^{q-a}}\,\phi p\,(p+s+\theta)\,(\delta^p f^{p+s})\,(\delta^{q-a}f^{-p-s-\theta-}$$

equivalent to

$$\frac{1}{[\alpha]^\alpha}\,\delta^{\alpha+1}f.S\,\frac{1}{[p]^p\,[q]^q}$$

$$\{\phi(p+\alpha+1)(p+s+\alpha+1)(\delta^p f^{p+s+\alpha})(\delta^q f^{-p-s-\alpha-\theta-1})$$
$$-\,\phi p.(p+s+\theta)(\delta^p f^{p+s})(\delta^q f^{-p-s-\theta-1})\},$$

in which last expression $p+q=k-\alpha-1$. By repeating the reduction (j) times, the general term becomes

$$\frac{1}{k(k-\alpha-1)(k-\alpha-\beta-2)\dots}\,\frac{1}{[\alpha]^\alpha\,[\beta]^\beta\dots}\,\delta^{\alpha+1}f.\delta^{\beta+1}f\dots\times$$

$$\frac{1}{[p^p]\,[q]^q}\,\Sigma\{(-)^{j-j'}\phi(p+\alpha+\beta\dots+j')[p+s+\alpha+\beta\dots+j']^{j'}\times$$
$$[p+s+\theta+\alpha+\beta\dots+j-1]^{j-j'}(\delta^p f^{p+s+\alpha+\beta\dots})(\delta^q f^{-p-s-\theta-j-\alpha-\beta\dots})\},$$

where the sums $\alpha+\beta\dots$ contain j' terms, j' being less than j or equal to it, and Σ extends to all combinations of the quantities $\alpha,\beta\dots$ taken j' and j' together (so that the summation contains 2^j terms). Also $p+q=k-\alpha-\beta\dots(j$ terms$)-j$, and the products $k(k-\alpha-1)(k-\alpha-\beta-2)\dots$ and $[\alpha]^\alpha[\beta]^\beta,\dots$ $\delta^{\alpha+1}f\,\delta^{\beta+1}f\dots$ contain each of them j terms. Suppose the reduction continued until $k-\alpha-\beta\dots(j$ terms$)-j=0$, then the only values of p,q are $p=0$, $q=0$; and the general term of

$$S\,\frac{\phi p}{[p]^p\,[q]^q}\,\{(\delta^p f^{p+s})(\delta^q f^{-p-s-\theta})\}$$

becomes

$$\frac{1}{k(k-\alpha-1)(k-\alpha-\beta-2)\dots}\cdot\frac{1}{[\alpha]^\alpha\,[\beta]^\beta\dots}\,\delta^{\alpha+1}f.\delta^{\beta+1}f\dots f^{-s-j-\theta}\times$$

$$\Sigma\{(-)^{j-j'}\phi(\alpha+\beta\dots+j')[s+\alpha+\beta\dots+j']^{j'}[s+\theta+\alpha+\beta\dots+j-1]^{j-j'}\}.$$

If $\theta=0$, the general term reduces itself to

$$\frac{1}{k(k-\alpha-1)(k-\alpha-\beta-2)\dots}\,\frac{1}{[\alpha]^\alpha\,[\beta]^\beta\dots}\,\delta^{\alpha+1}f.\delta^{\beta+1}f\dots f^{-s-j}.$$

$$\Sigma\{(-)^{j-j'}(s+\alpha+\beta\dots+j').\phi(\alpha+\beta\dots+j').[s+\alpha+\beta\dots+j-1]^{j-1}\};$$

whence finally, if $\phi p=\dfrac{1}{p+s}$, the general term of

$$S\,\frac{1}{(p+s)[p]^p\,[q]^q}\,\{(\delta^p f^{p+s})(\delta^q f^{-p-s})\}$$

becomes

$$\frac{1}{k(k-\alpha-1)(k-\alpha-\beta-2)\dots}\,\frac{1}{[\alpha]^\alpha\,[\beta]^\beta\dots}$$
$$\delta^{\alpha+1}f.\delta^{\beta+1}f\dots f^{-s}\,\Sigma\{(-)^{j-j'}[s+\alpha+\beta\dots+j-1]^{j-1}\}.$$

And it is readily shewn that the sum contained in this formula vanishes, which proves the equation in question.

IV.

The demonstration of the equation (11) is much sim
We have

$$\delta^{t-1}(f^{s-1}\delta f) = \Sigma\left\{\frac{[t-1]^{n-1}}{[n-1]^{n-1}}\,\delta^{n-1}(f^{s-t-1}\delta f)\,.\,(\delta^{t-n}f^{t})\right\}$$

i.e. $$\delta^{t} f^{s} = \frac{s}{s-t}\,\Sigma\left\{\frac{[t-1]^{n-1}}{[n-1]^{n-1}}(\delta^{n} f^{s-t})(\delta^{t-n}f^{t})\right\},$$

where n extends from $n=1$ to $n=t$. Similarly

$$\delta^{t-n}f^{t} = \frac{t}{n}\,\Sigma\left\{\frac{[t-n-1]^{p-1}}{[p-1]^{p-1}}(\delta^{p} f^{n})(\delta^{t-n-p}f^{t-n})\right\},$$

$$\delta^{t-n-p}f^{t-n} = \frac{t-n}{p}\,\Sigma\left\{\frac{[t-n-p-1]^{q-1}}{[q-1]^{q-1}}(\delta^{q} f^{p})(\delta^{t-n-p-q}f^{t-n-p}\right.$$

&c.

Or substituting successively, and putting $t-n-p-q=r$

$$\delta^{t}.f^{s} = \frac{s}{s-t}\,\Sigma\left\{\frac{[t]^{t-r-1}}{[n]^{n}[p]^{p}(q+r)[q-1]^{q-1}}(\delta^{n} f^{s-t})(\delta^{p} f^{p})(\delta^{q} f\right.$$

&c. And the last of these corresponding to a zero
of the last of the quantities $n, p, q\ldots$ is evidently th
quired equation (11).

V.

The formula (18) in my paper on Lagrange's theo
(*Journal*, vol. III. p. 283) is incorrect. I propose at pre
after giving the proper form of the formula in question
develope the result of the substitution indicated at the
clusion of the paper. It will be convenient to call to m
the general theorem, that when any number of vari
$x, y, z\ldots$ are connected with as many other variables $u, v,$
by the same number of equations (so that the variabl
each set may be considered as functions of those of the o
set) the quotient of the expressions $dxdy\ldots$ and $dudv\ldots$
equal to the quotient of two determinants formed with
functions which equated to zero express the relations
tween the two sets of variables; the former with the
ferential coefficients of these functions with respect to $u,$
the latter with the differential coefficients with respec
$x, y\ldots$ Consequently the notation $\dfrac{dxdy\ldots}{dudv\ldots}$ may be
sidered as representing the quotient of these determin

This being premised, if we write

$$x - u - h\theta(x, y\dots) = 0,$$
$$y - v - k\phi(x, y\dots) = 0.$$

The formula in question is

$$F(x, y\dots)\,.\,\frac{dxdy\dots}{dudv\dots} = \delta_u^{\,h\delta_h}\delta_v^{\,k\delta_k}\dots e^{h\theta+k\phi\dots}F,$$

if for shortness the letters $\theta, \phi\dots F$ denote what the corre-sponding functions become when $u, v\dots$ are substituted for $x, y\dots$ Let $\frac{1}{\Delta}$ denote the value which $\frac{dxdy\dots}{dudv\dots}$, considered as a function of $x, y\dots$, assumes when these variables are changed into $u, v\dots$, we have

$$\nabla = \begin{vmatrix} 1 - h\delta_u\theta, & -h\delta_v\theta\dots \\ -k\delta_u\phi, & 1 - k\delta_v\phi\dots \\ \vdots & \end{vmatrix}$$

And by changing the function F, we obtain

$$F(x, y\dots) = \delta_u^{\,h\delta_h}\delta_v^{\,k\delta_k}\dots e^{h\theta+k\phi\dots}F\nabla;$$

where, however, it must be remembered that the $h, k\dots$, in so far as they enter into the function ∇, are not affected by the symbols $h\delta_h$, $k\delta_k\dots$ In order that we may consider them to be so affected, it is necessary in the function ∇ to replace h, k, &c. by $\frac{h}{\delta_u}$, $\frac{k}{\delta_v}$, &c. Also, after this is done, observing that the symbols $h\delta_u\theta$, $h\delta_v\theta\dots$ affect a function $e^{h\theta+k\phi+\dots}F$, the symbols $h\delta_u\theta$, $h\delta_v\theta\dots$ may be replaced by δ_u^{θ}, $\delta_v^{\theta}\dots$, where the θ is not an index but an affix, denoting that the differentiation is only to be performed with respect to $u, v\dots$ so far as these variables respectively enter into the function θ. Transforming the other lines of the determinant in the same manner, and taking out from $\delta_u^{\,h\delta_h}$, $\delta_v^{\,k\delta_k}\dots$ the factor $\delta_u\delta_v\dots$ in order to multiply this last factor into the determinant, we obtain

$$F(x, y\dots) = \delta_u^{\,h\delta_h - 1}\,.\,\delta_v^{\,k\delta_k - 1}\dots e^{h\theta+k\phi\dots}F\Box;$$

where

$$\Box = \begin{vmatrix} \delta_u - \delta_u^{\theta}, & -\delta_u^{\phi}\dots \\ -\delta_v^{\theta}, & \delta_v - \delta_v^{\phi} \\ \vdots & \end{vmatrix}$$

in which expression δ_u, $\delta_v\dots$ are to be replaced by

$$\delta_u^{\gamma} + \delta_u^{\theta} + \delta_u^{\phi}\dots \quad \delta_v^{\gamma} + \delta_v^{\theta} + \delta_v^{\phi}\dots$$

The complete expansion is easily arrived at by induction, and the form is somewhat singular. In the case of a single

variable u we have $\square = \delta_u{}^F$, in the case of two variables $\square = \delta_u{}^F \delta_v{}^F + \delta_u{}^F \delta_v{}^\theta + \delta_u{}^\phi \delta_v{}^F$. Or writing down only the affixes in the case of a single variable we have F; in the case of two variables FF, $F\theta$, ϕF; and in the case of three variables FFF, ϕFF, χFF $F\chi F$, $F\theta F$, $FF\theta$, $FF\phi$, $F\theta\theta$, $F\theta\phi$, $F\chi\theta$, $\phi F\phi$, $\chi F\phi$, $\phi F\theta$, $\chi\chi F$, $\phi\chi F$, $\chi\theta F$; where it will be observed that θ never occurs in the first place, nor ϕ in the second place, nor θ, ϕ (in any order) in the first and second places, &c., nor θ, ϕ, χ (in any order) in the first, second, and third places. And the same property holds in the general case for each letter and binary, ternary, &c. combination, and for the entire system of letters, and the system of affixes contains every possible combination of letters not excluded by the rule just given. Thus in the case of two letters, forming the system of affixes FF, $F\theta$, ϕF, θF, $F\phi$, $\theta\phi$, $\phi\theta$, the last four are excluded, the first three of them by containing θ in the first place or ϕ in the second place, the last by containing ϕ, θ in the first and second places: and there remains only the terms FF, $F\theta$, ϕF forming the system given above. Substituting the expanded value of $\square$ in the expression for $F(x, y...)$, the equation may either be permitted to remain in the form which it thus assumes, or we may, in order to obtain the finally reduced form, after expanding the powers of h, k..., connect the symbols $\delta_u{}^\theta$, $\delta_u{}^\phi ... \delta_u{}^F$, &c. with the corresponding functions θ, $\phi ... F$, and then omit the affixes; thus, in particular, in the case of a single variable the general term of Fx is

$$\frac{h^p}{[p]^p}\,\delta_u{}^{p-1}\,(\theta^p \delta_u F),$$

(the ordinary form of Lagrange's theorem). In the case of two letters the general term of $F(x, y)$ is

$$\frac{h^p k^q}{[p]^p\,[q]^q}\,\delta_u{}^{p-1}\delta_v{}^{q-1}\,\{\theta^p\phi^q\delta_u\delta_v F + \phi^q\delta_v\theta^p\delta_u F + \theta^p\delta_u\phi^q\delta_v F\},$$

(see the *Mécanique Céleste*, tom. I. p. 176). In the case of three variables, the general term is

$$\frac{h^p k^q l^r}{[p]^p\,[q]^q\,[r]^r}\,\delta_u{}^{p-1}\delta_v{}^{q-1}\delta_w{}^{r-1}\,\{\theta^p\phi^q\chi^r\,\delta_u\delta_v\delta_w F + ...\},$$

the sixteen terms within the { } being found by comparing the product $\delta_u\delta_v\delta_w$ with the system FFF, ϕFF, &c. given above, and then connecting each symbol of differentiation with the function corresponding to the affix. Thus in the first term the δ_u, δ_v, δ_w, each affect the F, in the second term the δ_u affects ϕ^q, and the δ_v and δ_w each affect the F,

nd so on for the remaining terms. The form is of course educible from Laplace's general theorem, and the actual evelopment of it is given in Laplace's Memoir in the *Hist. l'Acad.* 1777. I quote from a memoir by Jacobi which I ke this opportunity of referring to, "De resolutione equanum per series infinitas," *Crelle*, tom. VI. p. 257, founded a preceding memoir, "Exercitatio Algebraica circa disptionem singularem fractionum quæ plures variabiles volvunt," tom. V. p. 344.

one Buildings,
April 6, 1850.

ON A DOUBLY INFINITE SERIES.

By Arthur Cayley.

The following completely paradoxical investigation of the operties of the function Γ (which I have been in possession for some years) may perhaps be found interesting from its nnexion with the theories of expansion and divergent series.

Let $\Sigma_r \phi r$ denote the sum of the values of ϕr for all eger values of r from $-\infty$ to ∞. Then writing

$$u = \Sigma_r [n-1]^r x^{n-1-r} \ldots\ldots\ldots\ldots (1)$$

here n is any number whatever), we have immediately

$$\frac{du}{dx} = \Sigma_r [n-1]^{r+1} x^{n-2-r} = \Sigma_r [n-1]^r x^{n-1-r} = u;$$

. $$\frac{du}{dx} = u, \text{ or } u = C_n e^x,$$

he constant of integration being of course in general a nction of n). Hence

$$C_n e^x = \Sigma_r [n-1]^r x^{n-1-r} \ldots\ldots\ldots\ldots (2).$$

e^x is expanded in general in a *doubly infinite necessarily ergent series of fractional powers of x*, (which resolves elf however in the case of n a positive or negative integer, o the ordinary singly infinite series, the value of C_n in this e being immediately seen to be Γn).

The equation (2) in its general form is to be considered a definition of the function C_n. We deduce from it

$$\Sigma_r [n-1]^r (ax)^{n-1-r} = C_n e^{ax},$$

$$\Sigma_{r'} [n'-1]^{r'} (ax')^{n'-1-r'} = C_{n'} e^{ax'};$$

$$\vdots$$

and also

$$\Sigma_k\,[n+n'\,..\,-1]^k\,\{a\,(x+x'\,.\,.)\}^{n+n'..-1-k} = C_{n+n'..}\,e^{a\,(x+x'..)}.$$

Multiplying the first set of series, and comparing with this last,

$$C_{n+n'..}\,\Sigma_{r,r'..}\,[n-1]^r\,[n'-1]^{r'}\,.\,. = x^{n-1-r}x'^{n'-1-r'}\,.\,.$$
$$C_n C_{n'}\,.\,.\,.\,.\,[n+n'\,.\,.-1]^k\,(x+x'\,.\,.)^{n+n'..-1-k}\;\ldots\ldots(3),$$

(where r, r' denote any positive or negative integer numbers satisfying $r+r'+\,..=k+1-p$, p being the number of terms in the series n, n'...). This equation constitutes a multinomial theorem of a class analogous to that of the exponential theorem contained in the equation (2).

In particular

$$C_{n+n'..}\,\Sigma_{r,r'..}\,[n-1]^r\,[n'-1]^{r'}\,.\,. = C_n C_{n'}\,.\,.\,[n+n'\,.\,.-1]^k\,p^{n+n'...-1-k}$$
$$\ldots\ldots\ldots(4).$$

And if $p=2$, writing also m, n for n, n', and $k-1-r$ for r',

$$C_{m+n}\Sigma_r\,[m-1]^r\,[n-1]^{k-1-r} = C_m C_n\,[m+n-1]^k\,2^{m+n-1-k}\,\ldots\ldots(5).$$

Or putting $k=0$ and dividing,

$$C_m C_n \div C_{m+n} = \frac{1}{2^{m+n-1}}\,\Sigma_r\,[m-1]^r\,[n-1]^{-1-r}\,\ldots\ldots(6).$$

Now the series on the second side of this equation is easily seen to be convergent (at least for positive values of m, n). To determine its value write

$$\mathrm{F}(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx\,;$$

then $\mathrm{F}(m,n) = \int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}\,dx + \int_0^{\frac{1}{2}} x^{n-1}(1-x)^{m-1}\,dx.$

And by successive integrations by parts, the first of these integrals is reducible to $\frac{1}{2^{m+n-1}}\,\Sigma_r\,[m-1]^r\,[n-1]^{-1-r}$, r extending from -1 to $-\infty$ inclusively, and the second to

$$\frac{1}{2^{m+n-1}}\,\Sigma_r\,[m-1]^r\,[n-1]^{-1-r},$$

r extending from 0 to ∞; hence

$$\mathrm{F}(m,n) = \frac{1}{2^{m+n-1}}\,\Sigma_r\,[m-1]^r\,[n-1]^{-1-r},$$

or $$C_m C_n \div C_{m+n} = \mathrm{F}(m,n)\,\ldots\ldots\ldots\ldots(7),$$

which proves the identity of C_m with the function $\Gamma(m)$. [Substituting in two of the preceding equations, we have

$$\Gamma n\Gamma n'\,.\,.\div\Gamma(n+n'\,.\,.) = \frac{1}{\text{[illegible]}}\,\Sigma_{r,r'..}[n-1]^r[n'-1]^{r'}\,..$$
$$\ldots\ldots(8),$$

where, as before, p denotes the number of terms in the series n, n'.. and $r + r' + \ldots = k + 1 - p$), the first side of each equation is, it is well known, reducible to a multiple finite integral by means of a theorem of M. Dirichlet. And

$$F(m, n) = \frac{1}{[m + n - 1]^k \cdot 2^{m+n-1-k}} \Sigma_r [m - 1]^r [n - 1]^{k-r} \ldots . (9),$$

where r extends from $-\infty$ to $+\infty$, and k is arbitrary. By giving large negative values to this quantity very convergent series may be obtained for the calculation of $F(m, n)$].

LAWS OF THE ELASTICITY OF SOLID BODIES.

By W. J. Macquorn Rankine, Civil Engineer, F.R.S.E., F.R.S.A., &c.

[Read to the British Association for the Advancement of Science, Section A, at Edinburgh, 1st August, 1850.]

Introduction.

1. The science of the Elasticity of Solid Bodies, considered with reference to its most important application, the determination of the strength of structures, consists of three parts.

First. The investigation of what may be specially termed the *Laws of Elasticity;* that is to say, the mutual relations which must exist between the elasticities of different kinds possessed by a given solid, and between the different values of those elasticities in different directions.

Secondly. The integration of the equations of equilibrium and motion of the particles of an elastic solid. The results of this process enable us to determine the relative displacements of the particles from their natural positions, in a solid body of a given material and figure, subjected to a given combination of forces.

Thirdly. The application of the results derived from the first two branches of the theory to our experimental knowledge of the pressures and relative displacements to which particles of known materials may safely be subjected in practice. This enables us to compute the strength of actual structures.

Notwithstanding the great amount of attention which has been paid to the *strength of materials*, and the numerous

and elaborate experiments which have been made respecting it, few examples exist of the sound application of physical and mathematical principles to practice in connection with this subject. This has arisen chiefly from the fact, that the first and second branches of the inquiry have to a great extent been carried on without reference to their application to the third, and the third conducted without regard to the principles of the first and second. The results of investigation on correct principles, into the theory of elasticity, have been limited in their applications, with a few exceptions, to the laws of the propagation of vibratory movements; and those few exceptions relate almost exclusively to bodies of equal elasticity in all directions; a class which excludes many of the most useful materials of construction. On the other hand, when it has been found necessary to adopt theoretical principles, for the purpose of reducing the results of experiments on the strength and elasticity of materials to a system, assumptions have often been made, with a view chiefly to simplicity in calculation, of a kind inconsistent with the real nature of elastic bodies.

3. The present inquiry relates to the First Part of the Theory of Elasticity, viz. the laws of the relations which must exist between the elasticities of different kinds possessed by a given substance, and between their various values in different directions.

§. I. *Composition and Resolution of Strains and Molecular Pressures.*

4. At the outset of the inquiry two preliminary problems present themselves: the composition and resolution of relative molecular displacements; and the composition and resolution of pressures, such as the parts of elastic bodies exert upon each other. The former is a question of pure Geometry; the latter, of pure Statics. They are usually considered simultaneously, on account of the analogy which exists between their solutions. This is not the result of the physical connexion between the two classes of phenomena, and it would still exist although there were no such physical connexion; it is merely a consequence of the analogy between forces in Statics and straight lines in Geometry.

Those two problems have been so fully investigated by MM. Cauchy, Lamé, and Clapeyron, as to leave nothing

ther to be done. The theorems and formulæ which they obtained are many and important. In the present r I shall state those principles and results only to h there will be occasion to refer in the sequel.

5. It is desirable that some single word should be as-ed to denote the state of the particles of a body when placed from their natural relative positions. Although word *strain* is used in ordinary language indiscrimi-ly to denote Relative Molecular Displacement, and force by which it is produced, yet it appears to me it is well calculated to supply this want. I shall there-use it, throughout this paper, in the restricted sense *Relative Displacement of Particles*, whether consisting Dilatation, Condensation, or Distortion; while under the m *Pressure* I shall include every kind of force which ts between elastic bodies, or the parts of an elastic body, the cause or effect of a state of strain, whether that rce is tensile, compressive, or distorting.

The nature and magnitude of a simple and uniform strain are defined by three things.

First. The direction of the lines along which the particles of the body are displaced from their natural position.

Secondly. The direction along which the rate of variation of the displacement from point to point is a maximum. This direction is normal to a series of planes of equal displace-ment, and may be called the *strain-normal.*

Thirdly. The amount of that rate of variation; being the differential coefficient of the displacement with respect to dis-tance along the strain-normal.

6. A strain may be resolved into three components, in which the directions of displacement shall be respectively arallel to three rectangular axes, while the strain-normal mains unchanged, by multiplying its amount by the direc-n-cosines of the total displacement.

Each of these three components may itself be resolved o three components, in which, the direction of displace-nt remaining unchanged, the strain-normals are respec-ely parallel to the three axes, by multiplying its amount the direction-cosines of the original strain-normal.

Thus every strain is reducible to nine components.

These nine components, however, are equivalent to but distinct strains. If we consider the strains as thus re-ed to three rectangular axes, we shall find that they of two kinds: *longitudinal*, that is to say, strains of linear nsion or condensation, where the displacements are pa-

rallel to the strain-normals; and *transverse*, or strai
distortion, when these directions are at right angles. '
if x, y, z denote the three rectangular axes, and ξ,
small molecular displacements respectively parallel to t
then

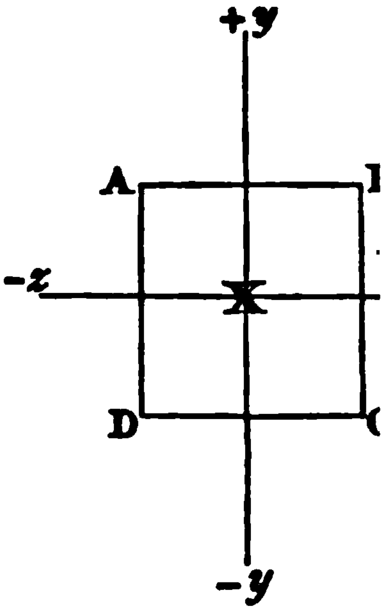
$$\frac{d\xi}{dx},\ \frac{d\eta}{dy},\ \frac{d\zeta}{dz},$$

are longitudinal strains, which are dilatations when po
and condensations when negative. I shall denote
respectively by N_1, N_2, N_3;
their sum, when positive, is the *cubic* dilatation of the
ticles, and when negative, the cubic condensation.

Transverse strains, or distortions, are represented b
six differential coefficients of the displacements with re
to axes at right angles to them; viz.

$$\frac{d\eta}{dz},\ \frac{d\zeta}{dy};\ \frac{d\zeta}{dx},\ \frac{d\xi}{dz};\ \frac{d\xi}{dy},\ \frac{d\eta}{dx}.$$

Let the axis of x be perpendicular to the plane o
paper. Let $ABCD$ be the section, by the plane yz, of a prism which in its natural state is square, and has its faces normal to the axes of y and z. A distortion in the plane yz, relatively to these axes, is measured by the deviation from rectangularity of this originally square section, that deviation being considered positive which makes the angles B and D acute. Now so far as the positions of the particles in this prism relatively to each other are



concerned, it is immaterial whether that deviation from
angularity is produced by keeping the sides AD an
parallel to their original positions, and giving angular m
to AB and DC—a change represented by $\frac{d\eta}{dz}$; or by ke
AB and DC parallel to their original positions, and g
angular motion to AD and BC—a change represent
$\frac{d\zeta}{dy}$; or by combining those two operations: so that th
transverse strain in the plane yz is represented by th
of these two coefficients,

$$\frac{d\eta}{dz} + \frac{d\zeta}{dy} = 2T_1.$$

[redacted]

Similar reasoning gives, for the the total distortion in the plane xz,

$$\frac{d\zeta}{dx} + \frac{d\xi}{dz} = 2T_2,$$

in the plane xy,

$$\frac{d\xi}{dy} + \frac{d\eta}{dx} = 2T_3.$$

The factor 2 is used in these expressions for the sake of convenience in the employment of certain formulæ, to be afterwards quoted.

The halved-differences of the pairs of differential coefficients,

$$\frac{1}{2}\left(\frac{d\eta}{dz} - \frac{d\zeta}{dy}\right); \quad \frac{1}{2}\left(\frac{d\zeta}{dx} - \frac{d\xi}{dz}\right); \quad \frac{1}{2}\left(\frac{d\xi}{dy} - \frac{d\eta}{dx}\right),$$

represent rotations of the prism as a whole, round the axes of x, y, z respectively; which have no connexion with the positions of its particles relatively to each other.

The component strains into which all others can be resolved with respect to a given set of axes, are thus reduced to six, three longitudinal and three transverse.

7. A pressure, like a strain, is defined by three things.

1st. The direction of the pressure.

2nd. The position of the surface at which the pressure is exerted.

3rd. The amount of the pressure as expressed in units of force per unit of area of the surface of action.

A pressure on a plane, in whatsoever direction it may act, may be resolved into three rectangular components, one normal to the plane, and two tangential. The normal pressure may be compressive or tensile: when compressive, it is considered as positive; when tensile, negative.

In an elastic solid which is *in equilibrio*, let a cube be conceived to exist with its faces normal to the axes of coordinates, and let the pressures throughout its extent be uniform. This cube exerts on the matter round it, and is reacted on by three pairs of normal pressures, at the faces respectively normal to the axes of

$$x, \quad y, \quad z,$$

which may be denoted by

$$P_1, \quad P_2, \quad P_3,$$

the pressures at opposite faces being equal.

Let $ABCD$ represent the section of this cube by the plane yz. On the faces AB and CD, parallel to xz, let

a pair of tangential forces act in the directions denoted by the order of the letters, tending to produce distortion by making the angles B and D acute and A and C obtuse. Let a pair of forces of similar tendency act on the faces CB and AD, parallel to zy. These two pairs of forces are equal and opposite to those which the cube, in consequence of the transverse displacements of its particles, exerts on the surrounding portion of the solid. No displacement of the relative situations of a system of particles can give the system a tendency to revolve as a whole round an axis. Such a tendency must exist in the cube unless the tangential faces on the forces AB, CD are equal to those on the faces CB, AD.

Therefore *the tangential pressure parallel to* z, *on a plane normal to* y, *is equal to the tangential pressure parallel to* y, *on a plane normal to* z: a theorem first proved by Cauchy.

The common value of those forces may be denoted by

$$Q_1,$$

as they are both perpendicular to x.

Similar reasoning shews that the two pairs of tangential forces perpendicular to y have one common value

$$Q_2.$$

In like manner, those perpendicular to z may be denoted by

$$Q_3.$$

Thus the pressures exerted by and on the cube are reduced to six, three normal and three tangential.

8. The composition of pressures applied to different planes, and their reduction to new axes, depends on the following principle.

Conceive a small triangular pyramid, with its apex at the origin of rectangular coordinates, its sides being formed by the three coordinate planes, and its base by a plane in any given direction intersecting them. Let pressures, in one given direction, act on the three sides, and be balanced by a pressure in the same direction on the base. Each of the three sides is equal to the base multiplied by the cosine of the angle between the normal to the base and the normal to the side in question. Therefore the total pressure on the base is equal to the sum of the pressures on the sides, each multiplied by the cosine of the angle between the normal to the side in question and the normal to the base. If the normal to this base is one of three new axes of rectangular coordinates, the total pressure thus found may be reduced to normal [illegible] tial pressures by multiplying it by its di[illegible] pect to the new axes.

9. I annex, for convenience of reference, the general rmulæ which have been deduced from this principle.

Let x, y, z be rectangular axes of coordinates, and P_1, P_2, P_3, Q_1, Q_2, Q_3 normal and tangential pressures hich act as shewn in the following table:

Normals.	*Planes.*	*Pressures parallel to* x	y	z
x	yz	P_1	Q_3	Q_2
y	zx	Q_3	P_2	Q_1
z	xy	Q_2	Q_1	P_3

Let R_1, R_2, R_3 be the rectangular components of the total essure at a plane, the direction-cosines of whose normal re a_1, a_2, a_3.

Then
$$\left.\begin{aligned} R_1 &= a_1P_1 + a_2Q_3 + a_3Q_2,\\ R_2 &= a_1Q_3 + a_2P_2 + a_3Q_1,\\ R_3 &= a_1Q_2 + a_2Q_1 + a_3P_3,\end{aligned}\right\}\dots\dots\dots\dots(1).$$

Let this normal be taken as the axis of x' in a new t of rectangular axes $x'y'z'$, which make with the original es the angles whose cosines are given in the following ble:

Original Axes.	*New Axes.* x'	y'	z'	
x	a_1	b_1	c_1	Direction-cosines.
y	a_2	b_2	c_2	
z	a_3	b_3	c_3	

Let $P_1' P_2' P_3'$, $Q_1' Q_2' Q_3'$ be the normal and tangential pressures, as reduced to the new axes: then

$$\left.\begin{aligned}
&= P_1a_1^2 + P_2a_2^2 + P_3a_3^2\\
&\quad + 2Q_1a_2a_3 + 2Q_2a_3a_1 + 2Q_3a_1a_2,\\
&= P_1b_1^2 + P_2b_2^2 + P_3b_3^2\\
&\quad + 2Q_1b_2b_3 + 2Q_2b_3b_1 + 2Q_3b_1b_2,\\
&= P_1c_1^2 + P_2c_2^2 + P_3c_3^2\\
&\quad + 2Q_1c_2c_3 + 2Q_2c_3c_1 + 2Q_3c_1c_2,\\
&= P_1b_1c_1 + P_2b_2c_2 + P_3b_3c_3\\
&\quad + Q_1(b_2c_3 + b_3c_2) + Q_2(b_3c_1 + b_1c_3) + Q_3(b_1c_2 + b_2c_1),\\
&= P_1c_1a_1 + P_2c_2a_2 + P_3c_3a_3\\
&\quad + Q_1(c_2a_3 + c_3a_2) + Q_2(c_3a_1 + c_1a_3) + Q_3(c_1a_2 + c_2a_1),\\
&= P_1a_1b_1 + P_2a_2b_2 + P_3a_3b_3\\
&\quad + Q_1(a_2b_3 + a_3b_2) + Q_2(a_3b_1 + a_1b_3) + Q_3(a_1b_2 + a_2b_1),
\end{aligned}\right\}\dots(2).$$

By the substitution of N for P and T for Q, the for given above are made applicable to the reduction of *str* to new axes of coordinates.

I shall not here recapitulate the many elegant and im tant theorems which MM. Cauchy and Lamé and Clape have deduced from those equations, as they do not r to the branch of the theory of elasticity of which this p treats.

I may mention that in their memoir in the seventh vo of Crelle's *Journal*, MM. Lamé and Clapeyron have N and T to denote *pressures*, and have expressed *str* simply by the differential coefficients $\frac{d\xi}{dx}$, &c.

§ II. *Physical Relations between Pressures and Strain far as they are independent of Hypotheses respe the Molecular constitution of matter.*

10. In almost all investigations which have hitherto b made respecting the elasticity of bodies which have diffe degrees of elasticity in different directions, it has been practice to take some hypothesis as to the molecular stitution of solid bodies as the basis of calculation f the outset of the inquiry. It appears to me, however, the more philosophical course is, to ascertain in the place what conclusions can be attained as to the Law Elasticity without the aid of any such hypothesis, afterwards to enquire how far the theory can be simplif and what additional results can be gained by introdu suppositions respecting the ultimate constitution of matte

For the present, therefore, I shall make no assump as to the questions, whether bodies are systems of phy points, or of atoms of definite bulk and figure, or continuous, or have a constitution intermediate bet those three; and I shall use the word *particle* in its sense of a *small part*.

11. I shall restrict the present inquiry to homog bodies possessing a certain degree of symmetry in molecular actions, which consists in this: that the upon any given particle of the body, of any tw particles situated at equal distances from it within the of molecular action, in opposite directions, shall b and opposite.

Substances may possess higher degrees of molecu metry, but this is the lowest.

The statement that a body is *homogeneous* means, when applied to molecular action, that the mutual action of a pair of particles, situated at a given distance from each other in a given direction, shall be equal to that of any other pair of particles equal to the first, situated at an equal distance from each other in a parallel direction.

12. It is known by observation, that strains and pressures are physically connected. It is also known by observation, that the pressure with which a strain is connected consists in a tendency of the body to recover its natural state, and is opposite or nearly opposite in direction to the strain; thus longitudinal condensation is accompanied with positive normal, or nearly-normal pressure; longitudinal dilatation, with negative normal, or nearly-normal pressure; and distortion in a given plane, with tangential pressure in the same plane, of opposite sign.

It is known by experiment, that when a pressure and the strain with which it is connected are given in direction, and when the strain does not exceed a certain limit, being in most cases the utmost limit to which a structure can be strained without danger to its permanency, the pressure and the strain are sensibly proportional to each other. The quantity by which a strain is to be multiplied to give the corresponding pressure is a *Coefficient of Elasticity*, and is expressed, like a pressure, by a certain number of units of force per unit of surface.

I have said that a strain and the corresponding pressure referred to the same plane are opposite or *nearly* opposite in direction; for they are not of necessity exactly opposite for all directions of strain, except in substances which are possessed of the highest degree of molecular symmetry; that is to say, which are equally elastic in all directions. For those having lower degrees of symmetry, the following proposition is true.

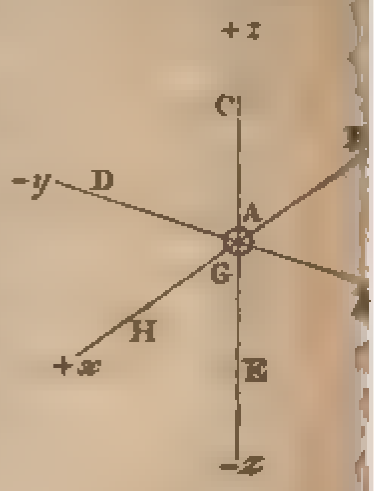
THEOREM I. *In an elastic substance which is homogeneous and symmetrical with respect to molecular action, there are three directions at right angles to each other, in which a longitudinal strain produces an exactly normal pressure on a plane at right angles to the direction of the strain.*

Those three directions are called *Axes of Elasticity*. The proposition is equivalent to an assertion, that *the lowest degree of symmetry of molecular action necessarily involves symmetry with respect to three rectangular coordinate planes.*

This theorem has been often demonstrated for sy[...] atoms. But it is easily seen that the truth of th[...] monstrations depends, not on the special hypothese[...] they involve, but on the fundamental condition of sy[...]

The following demonstration involves no hypothe[...]

Let a point O in the interior of a body be assume[...] origin of rectangular coordinates, the axes being considered as fixed, and the body as moveable angularly in all directions about the origin. Space round O is divided by the coordinate planes into eight similar indefinitely-extended rectangular three-sided pyramids. Let those pyramids be designated as follows, according to the signs of the coordinates comprised in them.



Signs of x	y	z	Designation of Pyramid.
+	+	+	*A*
−	+	+	*B*
−	−	+	*C*
+	−	+	*D*
+	+	−	*E*
−	+	−	*F*
−	−	−	*G*
+	−	−	*H*

To express the relative situations of these pyra[...] taken in pairs, let the following terms be used :

Diametrically opposite—when the pyramids touch [...] apex only : comprising the following pairs,

A, G ; *B, H* ; *C, E* ; *D, F*.

Diagonally opposite—when they touch at an edg[...] prising the pairs

A, H ; *D, E* ; *B, G* ; *C, F* ;

A, F ; *B, E* ; *D, G* ; *C, H* ;

A, C ; *B, D* ; *E, G* ; *F, H*.

Contiguous—when they touch in a face: comprising the pairs

$$A, B;\ D, C:\ H, G:\ E, F:$$
$$A, D;\ B, C:\ F, G:\ E, H:$$
$$A, E;\ B, F;\ C, G:\ D, H.$$

Each pair of contiguous pyramids forms a rectangular wedge, which has an *opposite wedge* touching it along the edge, and a *contiguous wedge* touching it at each of its two faces.

The pairs of opposite wedges are

$$AB,\ GH;\ \ CD,\ EF:$$
$$AD,\ FG;\ \ BC,\ EH:$$
$$AE,\ CG;\ \ BF,\ DH.$$

The pairs of contiguous wedges are

$$AB, CD;\ \ CD, GH;\ \ GH, EF;\ \ EF, AB:$$
$$AD, BC;\ \ BC, FG;\ \ FG, EH;\ \ EH, AD:$$
$$AE, BF;\ \ BF, CG;\ \ CG, DH;\ \ DH, AE.$$

According to the condition of symmetry already stated, the portions of matter comprised in any pair of diametrically-opposite pyramids must be symmetrical in their actions on a particle placed at O, or on any pair of equal particles symmetrically placed with respect to O, whatsoever may be the angular position of the body with respect to the axes.

Suppose the body to receive a longitudinal strain in the direction of the axis of z. Let a small circular area ω be conceived to exist in the plane of xy, with its centre at O: and let this area be the base of a cylinder extending indefinitely in a negative direction along the axis of z, and denoted by ωz. The pressure on the plane xy is proportional and parallel to the resultant of the actions of the four pyramids A, B, C, D, on the cylinder ωz, divided by the area ω. The action of each of those pyramids consists of a normal component parallel to z, and a tangential component parallel to the plane xy. In order that the total pressure may be normal, those tangential actions must balance each other; which can only be the case when the tangential action of the wedge AB parallel to the axis of y is equal and opposite to that of the contiguous wedge CD, and the tangential action of the wedge BC parallel to the axis of x is equal and opposite to that of the contiguous wedge AD.

The pair of contiguous wedges AB, CD, touch in the plane of xz, having the axis of x for their common edge. If the actions of this pair of wedges on ωz, when longitudinally

[illegible] unsymmetrical, this cannot arise from [illegible] of these wedges, which are [illegible] symmetrical with respect to each layer of particles in [illegible] the particles occupying the [illegible] through a right angle [illegible] the particles which formerly [illegible] and the particles which formerly oc- [illegible] of two pyramids, diametrically [illegible] symmetrical to A and B) [illegible] the body with respect [illegible] of the tangential [illegible] wedges AB and CD, on &c, [illegible] *equal* [illegible] *opposite in direction* [illegible] change will have been [illegible], so that the value [illegible] passed through zero. Therefore, whatever may be the situation of the axis of x amongst the particles of the body, it is possible, by rotating the body about that axis, to find a position in which the tangential actions of the wedges AB and CD, parallel to y, on the cylinder &c. shall balance each other. And by similar reasoning it may be proved, that whatever may be the situation of the axis of y amongst the particles of the body, it is possible, by rotating the body about that axis, to find a position in which the tangential actions of the wedges BC and AD, parallel to x, on the cylinder &c. shall balance each other.

Therefore by combining rotations about the axes of x and y, it is possible to find a position of the solid with respect to the axes of coordinates, such, that the tangential actions of the four pyramids A, B, C, D, on the cylinder &c. arising from a longitudinal strain along z, shall be in equilibrio, and that the total pressure on xy shall be normal.

The direction, with respect to the solid, which fulfils this condition, is called an *Axis of Elasticity*.

Let $zO+z$, being now an axis of elasticity, be considered as fixed in the solid.

From the manner in which the two pairs of wedges AB, CD and BC, AD, are composed of the four pyramids A, B, C, D, it is clear that the actions of the pair of diagonally opposite pyramids A, C, are symmetrical, and also those of the diagonally-opposite pyramids B, D. From this and the symmetry of the actions of diametrically-opposite pyramids it follows, that the actions of the four pairs of contiguous pyramids, A, E; D, H; B, F; C, G, are sym-

metrical, and also those of the two pairs of diagonally-opposite pyramids, E, G; F, H. This symmetry of action (subject to the condition of symmetry of strain) is not disturbed by rotation about the axis of z.

Let the small circular area ω be now conceived to exist in the plane yz, and let the cylinder of which it is the base extend in a negative direction along the axis of x, and be called the cylinder ωx. Let the solid receive a longitudinal strain along the axis of x. The action of A on ωx is symmetrical to that of E, and the action of D to that of H; therefore the tangential actions of the wedges AD, EH, parallel to z, balance each other. It remains only to make the tangential actions of the wedges AE, DH, parallel to y, on ωx, balance each other; which is to be done by rotation about the axis of z.

The solid is now in such a position that x, as well as z, is an axis of elasticity.

The pairs of contiguous pyramids are now all molecularly symmetrical about their common faces. Therefore the pairs of continuous wedges AB, EF; AE, BF, are symmetrical in their actions on a cylinder ωy, when longitudinally strained along y.

Therefore y also is an axis of elasticity; and the theorem is proved.

It is not necessary to the existence of rectangular axes of elasticity that the body should be homogeneous (in the sense in which I have used the word) throughout its whole extent, but only round each point throughout a space which is large as compared with the sphere of appreciable molecular action of each particle. Hence the rectangular axes of elasticity may vary in direction at different points of the same body; and some, or all of them, may follow the course of a system of curves, as they do in a rope, a piece of bent timber, or a curved bar of fibrous metal.

13. The axes of elasticity are evidently those which ought to be selected as axes of coordinates, for the resolution of all pressures and strains, in researches on the laws of elasticity. The strains and pressures being so resolved, we shall have the expression

$$-A_1 N_1$$

for *part* of the normal pressure on the plane yz; A_1 being the *coefficient of longitudinal elasticity* for the axis of x. But this is not the whole of that pressure; for it is known by observation, that the normal pressure on a given plane is augmented by condensation, and diminished by dilatation

of the particles, in a direction parallel to the given plane as well as normal to it. The normal pressure P_1 on yz, therefore, depends not only on the longitudinal strain N_1 along x, but also on the longitudinal strains N_2 and N_3 along y and z. Applying similar reasoning to the other normal pressures, they are found to be represented as follows:

$$\begin{array}{lll} \textit{Axis.} & \textit{Plane.} & \\ x \ldots\ldots & yz \ldots\ldots & P_1 = -A_1N_1 - B_3N_2 - B_2'N_3 \\ y \ldots\ldots & zx \ldots\ldots & P_2 = -B_3'N_1 - A_2N_2 - B_1N_3 \\ z \ldots\ldots & xy \ldots\ldots & P_3 = -B_2N_1 - B_1'N_2 - A_3N_3 \end{array} \Bigg\} \ldots (3).$$

The tangential pressures are represented, in terms of the distortions, in the following manner:

$$\begin{array}{ll} \textit{Plane.} & \\ yz \ldots\ldots & Q_1 = -2C_1T_1 \\ zx \ldots\ldots & Q_2 = -2C_2T_2 \\ xy \ldots\ldots & Q_3 = -2C_3T_3 \end{array} \Bigg\} \ldots\ldots\ldots\ldots (4).$$

These six equations are merely the representation of observed facts, framed with regard to the principle of axes of elasticity.

They contain *twelve* coefficients of elasticity, which may be thus classified:

A_1, A_2, A_3, are the coefficients of *longitudinal elasticity* for the axes of x, y, z, respectively;

B_1, B_1', are the coefficients of *lateral* elasticity in the plane of yz: the former expressing the effect of a strain along z in producing normal pressure parallel to y; the latter, the effect of a strain along y in producing normal pressure parallel to z.

B_2, B_2', are the coefficients of lateral elasticity in the plane of zx, and

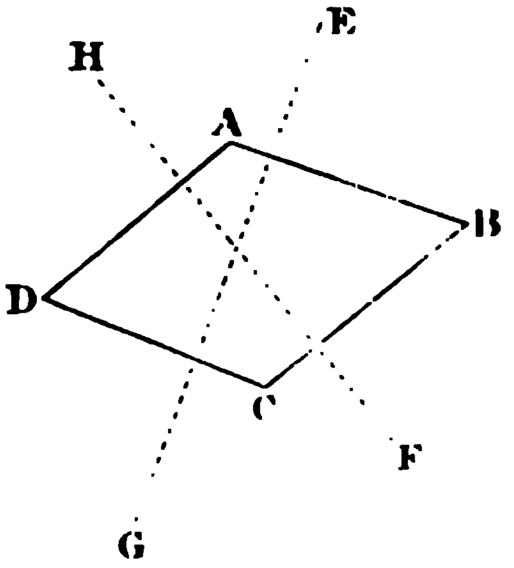
B_3, B_3', in the plane of xy.

C_1, C_2, C_3, are the coefficients of *transverse or tangential elasticity*, or of *rigidity*, in the planes of yz, zx, and xy, respectively. The possession of this species of elasticity is the property which distinguishes solids from fluids, and is that upon which the strength and stability of solid structures entirely depend. When a beam, or any other portion of a solid structure, *takes a set*, as it is called, (or undergoes permanent alteration of figure,) it is the *rigidity* which has been overstrained, and has given way. So far as I am aware, however, it has not hitherto been *directly* referred to in researches on the strength of materials, except in those relative to torsion.

The principal object of the present inquiry is to determine what mutual relations must necessarily exist amongst those twelve coefficients of elasticity in each substance.

14. The three coefficients of rigidity, so far as we have as yet seen, represent the elasticity called into play by three kinds of distortion, measured respectively by the alteration of the angles of the three rectangular sections of a cube whose faces are normal to the three axes of elasticity. I shall now however prove, that the tangential pressures produced by equal distortions are equal, so long as the plane in which the distortion takes place is unchanged, and are not altered by any change of the direction, in that plane, of the sides of the figure on which the distortion is measured: that is to say—

THEOREM II. *The coefficient of rigidity is the same for all directions of distortion in a given plane.*



Let $ABCD$ be the section at right angles to the edges of a rhombic prism having any angles; and GE and FH two lines normal respectively to the faces of the prism. Let this prism undergo a small alteration in the angles of its section $ABCD$.

Whether we estimate the *distortion* so produced, as a transverse displacement of the particles in lines parallel to AB, and varying along the strain-normal GE, or as a transverse displacement of the particles in lines parallel to AD, and varying along the strain-normal FH, the result, so far as the relative *transverse* displacements of the particles are concerned, will be the same.

Also, the tangential pressures are the same at the pair of faces AB and CD, and at the pair of faces BC and AD; for otherwise a relative displacement among the particles of a body would produce a force tending to make it revolve as a whole round an axis; which is impossible.

Therefore the tangential forces produced by equal transverse displacements relatively to two strain-normals which make any angle with each other are equal, provided the displacements are in the same plane with the normals; therefore the coefficient of rigidity is the same for all directions in a given plane.

15. This theorem leads to another, which expresses the relations between the twelve coefficients of elasticity, as far as it is possible to determine them independently of hypotheses respecting the constitution of matter.

THEOREM III. *In each of the coordinate planes of elasticity, the coefficient of rigidity is equal to one-fourth part*

[redacted]

of the sum of the two coefficients of longitudinal elasi for the axes which lie in that plane, diminished by one-fc part of the sum of the two coefficients of lateral elasticit the same plane.

For example, let the plane be that of yz, in which coefficient of rigidity is C_1, those of longitudinal elasti A_2 and A_3, and those of lateral elasticity, B_1 and B_1'.

Let $2T_1'$ represent a distortion in the plane yz, rel to two new axes y', z' in the same plane, and let the a $\widehat{yy'} = \theta$. Let this distortion be resolved with respect to original axes, according to equation (2). Then

$$N_1 = 0; \quad N_2 = -2T_1' \cos\theta \sin\theta; \quad N_3 = 2T_1' \cos\theta \sin$$

$$2T_1 = 2T_1'(\cos^2\theta - \sin^2\theta); \quad T_2 = 0; \quad T_3 = 0.$$

The corresponding pressures referred to the original axes,

$$P_1 = -2T_1'(B_3 - B_3')\cos\theta\sin\theta,$$

$$P_2 = -2T_1'(-A_2 + B_1)\cos\theta\sin\theta,$$

$$P_3 = -2T_1'(-B_1' + A_3)\cos\theta\sin\theta,$$

$$Q_1 = -2C_1T_1'(\cos^2\theta - \sin^2\theta), \quad Q_2 = 0, \quad Q_3 = 0.$$

Let us now determine, according to equation (2), the above pressures, the *tangential* pressure Q_1' as refe to the new axes. Then

$$Q_1' = -2T_1'\{C_1 + \cos^2\theta\sin^2\theta(A_2 + A_3 - B_1 - B_1' - 4C$$

But by the preceding theorem we have also

$$Q_1' = -2T_1'C_1$$

for all values of θ; which cannot be true unless the efficient of $\cos^2\theta\sin^2\theta$ in the first value of Q_1' is $= 0$. sequently

$$C_1 = \frac{1}{4}(A_2 + A_3 - B_1 - B_1').$$

By applying similar reasoning to the planes of zx and xy it is also proved that

$$C_2 = \frac{1}{4}(A_3 + A_1 - B_2 - B_2'),$$

$$C_3 = \frac{1}{4}(A_1 + A_2 - B_3 - B_3') \qquad \ldots\ldots\ldots($$

being the algebraical statement of the theorem enunciate

Thus the number of independent coefficients of elast is reduced to *nine* ... ree are functi

and this is the utmost reduction of their number which can be made without the aid of suppositions as to the constitution of matter.

The determination by experiment of nine constants for each substance is an undertaking almost hopeless; it is therefore desirable to ascertain whether, by the introduction of some probable hypothesis, their number can be further reduced.

§ III. *Results of the Hypothesis of Atomic Centres.*

16. Almost all the investigations of the laws of elasticity which have hitherto appeared, are founded on the hypothesis of Boscovich: that matter consists of physical points or centres of force, or of atoms acting as if their masses were concentrated at their centres; which physical points or atoms occupy space, and produce the phenomena of elasticity, because the forces which act between them, and which depend on their relative distances and positions, tend to make them remain in certain relative positions, and at certain distances apart.

Although the results of this supposition are not verified by all solid substances; still it seems probable that its errors are to be corrected, not by rejecting it, but by combining it with another, to which I shall afterwards refer.

I shall now, therefore, shew to what extent the laws of elasticity are simplified by adopting Boscovich's supposition of atomic centres of force, acting on each other by attractive and repulsive forces along the lines joining them. It will be seen, that in consequence of the course adopted, of determining in the first place the necessary relations between the coefficients of elasticity which must exist independently of all special hypotheses, this investigation is almost entirely freed from the algebraical intricacy in which it would otherwise be involved.

17. All the consequences peculiar to this hypothesis flow from the following single theorem, in which the term *perfect solid* is used to denote a body whose elasticity is due entirely to the mutual attractions and repulsions of atomic centres of force.

THEOREM IV. *In each of the coordinate planes of elasticity of a perfect solid, the two coefficients of lateral elasticity, and the coefficient of rigidity, are all equal to each other.*

Take, for example, the plane of *yz*. The proposition enunciated is equivalent to the *assertion*, that the tangential

pressure parallel to y at the plane of xy, produced by a given transverse strain $2T_1 = \frac{d\eta}{dz}$, which consists in a displacement of the atomic centres parallel to y and varying with z, is equal to the normal pressure parallel to z at the same plane xy, produced by a longitudinal strain $N_2 = \frac{d\eta}{dy}$, which consists in condensing or dilating the atomic centres in a direction parallel to y, provided that longitudinal strain is equal in amount to the transverse strain.

The pressure on a given area of the plane xy, is the effect of the joint actions of the atomic centres on the negative side of that plane upon the atomic centres on the positive side.

In the natural or unstrained condition of the body, this pressure is null; shewing that those forces neutralize each other. When the body is strained therefore, the pressure is the resultant of the *variations* of all those forces, arising from the displacements of the atomic centres from their natural relative positions.*

Let m and μ denote a pair of atomic centres, m being situated on the positive side of the plane xy, and μ on the negative side. The force acting between m and μ is supposed to act along the line joining them, and to be a function of its length. When the relative displacement of the atoms is very small as compared with their distance, the variation of this force will be sensibly proportional to the variation of distance, multiplied by some function of the distance. It may therefore be denoted by

$$\phi r . dr,$$

where r denotes the distance (μm). Let this line make with the axes the angles α, β, γ.

Let the strain to be considered, in the first place, be transverse; the displacements being parallel to y and varying with z; the rate of variation being

$$\frac{d\eta}{dz} = 2T_1,$$

and the force to be estimated being in the direction y. Then the displacement of m relatively to μ is

$$\Delta\eta = 2T_1 r \cos\gamma.$$

The variation of their distance apart is

$$\delta r = \cos\beta\Delta\eta = 2T_1 r \cos\beta\cos\gamma.$$

---

* Small quantities of [illegible]tively to the strains T_1, &c., are here neglected.

...he variation of the force acting between them is

$$\phi r \cdot \delta r = 2T_1 r\phi r \cdot \cos\beta \cos\gamma.$$

...nd the component of that variation parallel to y, which ...rms the part of the tangential pressure due to the action ... μ on m, is

$$\cos\beta\phi r \cdot \delta r = 2T_1 r\phi r \cdot \cos^2\beta \cos\gamma \ldots\ldots (a).$$

Next, let the strain be longitudinal, parallel to y, and ...enoted by

$$N_2 = \frac{d\eta}{dy}.$$

...hen the displacement of m relatively to μ is

$$\Delta\eta = N_2 r \cos\beta.$$

...he variation of their distance apart is

$$\delta r = \cos\beta\Delta\eta = N_2 r \cos^2\beta.$$

...he variation of the force acting between them is

$$\phi r \cdot \delta r = N_2 r\phi r \cos^2\beta.$$

...nd the component of that variation parallel to z, which ...rms the part of the normal pressure on the plane xy due ... the action of μ on m, is

$$\cos\gamma\phi r \cdot \delta r = N_2 r\phi r \cos^2\beta \cos\gamma \ldots\ldots\ldots (b).$$

On comparing the expressions (a) and (b) it will be seen ...t the quantities by which $2T_1$ and N_2 are multiplied are ...entical. Therefore the tangential force in the direction y ... the plane xy produced by a distortion in the plane yz, ...d the normal force in the direction z produced by a longi...dinal strain along y, are equal when the strains are equal, ...*r each pair of atomic centres*. They are therefore equal ... a perfect solid, because its elasticity is wholly due to the ...utual actions of atomic centres; and the theorem is proved ... the plane yz, and may in the same manner be proved ... the other coordinate planes of elasticity. It is expressed ...gebraically as follows:

$$\begin{array}{l} \textit{Plane} \\ yz \ldots B_1 = B_1' = C_1 \\ zx \ldots B_2 = B_2' = C_2 \\ xy \ldots B_3 = B_3' = C_3 \end{array} \Bigg\} \ldots\ldots\ldots\ldots (6).$$

18. The combination of these equations with the equa...ns (5) of Theorem III. leads immediately to the follow... ...ults:

$$\left.\begin{aligned} C_1 &= \frac{A_2 + A_3}{6} \\ C_2 &= \frac{A_3 + A_1}{6} \\ C_3 &= \frac{A_1 + A_2}{6} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (7).$$

$$\left.\begin{aligned} A_1 &= 3(C_2 + C_3 - C_1) \\ A_2 &= 3(C_3 + C_1 - C_2) \\ A_3 &= 3(C_1 + C_2 - C_3) \end{aligned}\right\} \ldots\ldots\ldots\ldots (8).$$

that is to say,

THEOREM V. *In each of the three coordinate planes of elasticity of a perfect solid, the coefficient of rigidity is equal to one-sixth part of the sum of the two coefficients of longitudinal elasticity;*

and consequently,

For each axis of elasticity of a perfect solid, the coefficient of longitudinal elasticity is equal to three times the sum of the two coefficients of rigidity for the coordinate planes which pass through that axis, diminished by three times the coefficient of rigidity for the plane normal to that axis.

We have now arrived at the conclusion, that in a body whose elasticity arises wholly from the mutual actions of atomic centres, all the coefficients of elasticity are functions of the three coefficients of rigidity. Rigidity being the distinctive property of solids, a body so constituted is properly termed a *perfect solid.*

When the three coefficients of rigidity are equal, the body is a perfect solid, equally elastic in all directions. The equations 6 and 8 become

$$A = 3C; \quad B = C,$$

agreeing with the results deduced by various mathematicians from the hypothesis of Boscovich.

§ IV.—*Results of the Hypothesis of Molecular Vortices.*

19. The great and obvious deviations from the laws of elasticity as deduced from the hypothesis of Atomic Centres, which many substances present, render some modification of it essential.

Supposing a body to consist of a continuous fluid, diffused through space with perfect uniformity as to density and all

other properties, such a body must be totally destitute of rigidity or elasticity of figure, its parts having no tendency to assume one position as to *direction* rather than another. It may, indeed, possess elasticity of *volume* to any extent, and display the phenomena of cohesion at its surface and between its parts. Its longitudinal and lateral elasticities will be equal in every direction; and they must be equal to each other by equation (5), which becomes

$$0 = A - B; \quad C = 0.$$

If we now suppose this fluid to be partially condensed round a system of centres, there will be forces acting between those centres, greater than those between other points of the body. The body will now possess a certain amount of rigidity; but less, in proportion to its longitudinal and lateral elasticities, than the amount proper to the condition of perfect solidity. Its elasticity will, in fact, consist of two parts, one of which, arising from the mutual actions of the centres of condensation, will follow the laws of perfect solidity; while the other will be a mere elasticity of volume, resisting change of bulk equally in all directions.

In a paper on the Mechanical Action of Heat, in connexion with the elasticity of gases and vapours, (*Transactions of the Royal Society of Edinburgh*, Vol. xx. Part 1.), I have attempted to develope some of the consequences of a supposition of this kind, called the Hypothesis of Molecular Vortices.* It assumes, that each atom of matter consists of a nucleus or central physical point, enveloped by an elastic atmosphere, which is retained in its position by forces attractive towards the atomic centre, and which, in the absence of heat, would be so much condensed round that centre as to produce the condition of perfect solidity in all substances: that the changes of condition and elasticity due to heat arise from the centrifugal force of revolutions among the particles of the atmospheres, diffusing them to a greater distance from their centres, and thus increasing the elasticity which resists change of volume alone, at the expense of that which resists change of figure also; and that the medium which transmits light and radiant heat consists of the nuclei of the atoms, of small mass, but exerting intense forces, vibrating independently, or almost independently, of their atmospheres; *absorption* being the communication of that motion to the atmosphere, so that it is lost by the nuclei.

* An abstract of that paper is published in Poggendorff's *Annalen* for 1850, No. ix.

In the equations of the propagation of light, Mr. Green effects a apparent reduction in the number of coefficients by introducing th supposition that the vibrations are of necessity wholly tangential each wave-front. But this supposition is quite at variance with th nature of elastic solids, and is obviously intended by the author merely an assumption for the purpose of facilitating calculation, a obtaining approximately true results, in the case of luminiferous u dulations.

Mr. M'Cullagh's researches on the propagation of light (*Trans. R Irish Acad.* XXI.) involve a similar assumption.

The result peculiar to the investigations contained in the prese paper, is the establishment of certain mutual relations amongst the d ferent elasticities of a given substance, whereby the *six* coefficien of Poisson and Cauchy are reduced to functions of *three*, and the n coefficients of Mr. Green to functions of *four*: the former representi the condition of a medium whose elasticity is wholly due to the mut actions of atomic centres; the latter, that of a substance whose conditi is intermediate between those of a system of centres of force, and a continuous and uniformly diffused fluid.

General equations of vibratory movement, in the particular case uncrystallized media, agreeing with those of Mr. Green, are given Professor Stokes in his memoir on Diffraction (*Camb. Trans.* IX.). E two coefficients of Elasticity have the following values in the notati of this paper:

Prof. Stokes.	*This paper.*
a^2	$\frac{Ag}{D} = (3C + J)\frac{g}{D}$,
b^2	$= \frac{Cg}{D}$.

g denotes the accelerating force of gravity; and D, the weight of un of volume of the vibrating medium.

a and b, in Professor Stokes's paper, are the velocities of propagati of normal and tangential vibrations respectively.

In the researches of Poisson, Navier, Cauchy, Lamé, and othe on the elasticity of bodies equally elastic in all directions, the coefficie are often expressed in terms of two quantities, denoted by k and in the following expression for a normal pressure on the plane yz,

$$P_1 = -kN_1 - K(N_1 + N_2 + N_3):$$

k represents a species of longitudinal elasticity, under the condition th the volume remains unchanged; and K, an elasticity resisting change volume. Their values in the notation of this paper are as follows:

$$k = A - B = 2C; \quad K = B = C + J.$$

It is evidently impracticable to apply an analogous notation to bodi unequally elastic in different directions.

M. Wertheim has recently made a most elaborate and valuable seri of experiments on the elasticity of brass, glass, and caoutchouc, accordi to a method suggested by M. Regnault, for the purpose of determini the laws of elasticity of uncrystallized substances (*Ann. de Chim. et Phys.*, ser. III. tom. XXIII.). He concludes that for brass and glass, an

caoutchouc moderately strained, the following equation is nearly not exactly true, in the notation to which I have just referred,

$$k = K;$$

which, in the notation of this paper, is equivalent to the following,

$$J = C; \quad B = 2C; \quad A = 4C.$$

M. Wertheim has investigated the consequences which must follow in the solution of several problems connected with elasticity, if this law be universally true for solid bodies.

This supposition must be regarded as doubtful; and it is not, indeed, advanced by M. Wertheim as more than a conjecture. So far as our present knowledge goes, it seems more probable that the relations between C and J may be infinitely varied. If the effect of heat is to diminish C and increase J, there may be some temperature for each substance at which M. Wertheim's equation is verified. In the sequel I shall consider more fully the consequences to be deduced from M. Wertheim's experiments on this subject.

§ V. *Coefficients of Pliability, and of Extensibility and Compressibility, Longitudinal, Lateral, and Cubic.*

Examples of their Experimental Determination.

21. Coefficients of elasticity serve to determine pressures from the corresponding strains. We have now to consider the determination of strains from pressures.

To determine a distortion from the corresponding tangential pressure, it is sufficient to multiply, using the negative sign, by the reciprocal of the proper coefficient of rigidity. This reciprocal may be called a coefficient of *pliability*.

A similar process, however, cannot be applied to the calculation of longitudinal strains from normal pressures; because, as each normal pressure is a function of all the three longitudinal strains, so each longitudinal strain is a function of all the three normal pressures.

Let the longitudinal strains be represented in terms of the normal pressures by the following equations,

$$\left.\begin{aligned} N_1 &= -\mathfrak{a}_1 P_1 + \mathfrak{b}_3 P_2 + \mathfrak{b}_2 P_3 \\ N_2 &= \mathfrak{b}_3 P_1 - \mathfrak{a}_2 P_2 + \mathfrak{b}_1 P_3 \\ N_3 &= \mathfrak{b}_2 P_1 + \mathfrak{b}_1 P_2 - \mathfrak{a}_3 P_3 \end{aligned}\right\} \ldots\ldots\ldots(10).$$

Then the coefficients in these equations are found, by a process of elimination, to have the following values in terms of the coefficients of elasticity.

Let

$$K = 24\,(C_2^2 C_3 + C_2 C_3^2 + C_3^2 C_1 + C_3 C_1^2 + C_1^2 C_2 + C_1 C_2^2 - C_1^3 - C_2^3 - C_3^3)$$
$$- 52\,C_1 C_2 C_3 + J\{8\,(C_2 C_3 + C_3 C_1 + C_1 C_2) - 4\,(C_1^2 + C_2^2 + C_3^2)\}.$$

Then

$$a_1 = \frac{1}{K}\{\text{[illegible]}\}$$

$$a_2 = \frac{1}{K}\{\text{[illegible]}\}$$

$$a_3 = \frac{1}{K}\{\text{[illegible]}\} \quad \ldots (11)$$

$$b_1 = \frac{1}{K}\{\text{[illegible]} - 2(C_2 - C_3 - C_1)J\}$$

$$b_2 = \frac{1}{K}\{\text{[illegible]} - 2(C_3 - C_1 - C_2)J\}$$

$$b_3 = \frac{1}{K}\{\text{[illegible]} - 2(C_1 - C_2 - C_3)J\}$$

The above coefficients may be thus classified:

a_1 a_2 a_3 are the coefficients of *longitudinal extensibility* *compressibility* parallel respectively to the three axe elasticity.

b_1 b_2 b_3 are the coefficients of *lateral extensibility and* *pressibility* for the three coordinate planes of elasticity, ser to determine the effect of a normal pressure on those di sions of a body which lie at right angles to its direction.

From the manner in which the coefficient J enters the common denominator K, it is obvious that when coefficients of rigidity diminish without limit as comp with that of fluid elasticity, the six coefficients of li extensibility and compressibility increase *ad infinitum*.

In a body whose three coefficients of rigidity are diffe the coefficient of cubic compressibility, that is to say, quotient of the sum of the three longitudinal strains by mean of the three normal pressures, with the sign char has no fixed value unless some arbitrary relation be between those pressures. Let them be supposed, then, all equal; let their common value be P, and let the coeffi of cubic compressibility in this case be denoted by $\mathfrak{d}$: the

$$\mathfrak{d} = -\frac{N_1+N_2+N_3}{P} = a_1 + a_2 + a_3 - 2(b_1 + b_2 + b_3)$$

$$= \frac{1}{K}\{8(C_2C_3 + C_3C_1 + C_1C_2) - 4(C_1^2 + C_2^2 + C_3^2)\} \quad \ldots$$

Hence $\frac{1}{\mathfrak{d}} = J - 6(C_1 + C_2 + C_3)$

$$- \frac{196\,C_1C_2C_3}{8(C_2C_3 + C_3C_1 + C_1C_2) - 4(C_1^2 + C_2^2 + C_3^2)}$$

[redacted]

o that this coefficient is the sum of the three longitudinal icients of compressibility, diminished by twice the sum e three lateral coefficients. It does not, like them, in- *ad infinitum* when the rigidity vanishes; its ultimate in that case being

$$\frac{1}{J},$$

iprocal of the coefficient of fluid elasticity; as might een expected.

$C_2 = C_3$, so that the body is equally elastic in all ns round the axis of x, equations (11) and (12) take owing forms:

$$\left.\begin{array}{l}
\{12C_1C_3 - 6C_1^2 - C_3^2 + J(4C_3 - C_1)\} \\
8C_1^2 + 4C_1J) = \dfrac{2C_1 + J}{12C_1C_3 - 6C_1^2 - C_3^2 + J(4C_3 - C_1)} \\
\dfrac{1}{K}\{8C_3^2 - 9(C_1 - C_3)^2 + 4C_3J\} \\
6C_1C_3 - 3C_1^2 - C_3^2 + J(4C_3 - 2C_1)\} \\
\dfrac{1}{K}(2C_1C_3 + 2C_1J) = \dfrac{C_3 + J}{24C_1C_3 - 12C_1^2 - 2C_3^2 + J(8C_3 - 2C_1)} \\
16C_1C_3 - 4C_1^2) = \dfrac{4C_3 - C_1}{12C_1C_3 - 6C_1^2 - C_3^2 + J(4C_3 - C_1)} \\
12C_3 + 6C_1 - \dfrac{49C_3^2}{4C_3 - C_1}
\end{array}\right\} \quad \ldots. \text{(12A)}.$$

odies equally elastic in all directions the coefficients ressibility and extensibility take the following values:

$$\left.\begin{array}{ll}
\mathfrak{a} = \dfrac{2C + J}{5C^2 + 3CJ}; & \mathfrak{b} = \dfrac{C + J}{10C^2 + 6CJ}. \\
\mathfrak{d} = 3\mathfrak{a} - 6\mathfrak{b} = \dfrac{3}{5C + 3J}; & \therefore \dfrac{1}{\mathfrak{d}} = J + \frac{5}{3}C.
\end{array}\right\} \ldots \text{(12B)}.$$

bstances of this kind the coefficient of cubic compres- is the same, whether the three normal pressures are r unequal, being equal to the sum of the three longi- strains divided by the mean of the three normal es with the sign changed: that is to say,

$$\mathfrak{d} = -3\,\frac{N_1 + N_2 + N_3}{P_1 + P_2 + P_3}.$$

One of the most frequent errors in investigations respecting the elasticity and strength of materials, and the propagation of sound, has been to confound the coefficients of longitudinal elasticity with the reciprocals of the coefficients of longitudinal compressibility. The equations of this section shew clearly how widely these two classes of quantities may differ.

The reciprocal of the longitudinal extensibility, $\frac{1}{a}$, is what is commonly termed the *Weight of the Modulus of Elasticity*.

22. The following formula may be found useful in the determination of the coefficient J of fluid elasticity from experimental data.

Let us suppose that the three coefficients of rigidity of a substance, C_1, C_2, C_3, have been determined by experiments on torsion, and that some one of the coefficients of compressibility and extensibility in equation (11), or those derived from it, has also been determined by experiment. Let the actual value of this coefficient be called ϵ, and the value which it would have had, had J been $= 0$, ϵ_0. Also let K_0 denote the value which the denominator K would have had, had J been $= 0$, and let n be the factor by which J is multiplied in the numerator of ϵ, and m, in the denominator.

Then $$J = K_0 \cdot \frac{\epsilon_0 - \epsilon}{m\epsilon - n} \quad \ldots\ldots\ldots\ldots (13).$$

When applied to coefficients of longitudinal extensibility, this formula labours under the disadvantage, that a comparatively slight error in the experimental data may cause a serious error in the determination of J. Let us take, for example, an uncrystallized substance, and make successively the two following suppositions,

$$J = 0, \quad J = C:$$

it will be found that the results are, respectively,

$$a = \frac{1}{C} \times 0.4, \quad a = \frac{1}{C} \times 0.375,$$

being in the ratio of 16 : 15; so that any uncertainty in the experiments is in this case increased fifteen-fold in computing the value of $\frac{J}{C}$. Hence it appears, that without very great precision in the experiments, the coefficient of fluid elasticity cannot be satisfactorily determined by a comparison of the

s of longitudinal tension with those of torsion. It is
ally desirable, that the two sets of experiments should
de on the same piece of the material.

best data for calculations of this kind would be experi-
on cubic compressibility, in conjunction with experi-
on torsion; for as equations (12), (12A), and (12B)
in order to determine *J*, we have simply to subtract a
symmetrical function of the rigidities from the re-
il of the cubic compressibility. In the process of
tion, the errors in the experiments on rigidity are
lied, on an average, by $\frac{2}{3}$ only, while those of the
nents on compressibility sustain no augmentation what-

t after data of this kind, may be ranked experiments
gitudinal extensibility, as compared with the cubic
ibility or compressibility of the same piece of material.
method, suggested by M. Regnault and carried into
by M. Wertheim, I shall presently speak more fully.

re it possible to ascertain the velocity of sound in an
ted mass of an elastic material, along each of the axes
ticity, the coefficients of longitudinal elasticity could be
ined with great precision by the formula

$$A = \frac{v^2 D}{g},$$

e *v* is the velocity of sound, *D* the weight of unity
lume of the substance, and *g* the accelerating force of
y. But it is only practicable to determine the velocity
nd along prismatic or cylindrical rods; and, as I shall
in a subsequent paper, it is impossible, in the present
of our knowledge of the molecular condition of the
ficial particles of solid bodies, to assign theoretically
atio in which the velocity of sound along a rod is
an its velocity in an indefinitely extended mass. That
is only known empirically in a few cases, having va-
values lying between 1 and $\sqrt{\frac{2}{3}}$.

The experiments of M. Wertheim, on longitudinal
ubic extensibility (*Ann. de Ch. et de Phys.*, ser. III.
XIII.), were made upon brass and crystal; the results
calculated on the supposition that those substances are
geneous, and equally elastic in all directions. There
e no doubt of the correctness of this supposition with
t to well-annealed crystal; and with respect to brass,
probably very near *the truth.*

[illegible]

...periments of M. Wertheim were made on [illegible] In the following [illegible] are designated as M. Wertheim has [illegible] the coefficients [illegible] the mean results of numerous [illegible] expresses the fraction by which [illegible] is diminished or increased in bulk by [illegible] of one kilogramme on the square millimetre [illegible] [illegible] the number is [illegible]) [illegible] coefficients, and the coefficients of [illegible] by equation (14A), are given in [illegible] on the square millimetre.

Table of Coefficients calculated from M. Wertheim's experiments on the Extensibility of Brass and Crystal.

	Extensibilities per Kilogramme on the square Millimètre.		Reciprocals of the Extensibilities in Kilogrammes on the square Millimètre.		Coefficients of Elasticity in Kilogrammes on the square Millimètre.			
	Longitudinal. $\mathfrak{a}$	*Cubic.* $\mathfrak{b}$	$\frac{1}{\mathfrak{a}}$	$\frac{1}{\mathfrak{b}}$	*Rigidity.* C	*Fluid.* J	*Longitud.* A	*Lateral.* B
Brass...... Tube I.	0.0000939	0.0000904	10645.2	11058	3973	4436	16355	8409
" ...... " II.	0.0001015	0.0000942	9855.2	10620	3663	4515	15504	8178
" ...... " III.	0.0001035	0.0000979	9664.9	10216	3600	4216	15016	7816
Crystal ... Tube I.	0.0002596	0.0002803	3852.5	3569	1459.2	1136.6	5514.2	2595.8
" ... " II.	0.0002324	*	4302.6					
" ... " III.	0.0002873	0.0002568	3481.1	3894	1288.0	1747.0	5611.0	3035.0
" ... " IV.	0.0002258	0 0002543	4429.0	3933	1687.4	1120.9	6183.1	2803.3
" ... " V.	0.0002284	0.0002234	4379.1	4476	1637.7	1746.8	6659.9	3384.5

* Tube II. of Crystal was accidentally broken.

The various degrees of elasticity of the brass tubes are ascribed by M. Wertheim to the relative frequency with which they were subjected to wire-drawing, to reduce the thickness of metal. It may be observed, that this operation seems to increase the rigidity more than the fluid elasticity; a fact which might naturally have been expected.

The means of the three sets of results for brass are given in the following Table:

Coefficients of		Kilogrammes on the sq. Millim.	Lbs. Avoid. on the square Inch.
Rigidity	C	3745.3	5,327,100
Fluid Elasticity	J	4389.0	6,242,700
Longitudinal Elasticity	A	15625.0	22,224,000
Lateral Elasticity	B	8134.3	11,579,000
Reciprocals of Extensibilities.			
Longitudinal (or Weight of the Modulus of Elasticity)	$\frac{1}{\mathfrak{a}}$	10,054.4	14,301,000
Cubic	$\frac{1}{\mathfrak{b}}$	10,631.0	15,121,000

Coefficients of Extensibility and Compressibility.		Per Kilog. on the square Millim.	Per lb. on the square Inch.
Longitudinal.........	$\mathfrak{a}$	0.00009946	0.000,000,0699
Cubic	$\mathfrak{b}$	0.00009406	0.000,000,0661
Lateral	$\mathfrak{f}$	0.00003405	0.000,000,0239

The following result is calculated from the experiments of M. Savart on the torsion of brass wire (*Ann. de Chim. et de Phys.*, August 1829):

		Kilog. on the square Millim.	Lbs. on the square Inch.
Coefficient of Rigidity	C	3682	5,237,100
The difference being...........		63.3	90,000

Hence we see, that the rigidity of wire-drawn brass, as determined directly by torsion, differs from that calculated from the longitudinal and cubic extensibilities by only *one-sixtieth part*, being a very small discrepancy in experiments of this kind.

The following Table gives the means of the four sets of results, I., III., IV., V., for crystal:

	Kilog. on the square Millim.	Lbs. on the square Inch.
C	1518	2,159,100
J	1438	2,045,300
A	5992	8,522,600
B	2956	4,204,400
$\frac{1}{\mathfrak{a}}$	4039	5,746,100
$\frac{1}{\mathfrak{b}}$	3968	5,643,800

	Per Kilog. on the square Millim.	Per lb. on the square Inch.
$\mathfrak{a}$	0.0002476	0.0000001740
$\mathfrak{b}$	0.0002520	0.0000001772
$\mathfrak{b}$	0.0000818	0.0000000575

It is obvious that the above mean values for crystal are not to be relied upon as equally accurate with those for brass; for the wide discrepancies between the results of the experiments on the five crystal tubes shew that this substance, like every kind of glass, is subject to great variations in the physical properties of different specimens.

24. So far as I am aware, there is no substance, whose elasticity varies in different directions, for which experimental data as yet exist, adequate to determine the three coefficients of rigidity, and the coefficient of fluid elasticity.

Supposing the three coefficients of rigidity of a substance of this kind to be known by experiments on torsion, the process of MM. Regnault and Wertheim would readily furnish data for calculating the fluid elasticity.

For example: let a tension R per unit of area be applied to the ends of a tube whose axis is one of the axes of elasticity; say that of x. Let $\frac{\Delta U}{U}$ be the fraction by which its volume is increased, as before. Then

$$\left.\begin{aligned} \frac{\Delta U}{RU} &= -\frac{N_1 + N_2 + N_3}{P_1} = \mathfrak{a}_1 - \mathfrak{b}_2 - \mathfrak{b}_3 \\ &= \frac{1}{K}\{8C_1^2 - 2C_1(C_2 + C_3) - 6(C_2 - C_3)^2\} \end{aligned}\right\} \ldots(15).$$

Let the above equation be abbreviated into

$$\frac{\Delta U}{RU} = \frac{\phi(C)}{K_0 + mJ},$$

where $K = K_0 + mJ$, as in equation (13). Then

$$J = \frac{1}{m}\left\{\frac{RU}{\Delta U}\phi(C) - K_0\right\} \ldots\ldots\ldots (15\mathrm{A}).$$

The formulæ corresponding to equation (15) for tubes whose axes are parallel to y and z, are easily found by permutations of the indices 1, 2, 3. The sum of the three values of $\frac{\Delta U}{RU}$ thus obtained is obviously $= 0$.

[It may be remarked, with reference to § 17 of the preceding paper, that the effect of alterations of direction in the lines joining pairs of particles is not taken into account in the investigation of the elastic forces arising from the states of strain which are there considered. It appears to me that this effect, except for particular laws of force, will be of the same order as that which depends on the alterations of the mutual distances between the particles; and that if it be taken into account, the demonstration of Theorem IV. fails.

This objection occurred to me after the whole of the paper was in type, and I immediately suggested it to the author; but, as he was not convinced of the correctness of my view, he desired that the paper should be published as it stands, reserving additional explanations or modifications, if necessary, for the next Number of the *Journal*.—W. T.]

MATHEMATICAL NOTES.

I.—A Demonstration of Taylor's Theorem.

By Homersham Cox, B.A., Jesus College, Cambridge.

The following concise proof of Taylor's Theorem, with an expression for the remainder after any number of terms, appears to be free from any objectionable assumption.

If the quantities $f(a)$, $f'(a)$, $f''(a)$, &c. be finite, the following expression, in which n is a positive integer, a and h finite constant quantities,

$$\left.\begin{array}{l} f(a+x)-f(a)-f'(a).x-f''(a).\dfrac{x^2}{1.2}-\ldots f^{n-1}(a).\dfrac{x^{n-1}}{1.2.3..(n-1)} \\ -\dfrac{x^n}{h^n}\left\{f(a+h)-f(a)-f'(a).h-f''(a).\dfrac{h^2}{1.2}-\ldots f^{n-1}(a).\dfrac{h^{n-1}}{1.2.3..(n-1)}\right\} \end{array}\right\} \ldots. (1),$$

is zero when $x = 0$, and also when $x = h$.

A function which is zero at two limits cannot be always increasing nor always decreasing. Hence, if (1) and its derivatives be continuous, there is some value (x_1) of the variable between 0 and h, for which the differential coefficient

of (1) (*i.e.* its rate of increase) is zero. Or

$$f'(a+x)-f'(a)-f''(a).x-\ldots-\frac{n.x^{n-1}}{h^n}\{f(a+h)-f(a)-f'(a).h-\ldots\}\ldots(2)$$

is zero when $x=x_1$. (2) is also zero when $x=0$. Therefore, as before, there is another intermediate value (x_2) between x_1 and 0, for which the differential coefficient of (2) is zero. That differential coefficient is also zero when $x=0$.

Continuing the process to n differentiations of (1), we find that

$$f^n(a+x)-\frac{1.2\ldots n}{h^n}\{f(a+h)-f(a)-f'(a)h-\ldots\}$$

is zero when x has some intermediate value between 0 and h. This value may be expressed by θh, where θ is a proper fraction. Then

$$f^n(a+\theta h)-\frac{1.2\ldots n}{h^n}\{f(a+h)-f(a)-f'(a)h\ldots\}=0,$$

or $f(a+h)=f(a)+f'(a)h+f''(a)\dfrac{h^2}{1.2}$

$$+\ldots+f^{n-1}(a)\frac{h^{n-1}}{1.2.3\ldots n-1}+f^n(a+\theta h)\frac{h^n}{1.2\ldots n}.$$

If the last term of the second side of this equation decrease indefinitely as n increases,

$$f(a+h)=f(a)+f'(a)h+f''(a)\frac{h^2}{1.2}+\ldots \textit{ ad infinitum};$$

which is Taylor's Theorem.

II.—*Demonstration of the known Theorem, that the Arithmetic Mean between any number of Positive Quantities is greater than their Geometric Mean.*

By A. Thacker.

If x be a positive quantity, and n integral, we have, by the Binomial Theorem,

$$\left(1+\frac{x}{n}\right)^n=1+x+\frac{1-\frac{1}{n}}{2}.x^2+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{2.3}x^3+\ldots, \text{ and}$$

$$\left(1+\frac{x}{n-1}\right)^{n-1} = 1+x+\frac{1-\frac{1}{n-1}}{2}\cdot x^2+\frac{\left(1-\frac{1}{n-1}\right)\left(1-\frac{2}{n-1}\right)}{2.3}x^3\cdot$$

Here $1-\frac{1}{n} > 1-\frac{1}{n-1}$, $1-\frac{2}{n} > 1-\frac{2}{n-1}$, and so on; he every term involving n in the first series is greater than corresponding term in the second; and all the terms positive. Therefore, obviously,

$$\left(1+\frac{x}{n}\right)^n > \left(1+\frac{x}{n-1}\right)^{n-1} \text{ .... (1}$$

Now let there be n positive quantities, a_1, a_2, ... a_n, ranged in order of magnitude, a_1 being the least.

Then

$$\left(\frac{a_1+a_2+\ldots+a_n}{n}\right)^n = a_1^n\left\{1+\frac{a_1+a_2+\ldots+a_n-na_1}{na_1}\right\}^n$$

$$> a_1^n\left\{1+\frac{a_1+a_2+\ldots+a_n-na_1}{(n-1)a_1}\right\}^{n-1} \quad \text{by}$$

$$> a_1\left(\frac{a_2+a_3+\ldots+a_n}{n-1}\right)^{n-1}$$

In the same way

$$\left(\frac{a_2+a_3+\ldots+a_n}{n-1}\right)^{n-1} > a_2\left(\frac{a_3+a_4+\ldots+a_n}{n-2}\right)^{n-2}$$

$$\ldots\ldots > \ldots\ldots$$

$$\left(\frac{a_{n-1}+a_n}{2}\right)^2 > a_{n-1}a_n.$$

Hence, by multiplication we get

$$\left(\frac{a_1+a_2+\ldots+a_n}{n}\right)^n > a_1a_2\ldots a_n,$$

or $$\frac{a_1+a_2+\ldots+a_n}{n} > (a_1a_2\ldots a_n)^{\frac{1}{n}}. \qquad \text{Q. E. D.}$$

This proof seems simpler and more symmetrical tha one given by Cauchy, *Cours d'Analyse*, p. 458.

If there be n things, A, B, C, &c.—out of which a combination of m is to be taken. Some combinations have no *break*, as $ABCD\ldots$; some have one or more, as $AB \mid D \mid G \ldots$.

Let m_n stand for the number of ways in which m can be taken out of n. Then the number of ways in which m can be taken out of n, with neither more nor fewer than p breaks, is

$$p_{m-1} \times (p+1)_{n-m+1};$$

from which of course it follows that (0_k being unity)

$$m_n = 0_{m-1} \cdot 1_{n-m+1} + 1_{m-1} \cdot 2_{n-m+1} + 2_{m-1} \cdot 3_{n-m+1} + \ldots\ldots$$

A. DE M.

ON THE CONSTRUCTION OF THE NINTH POINT OF INTERSECTION OF TWO CURVES OF THE THIRD DEGREE, WHEN THE OTHER EIGHT POINTS ARE GIVEN.

By Thomas Weddle, F.R.A.S., Royal Military College, Sandhurst.

Denote the eight given points by 1, 2, 3 .... 8; also let $(12) = 0$ be the equation to the straight line passing through the two points 1, 2; and $(12345) = 0$,* the equation to the conic passing through the five points 1, 2, 3, 4, 5.

If the constants λ and μ be properly taken, the equations to two particular curves of the third degree passing through the eight given points, will be

$$(12345).(67) = \lambda.(12346).(57) \ldots\ldots (1),$$

and

$$(12345).(68) = \mu.(12346).(58) \ldots\ldots (2).$$

Divide the former equation by the latter, and reduce,

therefore $$\lambda.(57).(68) = \mu.(58).(67) \ldots\ldots\ldots (3),$$

which is evidently the equation to the conic passing through the points 5, 6, 7, 8, and the "ninth" point sought.

Again, let us interchange the points 1 and 8 in all that precedes. The equations to two particular curves of the third degree passing through the eight given points will thus be

$$(82345).(67) = \rho.(82346).(57) \ldots\ldots\ldots (4),$$

and

$$(82345).(61) = \sigma.(82346).(51) \ldots\ldots\ldots (5),$$

where ρ is such a constant as to make (4) pass through the point 1, and σ such as to make (5) pass through 7.

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* Of course, (12345) will $= (12).(34) + k\,(14).(23)$, where the [con]stant k is determined *so that the conic shall pass* through the point 5.

Divide (4) by (5); therefore

$$\rho\,.\,(57)\,.\,(61) = \sigma\,.\,(51)\,.\,(67)\;........\;(6),$$

which is the equation of the conic passing through the points 1, 5, 6, 7, and the "ninth" point.

Hence, the point sought will be the fourth point of intersection of the two conics (3) and (6), three of the points 5, 6, 7, of intersection being already known.

The method of finding this point algebraically, by eliminating x or y from the equations (3) and (6), (always bearing in mind that we know three roots of the resulting biquadratic), seems therefore evident enough, and I have only to shew how the point may be found geometrically.

Before doing so, I must however be permitted to state that I lay no claim to the preceding analysis. It was communicated to me by the Rev. T. P. Kirkman, in a letter dated April 16th, 1850; and I immediately inferred the construction about to be given. It would seem however that Mr. K. was not the first to solve this important problem, for in a subsequent letter to me he said, "I have heard from Dublin that Mr. Hart had made known in the University the method of finding the ninth point that I have given you about a month before I hit upon it. The proof he gives i pretty much the same as mine. Let

$$(94)\,.\,(23567) - (92)\,.\,(43567) = 0$$

be the curve through six points 234567. If this contain 1, 8, we have $\left(\frac{94}{92}\right)_1 = r$, a given ratio, and $\left(\frac{94}{92}\right)_8 = r_1$, a give ratio, so that the anharmonic ratio of the points 2481 is giver and 9 is on a given conic."

To facilitate the subsequent investigation, I shall suppos that, in the preceding analysis, the origin of coordinates ha been taken at P the intersection of the lines (81) and (72) that the absolute term in each of the equations (12345) = ((12346) = 0, (82345) = 0, and (82346) = 0, is + 1; and tha (67), (57), (68), (58), (61), and (51) have been multiplied b such constants that the values which these symbols take a any point (xy) are equal to the portions of the straight lin drawn through (xy) parallel to a line Z (assumed at plea sure), and intercepted between (xy) and the lines (67), (57 (68), (58), (61), and (51) respectively. Also by the notatio 12, (without the parenthesis), I mean the *distance* betwee the points 1 and 2, by $P2$ the distance between P and 2 &c.

In the conic (3), we know the four points 5, 6, 7, 8, an I shall now shew how a fifth point may be found.

Find the other points a and a'* in which the conics (12345) and (12346) are intersected by the straight line (81), and the other points b, b', in which the same conics are intersected by the straight line (72). Now, (1), the equation to determine λ is

$$\lambda = \frac{(12345)_8 \cdot (67)_8}{(12346)_8 \cdot (57)_8},$$

where $(12345)_8$, &c. mean the values which (12345), &c. take when the coordinates of the point 8 are substituted for x and y.

Now (*Mathematician*, vol. II. p. 74), $(12345)_8 = \dfrac{81 \cdot 8a}{P1 \cdot Pa}$, and $(12346)_8 = \dfrac{81 \cdot 8a'}{P1 \cdot Pa'}$; also $(67)_8$ and $(57)_8$ are the portions of the line drawn through the point 8 parallel to Z, and intercepted between 8 and the lines (67) and (57) respectively; and they are therefore known lines: therefore

$$\lambda = \frac{8a \cdot Pa' \cdot (67)_8}{8a' \cdot Pa \cdot (57)_8}.$$

Similarly,
$$\mu = \frac{7b \cdot Pb' \cdot (68)_7}{7b' \cdot Pb \cdot (58)_7};$$

therefore
$$\frac{\lambda}{\mu} = \frac{8a \cdot Pa' \cdot 7b' \cdot Pb \cdot (67)_8 \cdot (58)_7}{8a' \cdot Pa \cdot 7b \cdot Pb' \cdot (57)_8 \cdot (68)_7} \quad \ldots\ldots (7),$$

and this determines the ratio $\lambda : \mu$.

Through the point 8 draw a line parallel to Z, meeting the line (57) in c, and (67) in d; also suppose k to be the other point in which this line intersects the conic (3). At the point k we have

$$(58) = k8,\ (68) = k8,\ (57) = kc, \text{ and } (67) = kd;$$

therefore (3),
$$\lambda \cdot kc = \mu \cdot kd,$$

or,
$$kc : kd :: \mu : \lambda \quad \ldots\ldots\ldots\ldots (8).$$

Hence we may find a fifth point on the conic (3) in the following manner.

Join 8 and 1, and 7 and 2, by straight lines intersecting in P; and find a, b, the other points of intersection of these lines with the conic passing through the points 12345; also find the other points a', b' of intersection of the same lines with the conic passing through 12346. Through 8 and 7 draw any parallel lines, the former line meeting the l

* This can, of course, be done in several ways, one of which imme[diately] follows from Pascal's theorem.

(57) and (67) in c and d, and the latter meeting (58) and (68) in e and f. To $8c$, $8a$, and Pa' find a fourth proportional p; to $8d$, $8a'$, and Pa, a fourth proportional q; to $7f$, $7b'$, and Pb, a fourth proportional r; to $7e$, $7b$, and Pb', a fourth proportional s; and, finally, to q, p, and r, find a fourth proportional t. Divide cd in k, so that

$$kc : kd :: s : t,$$

then k will be a point on the conic (3).

In like manner a point k' may be found on the conic (6); and having thus found a fifth point on each conic, we may now proceed as follows.

Having given five points 5, 6, 7, 8, k on one conic, draw the tangents to the conic at the points 6 and 7, and let them intersect in L; also having given five points 1, 5, 6, 7, k' on the other conic, draw tangents at the points 6 and 7, intersecting in M; join LM; join also the points 5, 6 and the points 5, 7 by straight lines intersecting LM in A and B respectively; join A, 7 and B, 6 by straight lines intersecting in 9; then 9 is the fourth point of intersection of the two conics (3) and (6), and therefore the required ninth point of intersection of the two curves of the third degree.

The truth of this construction is easily seen, for the two conics (3) and (6) having the same inscribed quadrilateral 5697, the intersection of each pair of tangents at the opposite angles 6 and 7, must lie on the line joining the intersections of the opposite sides of the quadrilateral.

It will be observed, that if we multiply (2) by an arbitrary constant θ, and add the result to (1), we have

$$(12345) . \{(67) + \theta . (68)\} = (12346) . \{\lambda (57) + \theta\mu . (58)\}$$

for the general equation to curves of the third degree passing through the eight given points.

From the last equation we also infer that the equation to a curve of the third degree may be written

$$C . s = C' . s'$$

in an infinite number of ways; where C and C' are of the second degree, and s and s' of the first. This equation seems to have some analogy to the equation of a conic circumscribed about a quadrilateral, and several interesting inferences may be drawn from it.

Nov. 22, 1850.

ON THE THEORY OF LINEAR TRASFORMATIONS.

George Boole, Professor of Mathematics, Queens' College, Cork.

In the prosecution of any special branch of analysis, it is irable sometimes to pause and endeavour to take a concted view of the methods and the results already attained. ach a retrospect may serve both to afford an estimate of actual state of progress, and to indicate the direction in hich future effort may most usefully be engaged. In the esent paper I desire to contribute something towards the complishment of this object for the Theory of Linear Transmations. At the same time I wish to make some additions that theory, which, in the course of the brief survey which proposed, will readily fall into their proper place and conxion. I shall endeavour to shew, that whenever the transmation of the algebraic equation of a curve or surface is fected by the substitution of new axes of coordinates given character but unknown in position, the discovery both of e relations among the constants of the two equations, and of e linear relations among the variables as dependent upon ose constants, may be reduced to an application of the eory of homogeneous functions of the second degree.

What the main objects of a theory of linear transformations , and what are the principal methods that have been vised for their accomplishment, will appear from a conderation of the following well-known problem. Let x, y, d x', y', be two sets of rectangular axes in the same plane d having the same origin. Let also

$$ax^2 + 2bxy + cy^2 = 1 \ldots\ldots\ldots\ldots (1),$$

$$a'x'^2 + 2b'x'y' + c'y'^2 = 1 \ldots\ldots\ldots\ldots (2),$$

the two forms which the equation of a line of the second der assumes when referred to those sets of axes respectively. is required to determine the relations connecting a, b, c with $, b', c'$, as also the linear relation connecting x, y with x', y', and termining the relative position of the axes. In particular, it required to determine a', c' as functions of a, b, c, when $= 0$.

The method originally employed for the solution of this oblem, consisted in assuming for the linear relations the neral forms

$$x = \alpha x' + \beta y',$$

$$y = \alpha' x' + \beta' y',$$

substituting these values of x and y in the first (1), and equating the coefficients of x'^2, $x'y'$, and y'

result, with the corresponding coefficients in the first member of (2); and, finally, in eliminating a, β, a', β', from the three equations thus formed, and the three equations (necessary in a rectangular transformation)

$$\begin{aligned} a^2 + \beta^2 &= 1, \\ a'^2 + \beta'^2 &= 1, \\ aa' + \beta\beta' &= 0, \end{aligned}$$

leaving two final equations between a, b, c, and a', b', c', from which, if $b' = 0$, the values of a' and c' may be determined. By this method, and its obvious extension in the case of three or more variables, it has been found possible to transform any homogeneous equation of the second degree into another equation of the same degree, wanting the terms which contain the products of the variables xy, xz, &c.*

In a paper published in the *Cambridge Mathematical Journal*, vol. III. pp. 1, 106, it was shewn that the linear substitutions above described and the consequent eliminations might be entirely dispensed with, by employing in their stead the characteristic equation of rectangular transformations,

$$x^2 + y^2 = x'^2 + y'^2 \text{ (for curves)},$$

$$x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2 \text{ (for surfaces)}.$$

For the former of these cases, the following was the form which the problem assumed. Required to determine the relations connecting abc with $a'b'c'$ when we have simultaneously

$$ax^2 + 2bxy + cy^2 = a'x'^2 + 2b'x'y' + c'y'^2,$$

$$x^2 + y^2 = x'^2 + y'^2,$$

the ultimate relations between x, y and x', y' being linear. Required also the expression of the latter relations.

Exhibited under this form, the problem was seen to be a particular case of the following more general problem. Given any pair of homogeneous equations,

$$Q = Q' \quad \dots\dots\dots\dots\dots\dots \quad (3),$$

$$R = R' \quad \dots\dots\dots\dots\dots\dots \quad (4),$$

of any order n, and with any number of variables m, in each member, the m variables in the second members being connected by unknown linear relations with the variables in the first members: required the relations among the constant coefficients of the given equations; also the linear relations.

* The complete analysis of the corresponding problem for homogeneous equations of the nth degree will be given in a sequel to this paper.

Of the general solution of this problem contained in the
per referred to, the following is that part which refers
the determination of the relations among the constant
efficients.

Writing $$Q + hR = Q' + hR' \ldots\ldots\ldots\ldots(5),$$

d supposing $x, y\ldots$ to be the variables in the first member,
, y', &c. to be those in the second, form the two sets of
uations

$$\frac{d(Q+hR)}{dx} = 0, \qquad \frac{d(Q+hR)}{dy} = 0 \ldots\ldots (6),$$

$$\frac{d(Q'+hR')}{dx'} = 0, \qquad \frac{d(Q'+hR')}{dy'} = 0 \ldots\ldots (7).$$

om each set eliminate the variables, and there will result
o equations involving h, the coefficients of the one being
nctions of the constants entering into Q and R, those of
e other functions of the constants entering into Q' and R'.
e condition that those equations shall be equivalent de-
mines the relations between the two sets of constants.

Thus, in the well-known case

$$ax^2 + by^2 + cz^2 + 2dyz + 2exz + 2fxy = a'x'^2 + b'y'^2 + c'z'^2,$$
$$x^2 + y^2 + z^2 \qquad = x'^2 + y'^2 + z'^2,$$

e obtain from the first members, after multiplying the
cond equation by h, adding, and taking the first differ-
tial coefficients divided by (2),

$$\begin{aligned}(a+h)\,x + \quad fy + \quad ez &= 0,\\ fx + (b+h)\,y + \quad dz &= 0,\\ ex + \quad dy + (c+h)\,z &= 0.\end{aligned}$$

hence, on elimination,

$$(a+h)(b+h)(c+h) + 2\,\text{DEF.} - (a+h)\,d^2 - (b+h)\,e^2 - (c+h)\,f^2 = 0 \ldots (8):$$

d if in this equation we change a, b, c, to a', b', c', and
e, f, to 0, we get for the second member, the corresponding
uation $$(a'+h)(b'+h)(c'+h) = 0.$$

e condition that these equations shall be equivalent re-
vely to h requires that $a'b'c'$ shall be the values of h, (with
s changed) derived from the cubic equation (8).

t will be seen that this theory essentially depend
properties of that function of the constant coeff

$2(n-1)$ relations among the constants. Now the number of terms in Q and R together in this case will be $2(n+1)$. There will therefore be $2(n+1)$ relations between the constants of Q, R, &c. and those (four in number) of the linear system, whence, eliminating the latter, we have $2(n-1)$ relations among the constants of Q, R, Q', R'. Here also, then, the theory gives the number of relations required.

When both the number of the variables and the degree of the functions exceed 2, the theory gives a greater number of relations than is required. Whence it may be inferred that some of the relations thus given are independent.

Another property of the function $\theta(Q)$, intimately connected with the above and noticed in the same memoir, is the following. If Q be linearly transformed to R, then

$$E^{\frac{\gamma n}{m}}\theta(Q)=\theta(R)\ldots\ldots\ldots\ldots(9),$$

E being that function of the constants in the linear system which is obtained by equating to 0; the second members, and eliminating the variables, γ being the degree of $\theta(Q)$, n that of Q, and m the number of the variables.

Farther, it is noticed that, since by differentiating the linear equations we obtain precisely the same relations between the differentials dx, dy, ... and dx', dy', ... as exist between x, y, ... and x', y', ..., and since also the values of the differentials are independent of the values of the variables, we may treat the successive derived equations

$$\frac{dQ}{dx}dx+\frac{dQ}{dy}dy=\frac{dQ'}{dx'}dx'+\frac{dQ'}{dy'}dy'\ldots\ldots(10),$$

$$\frac{d^2Q}{dx^2}dx^2+2\frac{d^2Q}{dxdy}dxdy+\frac{d^2Q}{dy^2}dy^2=\frac{d^2Q'}{dx'^2}dx'^2+\&c.\ldots(11),$$

reduces to $\theta(R)$, where R is what Q' becomes after y_1, y_2....y_r are made to vanish. Hence we have

$$\phi_1,\phi_2\ldots\ldots\phi_r\theta(Q)=E^n\theta(R).$$

I have ascertained that Mr. Sylvester's result is reducible to the above form.

There cannot be a doubt that for the discovery of the actual relation in question, the above theorem is far more convenient than Mr. Sylvester's. It is, however, quite possible that the particular form under which Mr. Sylvester's theorem is exhibited, may render it a more convenient instrument for the researches to which he proposes to apply it. The theorem of mine upon which he has commented, ought not to be disconnected from the system of which it forms a part.

Mr. Sylvester has, I am assured, too much love for truth to feel offended observations which are made, not with a view to establish a claim of rity, for which I care little, but to place in its true light a mathematical tion, otherwise very likely to be misunderstood.

(we confine ourselves for illustration to the case of two variables,) as homogeneous equations among the differentials dx, dy, dx', dy', just as if the differential coefficients

$$\frac{dQ}{dx}, \quad \frac{d^2Q}{dx^2}, \text{ \&c.}$$

were absolutely constant. This principle is of great utility when the functions Q and R are not of the same degree. For instead of discussing the equations directly, we may commence by the discussion of the second differential equations, which will be of the second order, with reference to dx, dy, dx', dy'.

These are all the really essential principles involved in the memoir referred to. I desire here to add to them two others which I shall proceed to demonstrate.

1st. Let there be two variables, x and y, in the first members which are linearly connected with x' and y', the variables of the second members, by the equations

$$x = \lambda x' + \mu y',$$

$$y = \lambda' x' + \mu' y' \ldots\ldots\ldots\ldots\ldots (12).$$

From these we shall find

$$x' = \frac{\mu' x - \mu y}{\lambda\mu' - \lambda'\mu},$$

$$y' = -\frac{\lambda' x + \lambda y}{\lambda\mu' - \lambda'\mu}.$$

Now $\dfrac{d}{dx} = \dfrac{dx'}{dx}\dfrac{d}{dx'} + \dfrac{dy'}{dx}\dfrac{d}{dy'}$; whence we get

$$\frac{d}{dx} = \frac{1}{\lambda\mu' - \lambda'\mu}\left(\mu'\frac{d}{dx'} - \lambda'\frac{d}{dy'}\right),$$

$$\frac{d}{dy} = \frac{1}{\lambda\mu' - \lambda'\mu}\left(-\mu\frac{d}{dx'} + \lambda\frac{d}{dy'}\right).$$

Let us represent $\lambda\mu' - \lambda'\mu$ by E; the above system may then be put in the form

$$-\frac{d}{dx} = \frac{1}{E}\left(\lambda'\frac{d}{dy'} - \mu'\frac{d}{dx'}\right),$$

$$\frac{d}{dy} = \frac{1}{E}\left(\lambda\frac{d}{dy'} - \mu\frac{d}{dx'}\right) \ldots\ldots\ldots (13).$$

But this is precisely what the system (12) would become if therein we changed

$$x \text{ into } \frac{d}{dy}, \qquad y \text{ into } -\frac{d}{dx},$$

$$x' \text{ into } \frac{1}{E}\frac{d}{dy'}, \qquad y' \text{ into } -\frac{1}{E}\frac{d}{dx'}.$$

Now if this change is permissible in the linear system, it is permissible in any equation derived from that system. We may therefore state the following principle:

If $\phi(x, y) = \psi(x', y')$ in virtue of linear relations connecting x and y with x' and y', then is the equation

$$\phi\left(\frac{d}{dy}, -\frac{d}{dx}\right) = \psi\left(\frac{1}{E}\frac{d}{dy'}, -\frac{1}{E}\frac{d}{dx'}\right) \ldots\ldots (14)$$

symbolically true. The utility of this result will shortly appear.

Consider, secondly, the case of any *rectangular* transformation, whether on a plane or in space. Taking for illustration the latter case as being the more general, we have

$$\begin{aligned} x &= ax' + by' + cz', & x' &= ax + a'y + a''z, \\ y &= a'x' + b'y' + c'z', & y' &= bx + b'y + b''z, \\ z &= a''x' + b''y' + c''z', & z' &= cx + c'y + c''z. \end{aligned}$$

Whence we find

$$\frac{d}{dx} = a\frac{d}{dx'} + b\frac{d}{dy'} + c\frac{d}{dz'},$$

$$\frac{d}{dy} = a'\frac{d}{dx'} + b'\frac{d}{dy'} + c'\frac{d}{dz'},$$

$$\frac{d}{dz} = a''\frac{d}{dx'} + b''\frac{d}{dy'} + c''\frac{d}{dz'}:$$

from which it appears that the system of relations between the symbols $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$, and $\frac{d}{dx'}, \frac{d}{dy'}, \frac{d}{dz'}$, is precisely the same as that connecting x, y, z, with x', y', z'. Hence the following theorem:

If $\phi(\mathrm{x}, \mathrm{y}, \mathrm{z}) = \psi(\mathrm{x}', \mathrm{y}', \mathrm{z}')$ *wherein* x, y, z *and* x′, y′, z′ *are rectangular coordinates having the same origin, then is the equation*

$$\phi\left(\frac{\mathrm{d}}{\mathrm{dx}}, \frac{\mathrm{d}}{\mathrm{dy}}, \frac{\mathrm{d}}{\mathrm{dz}}\right) = \psi\left(\frac{\mathrm{d}}{\mathrm{dx}'}, \frac{\mathrm{d}}{\mathrm{dy}'}, \frac{\mathrm{d}}{\mathrm{dz}'}\right) \ldots\ldots (15)$$

symbolically true.

Or, in other words, *In any equation obtained by rectangular transformation, we may change the variables* x, y, *&c. into the corresponding operating symbols* $\frac{\mathrm{d}}{\mathrm{dx}}, \frac{\mathrm{d}}{\mathrm{dy}}$, *&c.*

Some illustrations of this principle will also be given in a subsequent part of this paper. I have thought it necessary to exhibit together what appear to me to be the really fundamental principles of the theory, and I shall now shew their application to an important department of the inquiry which was originated by Mr. Cayley.

Hyperdeterminant Functions.

A year or two after the publication of the paper to which reference has chiefly been made, Mr. Cayley was led to the discovery that, for all homogeneous functions with two variables of an order higher than the third, there exist other functions than $\theta(Q)$, possessing those properties which I had regarded as peculiar to it. For the function of the fourth order $Q = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4$, Mr. Cayley discovered the constant function $ae - 4bd + 3c^2$, as possessing the requisite characters. Thus, if that function be represented by $\phi(Q)$, then, in the transformation of the simultaneous equations

$$Q = Q', \quad R = R',$$

two of the relations among the constants are given by the condition that the equations

$$\phi(Q + hR) = 0, \quad \phi(R + hR') = 0,$$

shall be equivalent. Moreover,

$$E^{\frac{1}{2}\gamma}\phi(Q) = \phi(Q'),$$

γ being the degree at $\phi(Q)$ and $\phi(Q')$.

Subsequently it was discovered that the function

$$ace + 2bdc - ad^2 - eb^2 - c^3,$$

which we will represent by $\psi(Q)$, possesses the same properties; and thus, instead of the six equations among the constants given by the condition of the identity of

$$\theta(Q + hR) = 0, \text{ and } \theta(Q' + hR') = 0,$$

we have the two equations given by the condition of the identity of

$$\phi(Q + hR) = 0, \quad \phi(Q' + hR') = 0,$$

the three equations given by the condition of the identity of

$$\psi(Q + hR) = 0, \quad \psi(Q' + hR') = 0,$$

and the single equation $\dfrac{\phi(Q)^3}{\psi(Q)^2} = \dfrac{\phi(Q')^3}{\psi(Q')^2}$.

That the two sextuple systems are equivalent, may b proved by means of a relation also found to exist in t above case, viz.

$$\theta(Q) = \phi(Q)^3 - 27\psi(Q)^2.$$

To illustrate the advantage of the system to which M Cayley's researches led, I may be allowed to mention, a the result of actual examination, that the reduction of th homogeneous function Q to the form $a'x'^4 + 6c'x'^2y'^2 + e'y'^4$, is matter of great difficulty, when the equations derived fro $\theta(Q)$ are employed, but is made easy by the employment o the other system. The functions, $\theta(Q)$, which have bee called determinants, and the more recently discovered func tions which have been denominated hyperdeterminants, ent as constituent elements into the forms of solution of algebra equations, and the solution of the equation of the fourt degree may be effected by their aid in a manner similar t that which is employed for the cubic equation in the memoir, vol. III. p. 119.

In the first volume of this *Journal* (new series) p. 104, M Cayley has adopted a method for the discovery of hype determinants, founded upon the principle of separation o symbols. I would, however, venture to suggest the follow ing mode as more easy of application. Let (14) be a homo geneous equation of the n^{th} order; then we have

$$E^n\phi\left(\frac{d}{dy}, \quad -\frac{d}{dx}\right) = \psi\left(\frac{d}{dy'}, \quad -\frac{d}{dx'}\right)\ldots.(16),$$

by which we can operate, not only upon the equatio $\phi(x, y) = \psi(x', y')$ from which it is derived, but upon any other equation obtained from the same system of linear trans formations.

For example, let the given equation be

$$ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4 = a'x'^4 + \&c.\ldots(17),$$

we have, on changing x into $\dfrac{d}{dy}$, y into $-\dfrac{d}{dx}$, &c.,

$$a\frac{d^4}{dy^4} - 4b\frac{d^3}{dx^3}, \ \frac{d}{dy} + \&c. = \frac{1}{E^2}\left(a'\frac{d^4}{dy'^4} - \&c.\right)\ldots(18).$$

And operating with this result upon the given equation, we get

$$ae - 4bd + 3c^2 = \frac{1}{E^2}(a'e' - 4b'd' + 3c'^2).\ldots.(18).$$

Hence $ae - 4bd + 3c^2$ is a hyperdeterminant of the second degree.

To obtain one of the third degree, let us first consider any equation of the second degree,

$$ax^2 + 2bxy + cy^2 = a'x'^2 + \&\text{c.} \ldots\ldots\ldots (19);$$

the operating equation hence derived is

$$a\frac{d^2}{dy^2} - 2b\frac{d^2}{dxdy} + c\frac{d^2}{dx^2} = \frac{1}{E^2}\left(a'\frac{d^2}{dy'^2} - \&\text{c.}\right),$$

which, applied to the previous equation, gives

$$ac - b^2 = \frac{1}{E^2}(a'c' - b'^2) \ldots\ldots\ldots\ldots (20).$$

Now the equation (17) after two differentiations, gives

$$(ax^2 + 2bxy + cy^2)dx^2 + 2(bx^2 + 2cxy + dy^2)dxdy + (cx^2 + 2dxy + ey^2)dy^2$$
$$= (a'x'^2 + 2b'x'y' + c'y'^2)dx'^2 + \&\text{c.},$$

which, by a principle already stated in this paper, may be treated in all respects as a homogeneous equation of the second order, between dx, dy, and dx', dy'. Whence, by (27), we have

$$(ax^2 + 2bxy + cy^2)(cx^2 + 2dxy + ey^2) - (bx^2 + 2cxy + dy^2)^2 = \frac{1}{E^2}U,$$

U being what the first member would become if each of the letters were dashed. Perform on this equation expanded the operations represented by (18), and we get

$$ace + 2bcd - ad^2 - eb^2 - c^3 = \frac{1}{E^4}(a'c'e - a);$$

whence $ace + 2bcd - ad^2 - eb^2 - c^3$, is a hyperdeterminant of the third degree. And for all orders, these functions may thus be determined by the performance of a single operation upon a single function.

On applying this method to the homogeneous function of the fifth order,

$$ax^5 + 5bx^4y + 10cx^3y^2 + 10dx^2y^3 + 5exy^4 + fy^5.$$

I find, beside Mr. Cayley's function of the fourth order, viz. $AC - B^2$, wherein

$$A = 2(bf - 4ce + 3d^2),\quad B = af - 3be + 2cd,\quad C = 2(ae - 4bd + 3c^2),$$

the following function of the eighth order, viz.

$$A(\beta\delta - \gamma^2) + B(\gamma\beta - \alpha\delta) + C(\alpha\gamma - \beta^2),$$

wherein
$$\alpha = bdf - be^2 + 2cde + c^2f - d^3,$$
$$3\beta = adf - ae^2 - bcf + bde + c^2e - cd^2,$$
$$3\gamma = acf - ade - b^2f + bd^2 + bce - c^2d,$$
$$\delta = ace - ad^2 - b^2e + 2bcd - c^3.$$

It appears from the researches of Mr. Salmon and Cayley, that $\theta(Q)$ is made up of two parts, one being the al function of the fourth order squared, the other the al function of the eighth order with a numerical coeffici We may gather from what precedes, that the employmen the hyperdeterminant functions would not in any mat degree facilitate the transformation of homogeneous funct of the fifth degree, since these, like $\theta(Q)$, would intro equations of the eighth order. They are however in all c of great theoretical interest, and their discovery constit a most important step in the theory of the linear tr formation of functions of two variables.

Of Rectangular Transformations.

The remarkable property expressed by equation (15) ables us to deduce relations among the constants in any of rectangular transformation with a facility which is q peculiar.

Suppose that we have

$$ax^2 + 2bxy + cy^2 = a'x'^2 + 2b'x'y' + c'y'^2 \ldots (24$$

the two systems x, y, and x', y' being rectangular; ther have also
$$x^2 + y^2 = x'^2 + y'^2.$$

Now by (15), changing in these equations x, y, x', y' $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dx'}, \frac{d}{dy'}$, we have

$$a\frac{d^2}{dx^2} + 2b\frac{d^2}{dxdy} + c\frac{d^2}{dy^2} = a'\frac{d^2}{dx'^2} + \&c.,$$
$$\frac{d^2}{dx^2} + \frac{d^2}{dy^2} = \frac{d^2}{dx'^2} + \frac{d^2}{dy'^2}.$$

Operate with these equations upon (24), and we get

$$a^2 + 2b^2 + c^2 = a'^2 + 2b'^2 + c'^2,$$
$$a + c = a' + c',$$

which are the relations required, the system being equiva to the well-known system

$$ac - b^2 \ldots = a' + c'.$$

Let $q = q'$ be a homogeneous equation in which q is of the form

$$q = \Sigma \frac{1.2\ldots n}{1.2\ldots\alpha\ 1.2\ldots\beta\ 1.2\ldots\gamma}\, ax^{\alpha}y^{\beta}z^{\gamma},$$

the numerical factor of each coefficient being the same as the corresponding numerical coefficient given by the multinomial theorem, and q' being a function similar in form of x', y', z'. Then if in the above we change x, y, &c. into $\frac{d}{dx}, \frac{d}{dy}$, &c., and operate upon the given equation, we shall obtain the same result as if we rejected in the given equation the variables, and substituted for each literal factor its square. Thus from the equation

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3 = a'x'^3 + \text{\&c.}$$

we should get $\quad a^2 + 3b^2 + 3c^2 + d^2 = a'^2 + \text{\&c.}$

Also, confining our attention to three variables x, y, z, since

$$x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2,$$

we have
$$\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2} = \frac{d^2}{dx'^2} + \text{\&c.}$$

Operating with this equation upon the primitive i times in succession, we have

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)^i q = \left(\frac{d^2}{dx'^2} + \text{\&c.}\right)^i r \ldots\ldots (26);$$

wherein, by giving to i the values 1, 2, 3, &c. up to the greatest integer in $\frac{1}{2}n$, we obtain a system of equations, to each of which the previous rule may be applied.

Thus, in the rectangular transformation of a homogeneous function of the fifth degree represented by

$$ax^5 + 5bx^4y + 10cx^3y^2 + 10dx^2y^3 + 5exy^4 + fy^5 = a'x'^5 + \text{\&c.}\ldots(27),$$

we obtain first the equations

$$(a+c)x^3 + 3(b+d)x^2y + 3(c+e)xy^2 + (d+f)y^3 = (a'+c')x'^3\ldots(28),$$

$$(a + 2c + e)x + (b + 2d + f)y = (a' + 2c' + e')x' + \text{\&c.}\ldots(29),$$

and from the above three, the constant relations

$$a^2 + 5b^2 + 10c^2 + 10d^2 + 5e^2 + f^2 = a'^2 + \text{\&c.},$$

$$(a + c)^2 + 3(b + d)^2 + 3(c + e)^2 + (d + f)^2 = (a' + c')^2 + \text{\&c.},$$

$$(a + 2c + e)^2 + (b + 2d + f)^2 = (a' + 2c' + e')^2 + \text{\&c.}\ldots($$

To find the two remaining relations among the constant

may operate by means of (29) on (28) or (27). The diffic here and in other applications of the method is not in ducing results, but in ascertaining how far the results obtai are independent of each other. I cannot however but thi that this is a question, the further study of which will le to important results, with reference even to objects wh attainment is not directly sought,—the theory of functio equations for example.

We learn from what precedes, that in the rectangu transformation of a homogeneous function of the $2n^{\text{th}}$ ord whatever the number of the variables, there will exist n lations of the second, and one relation of the first or among the constants, and that when the proposed function i the order $2n+1$, there will exist $n+1$ relations of the seco order among the constants. For the function

$$ax^3+by^3+cz^3+3dx^2y+3exy^2+3fx^2z+3gxz^2+3hy^2z+3iyz^2+6k$$

transformed into $a'x'^3$ + &c., these relations will be

$$a^2+b^2+c^2+3(d^2+e^2+f^2+g^2+h^2+i^2)+6k^2=a'^2+\text{\&c.}$$

$$(a+e+g)^2+(b+d+i)^2+(c+f+h)^2=(a'+e'+g')^2+\text{\&}$$

On Transformations in Space generally.

The character of a system of rectilineal axes may be c sidered as determined when the angles xy, yz, zx are kno Two such systems of axes being given in character, but in mutual position, the problem of linear transformati may be considered as having reference to the two followi objects.

First, the discovery of the relations between the coeffici of the equations of a proposed curve or surface as referred the supposed systems of axes.

Secondly, the discovery of the linear equations connecti the two sets of coordinates, and thereby of the mutual positi of the axes in terms of the coefficients of the two equatio of the proposed curve or surface.

I propose here to shew that the former of these objects c as far as needful be effected by the theory of *quadr* functions, subject only to the inconvenience produced by occasional uncertainty as to the independence of results, a that the second of the above objects can always be effect without any such restriction.

Let $\cos yz=\alpha$, $\cos zx=\beta$, $\cos xy=\gamma$, *i.e.* let α be t angle which the axes y and z make with each other, and on for the rest. In like manner, let $\cos y'z'=\alpha'$, $\cos x'z'=$

$\cos x'y' = \gamma'$; then we have the relation

$$x^2 + y^2 + z^2 + 2\alpha yz + 2\beta xz + 2\gamma xy = x'^2 + y'^2 + z'^2 + \text{\&c.} \ldots (31),$$

an equation of the second order which determines the character of the transformation to be effected. The homogeneous equation, the relations among whose coefficients are to be determined, we will represent by

$$Q = Q' \ldots\ldots\ldots\ldots\ldots\ldots (32),$$

Q being a homogeneous function of x, y, z, Q' of x', y', z', both of the same order n.

Taking the second total differentials of (31) and (32), we have

$$dx^2 + dy^2 + dz^2 + 2\alpha dydz + 2\beta dxdz + 2\gamma dxdy = dx'^2 + \text{\&c.},$$

$$\frac{d^2Q}{dx^2}dx^2 + \frac{d^2Q}{dy^2}dy^2 + \frac{d^2Q}{dz^2}dz^2 + 2\frac{d^2Q}{dydz}dydz + 2\frac{d^2Q}{dxdz}dxdz$$
$$+ 2\frac{d^2Q}{dxdy} = \frac{d^2Q'}{dx'^2}dx'^2 + \text{\&c.},$$

and we may consider these as a pair of homogeneous equations of the second order, with reference to the elements dx, dy, dz, and dx', dy', dz', regarding the coefficients in the second equation, viz. $\frac{d^2Q}{dx^2}, \frac{d^2Q}{dy^2}$, &c. as relatively constant.

Accordingly there will result among the coefficients of the two equations, three relations, one of which will be (*Cambridge Journal*, vol. III. p. 13, 1st Series),

$$\frac{L\frac{d^2Q}{dx^2} + M\frac{d^2Q}{dy^2} + N\frac{d^2Q}{dz^2} + 2S\frac{d^2Q}{dydz} + 2T\frac{d^2Q}{dxdz} + 2U\frac{d^2Q}{dxdy}}{1 + 2\alpha\beta\gamma - \alpha^2 - \beta^2 - \gamma^2}$$
$$= \frac{L'\frac{d^2Q'}{dx'^2} + \text{\&c.}}{1 + 2\alpha'\beta'\gamma' + \text{\&c.}} \ldots\ldots\ldots\ldots (33),$$

wherein $L = 1 - \alpha^2$, $M = 1 - \beta^2$, $N = 1 - \gamma^2$, $S = \beta\gamma - \alpha$, $T = \gamma\alpha - \beta$, $U = \alpha\beta - \gamma$.

The above equation may be represented under the form

$$KQ = K'Q' \ldots\ldots\ldots\ldots\ldots (34),$$

wherein K represents the operative function

$$\frac{1}{1 + 2\alpha\beta\gamma - \alpha^2 - \beta^2 - \gamma^2}\left(L\frac{d^2}{dx^2} + M\frac{d^2}{dy^2} + \text{\&c.}\right) \ldots (35),$$

and K' a similar operative function relatively to x', y', z'.

As Q and Q' are functions of the n^{th} order, the equation (34) will, when the operations represented by K and K' performed, be of the $(n-2)^{th}$ order; and repeating operation, the equation

$$K^2Q = K'^2Q$$

will be of the $(n-4)^{th}$ order, and so on.

Whatever integer value n may therefore have, we eventually obtain by the above process an equation either the second or of the first order; of the second order if even, of the first order if n is odd. The coefficients of equation will be rational functions of the coefficients of original equation $Q = Q'$, and of α, β, γ.

If n be even, we thus obtain, in addition to the quadratic equation (31), another quadratic equation, which may employed as the former one has been to deduce a third quadratic equation, and so on in succession, as far as may necessary.* The relations which exist among the coefficients of these quadratic equations determined by the proper method, will express the required relations among the coefficients of the given equation.

If n is odd, the final equation in the first stage of the above process, being of the first order, cannot be directly employed to effect any subsequent reductions on the primitive equation (32). We may however effect the desired reduction in the following manner: The equation already found being of the first order, will, on being squared, give a quadratic function, whence, with (31), we shall have two quadratic functions. If then we take the second total differentials of these equations and of the primitive equation $Q = Q'$, we shall have three quadratic functions relatively to dx, dy, dz, dx', dy', dz'. Let these, for convenience, be represented by

$$x = x', \quad y = y', \quad z = z';$$

then the relations which connect the constants of these equations are determined by the conditions that the equations

$$\left.\begin{aligned} \theta(x + hy + kz) &= 0 \\ \theta(x' + hy' + kz') &= 0 \end{aligned}\right\} \dots\dots\dots\dots (36)$$

shall be equivalent relatively to h and k. Among the equations afforded by this condition, there is one which will involve at the same time the coefficients of all the three equations (*Journal*, vol. III. p. 109, 1st Series). This equation will give a formula of reduction similar to that which we

* At least for any useful purpose: *vide Sequel.* I have found that the equations are not independent throughout.

before obtained. Hence we shall have as before, representing the operating function by L and L',

$$LQ = L'Q', \quad L^2Q = L'^2Q', \text{ \&c.},$$

the last of which will be linear as before. The relations which connect the coefficients of these linear equations with those of (31) will be the relations sought, and it is evident that they will be expressed in rational functions of the original coefficients.

It appears from the above, that it is possible to deduce the relations which connect the constant coefficients of homogeneous functions of two and three variables, and of any integral order linearly transformed by methods derivable from the theory of quadratic functions, whenever the transformation in question represents a geometrical change of axes; the character of the two systems of axes, but not their relative position, being known. If the systems of axes are both rectangular, the operating equations $K = K'$ assigned in the outset of the above process will be

$$\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2} = \frac{d^2}{dx'^2} + \frac{d^2}{dy'^2} + \frac{d^2}{dz'^2} \dots\dots (37),$$

agreeing with a result previously arrived at by different considerations.

The solution of the problem which we shall proceed to consider, is not only theoretically possible, but may be practically accomplished with considerable facility.

Determination of the Linear Relations.

Given the two forms which the algebraic equation of a curve or surface assumes when referred to two distinct systems of rectilineal axes having a common origin, it is required to determine the linear equations by which the coordinates of the one system are connected with those of the other.

The proposed equations being cleared of radical signs, and reduced to rational and integral forms, an equation will readily be formed between the homogeneous functions of the highest order which they severally involve. This equation let us represent by $Q = Q'$.

Representing also by α, β, γ, the respective values of $\cos yz$, $\cos xz$, $\cos xy$, we shall have, as in the last example,

$$x^2 + y^2 + z^2 + 2\alpha yz + 2\beta xz + 2\gamma xy = x'^2 + \text{\&c.} \dots (38).$$

From these equations we shall, as in the last example (35), (37), obtain $KQ = K'Q', \quad K^2Q = K'^2Q'$, &c.,

whence, finally, we ol
first degree, accordin
degree.

If the degree of Q
above system will be
the other linear equ
method already expl
formed so as to prod
the solution to the foll

If the degree of Q
described will lead t
represent by

$$ax^2 + by^2 + cz^2 + 2d$$

Multiplying (38) by h
the two equivalent sys

$$(a + h)x \quad +$$
$$(f + h\gamma)x +$$
$$(c \ + h\beta)x +$$

and $$(a' + h)x' + ($$

&c.

eliminating the variabl
equation for determini
the second and third eq

$$\frac{x}{(b + h)(c + h) - (d + ha}$$

$$= \frac{}{(f + h\gamma)(d}$$

which we may represent

In like manner the second

$$\frac{x}{l}$$

and it is clear that whene
satisfied also, and *vice vers*

Now differentiate (38), and
root of (38), and we have

$$\frac{(x + \gamma y + \beta z)dx + (\gamma x + y}{(x^2 + y^2 + z^2 + 2ayz}$$

$$= \frac{(x' + \gamma' y' + \beta' z}{(x'^2 + y'^2 +}$$

The first member of the above equation is homogeneous and of the order 0 in respect of x, y, z, and the second is a similar function of x', y', z'. Hence we may substitute in them l, m, n for x, y, z, and l', m', n' for x', y', z': we may also, after doing this, replace dx, dy, dz by x, y, z, &c., since the relations are the same in form. Whence we have

$$\frac{(l+\gamma m+\beta n)x+(\gamma l+m+\alpha n)y+(\beta l+\alpha m+n)z}{(l^2+m^2+n^2+2\alpha mn+2\beta ln+2\gamma lm)^{\frac{1}{2}}}$$
$$=\frac{(l'+\gamma' m'+\beta' n')x'+\&c.}{(l'^2+m'^2+\&c.)^{\frac{1}{2}}}\ \dots\dots\dots (43).$$

In this equation l, m, n, l', m', n' are functions of h. By giving therein to h, then, its three values dependent on the solution of the cubic equation above referred to, we shall obtain three distinct equations, which will complete the linear system.

When the axes are rectangular, the results of the above theory are greatly simplified. The reduction of the proposed equation $Q = Q'$ to a quadratic, is effected by performing as often as necessary the operations represented by the equation

$$\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}=\frac{d^2}{dx'^2}+\frac{d^2}{dy'^2}+\frac{d^2}{dz'^2}.$$

Supposing the result to be of the form (39), we have

$$(a+h)(b+h)(c+h)+2def-(a+h)d^2-(b+h)e^2-(c+h)f^2=0\dots(44)$$

for the cubic determining h. The values of the subsidiary quantities l, m, n, l', m', n', are simply

$$l=(b+h)(c+h)-d^2,\quad m=de-f(c+h),\quad n=fd-(b+h)e,$$
$$l'=(b'+h)(c'+h)-d'^2,\quad m'=d'e'-f'(c'+h),\quad n'=f'd'-(b'+h)e',$$

and finally, for the general type of the linear system, we have

$$\frac{lx+my+nz}{(l^2+m^2+n^2)^{\frac{1}{2}}}=\frac{l'x'+m'y'+n'z'}{(l'^2+m'^2+n'^2)^{\frac{1}{2}}}\ \dots\dots (45);$$

wherein it is only necessary to give to h its three values in succession.

Let us suppose that the second member of (39) is of the form $a'x'^2+b'y'^2+c'z'^2$, then the three values of h determined by the cubic equation (44) are $-a'$, $-b'$, $-c'$. If the first of these be employed in (45), we obtain, on making

$$d'=0,\quad e'=0,\quad f'=0,$$
$$l'=(b'-a')(c'-a'),\quad m'=0,\quad n'=0,$$

whence (45) gives $\quad \dfrac{lx+my+nz}{(l^2+m^2+n^2)^{\frac{1}{2}}}=x'.$

When we employ the second value of h, viz. when $-b'$ is employed, the second member of (45) assumes the form $\frac{0}{0}$. Its real value however is y'. This is proved by deducing the values of l, m, n, l', m', n', not from the second and third equations of (40), but from the first and third, and proceeding as above; for as the three equations of the system (40) are not independent, it is obviously indifferent which two of them we employ. The final result is, that the second member of (45) becomes x', y', or z', according as we make $h = -a'$, or $-b'$, or $-c'$, whenever the second member of (39) is of the form $a'x'^2 + b'y'^2 + c'z'^2$.

Jan. 20, 1851.

ON THE REDUCTION OF THE GENERAL EQUATION OF THE n^{th} DEGREE (SEQUEL TO A MEMOIR ON THE THEORY OF LINEAR TRANSFORMATIONS).

By George Boole.

In the preceding memoir on the Theory of Linear Transformations, I have endeavoured to give a brief sketch of the history of this peculiar and rather isolated branch of analysis, and to exhibit in their due order and connexion the main principles upon which it rests, as far as they are at present known to me. In the present essay I pass onward to their most important application, the reduction, whenever possible, of the general equations of curves and surfaces of the n^{th} degree. The more immediately practical character of this object is my apology for making it the subject of a distinct paper. There may be minds to which the properties of hyperdeterminants and the laws of transformations geometrically inconceivable may present little that attracts, but to which it may still appear worth while to learn how to determine when the equation of an actual curve or surface is reducible to a proposed symmetrical form, and how to effect its reduction.

What is termed the reduction of the general equation of the second degree, viz.

$$ax^2 + by^2 + cz^2 + 2dyz + 2exz + 2fxy = 1,$$

consists in transforming it, by referring it to a new system of rectangular axes, to the form

$$a'x'^2 + b'y'^2 + c'z'^2 = 1,$$

an object the accomplishment of which [illegible] possible; but it is not always possible [illegible] reduction for the general equation [illegible]

$$ax^n + by^n + [illegible] = [illegible]$$

Whenever n is of an order greater [illegible] conditions among the constants [illegible] the above equation may, by [illegible] formations, be reduced to the form

$$a'x'^n + [illegible] = [illegible]$$

accordingly one of the final [illegible] transformations, with reference [illegible] of the following problem. [illegible] instance, whether the proposed [illegible] 2ndly. When it is so, to effect the [illegible] mining the values of [illegible] linear relations connecting [illegible] consequent mutual position [illegible]

I shall shew that the problem [illegible] solved, and with scarcely any difficulty [illegible] its solution in the case of [illegible] Two illustrations of the method [illegible] sponding reduction [illegible] other reductions having [illegible] ticular sets of terms, [illegible] involve odd powers of [illegible] possible at all, by similar [illegible] illustration will also be given [illegible] will each serve as the representative [illegible] I deem it unnecessary to [illegible] statement of theory, but shall at once proceed to the application.

Example I

It is required to ascertain when it is possible to reduce the general equation of the fourth degree,

$$ax^4 + by^4 + cz^4 + 4dx^3y + 4ex^3z + 6fx^2y^2 + 12gx^2yz + 6hx^2z^2 + 4ixy^3$$

$$+ 12jxy^2z + 12kxyz^2 + 4lxz^3 + 4my^3z + 6ny^2z^2 + 4oyz^3 = 1 \ldots (1),$$

to the form $$a'x'^4 + b'y'^4 + c'z'^4 = 1 \ldots\ldots\ldots\ldots (2),$$

by rectangular transformation, and to determine in such case the values of a', b', c', and the linear relations connecting the two sets of variables x, y, z, and x', y', z' together.

$$\frac{d^2Q}{dx^2} + \frac{d^2Q}{dy^2} + \frac{d^2Q}{dz^2} = \frac{d^2Q'}{dx'^2} + \frac{d^2Q'}{dy'^2}$$

substituting herein for Q and Q' their valu differentiations, and dividing by 12, we ha form

$$\alpha x^2 + \beta y^2 + \gamma z^2 + 2\delta yz + 2\epsilon xz + 2\zeta xy = a'x'^2$$

wherein

$$\alpha = a + f + h, \quad \delta = g + m +$$

$$\beta = b + f + n, \quad \epsilon = e + j +$$

$$\gamma = c + h + n, \quad \zeta = d + i + k$$

Now this being an equation of the second de as in the concluding part of the preceding me equation

$$(\alpha+h)(\beta+h)(\gamma+h) + 2\delta\epsilon\zeta - (\alpha+h)\delta^2 - (\beta+h)\epsilon^2 - (\gamma$$

whence the three values of h with signs cha ermine a', b', c'.

Further, if we write

$$= \zeta\delta - \epsilon(\beta + h), \quad q = \epsilon\zeta - \delta(\alpha + h), \quad r = (\alpha + h$$

have

$$\frac{px + qy + rz}{\sqrt{(p^2 + q^2 + r^2)}} = x' \text{ or } y' \text{ or } z' \ldots$$

ording as we make $h = -a'$ or ...
q, r, whence th...

s now only necessary to substitute in the second member (4), that is in the function $a'x'^4 + b'y'^4 + c'z'^4$, for a', b', c', ir determined numerical values, and for x', y', z' their ermined expressions as linear functions of x, y, z. If the nsformation is a possible one, the two members of the ation will become identical; if they do not become ntical, it may certainly be inferred that the transformation impossible

The logic of the above investigation may be thus stated:

1st. Either the proposed reduction is possible, or it is not sible.

2nd. If it is possible, then the equation $Q = Q'$ is true, d the linear relations (8) deduced from it are true, and refore satisfy the equations $Q = Q'$; hence, if the linear ations (8) do not satisfy the equation $Q = Q'$, the reduction not possible.

3rd. If the reduction is not possible, no set of linear re- ions representing a rectangular transformation will satisfy e equation $Q = Q'$, therefore the system (8) will not satisfy t equation; hence, if the linear relations do satisfy that ation, the reduction is possible.

Wherefore the satisfying of the equation $Q = Q'$ by the ear system (8), is a criterion of the possibility of the duction proposed; and that reduction being ascertained to possible, the values of the coefficients a', b', c' are truly igned by the above process.

It will be seen that we only investigate the relations among e constants a, b, c, &c. a', b', c', as far as is necessary to the termination of the latter set of quantities. For this purpose ee relations suffice, and in the present instance these ations are involved in the use which is made of the cubic uation (7). Theory, however, assigns twelve relations ong the constants of the original equation, so that after e determination of a', b', c', there still remain nine con- ions to be satisfied. The expression of these conditions, is by the preceding memoir shewn, is always theo- ically possible, or rather it is shewn to be always possible deduce a series of rational equations among the constants in ich the proposed conditions will certainly be involved, hough in a case so complex as the above great difficulty ght arise in determining how far the different results ained would be independent: fortunately it is quite un- essary to engage in so troublesome an i[illegible] The stitution of the linear values of x, y', z' [illegible] d mber of the equation $Q = Q'$, permits us a

whether the transformation is possible; and the single condition of the resulting identity of the two members supplies the place of the nine conditions among the constants to which reference has been made.

The above example will serve to illustrate the method which is to be employed whenever it is required to ascertain whether an equation of an even degree $2n$ is reducible to the form

$$a'x'^{2n} + b'y'^{2n} + c'z'^{2n} = 1.$$

If $Q = Q'$ be the equation connecting the transformed homogeneous functions, then will

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)^{n-1} Q = \left(\frac{d^2}{dx'^2} + \frac{d^2}{dy'^2} + \frac{d^2}{dz'^2}\right)^{n-1} Q'$$

give an equation of the second order, determining a', b', c', and the linear relations.

Example II.

To reduce if possible the equation

$$ax^3 + by^3 + cz^3 + 3dx^2y + 3ex^2z + 3fxy^2 + 6gxyz + 3hxz^2 + 3iy^2z + 3jyz^2 = 1 \ldots .(9),$$

to the form $\qquad a'x'^3 + b'y'^3 + c'z'^3 = 1 \ldots\ldots\ldots\ldots (10)$

by rectangular transformation.

Representing as before the respective first members of the above equations by Q and Q', the equation

$$\frac{d^2Q}{dx^2} + \frac{d^2Q}{dy^2} + \frac{d^2Q}{dz^2} = \frac{d^2Q'}{dx'^2} + \frac{d^2Q'}{dy'^2} + \frac{d^2Q'}{dz'^2},$$

gives, on dividing by 6 and assuming

$$a + f + h = \alpha, \quad b + d + j = \beta, \quad c + e + i = \gamma,$$

the equation $\quad \alpha x + \beta y + \gamma z = a'x' + b'y' + c'z'.$

Now the transformation being rectangular, we are permitted, according to the previous memoir, to change in the above equation x into $\dfrac{d}{dx}$, x' into $\dfrac{d}{dz}$, &c.: performing this change, and operating as the equation $Q = Q'$, we have

$$\alpha \frac{dQ}{dx} + \beta \frac{dQ}{dy} + \gamma \frac{dQ}{dz} = a' \frac{dQ'}{dx'} + b' \frac{dQ'}{dy'} + c' \frac{dQ'}{dz'};$$

which, on performi ng and dividing by 3,

gives $\alpha(ax^2 + 2dxy + 2exz + fy^2 + 2gyz + hz^2)$

$$+ \beta(dx^2 + 2fxy + 2gxz + by^2 + 2iyz + jz^2) = a'^2x'^2 + b'^2y'^2 + c'^2z'^2$$
$$+ \gamma(ex^2 + 2gxy + 2hxz + iy^2 + 2jyz + cz^2),$$

whence, if we write

$$\alpha a + \beta d + \gamma e = A, \quad \alpha f + \beta b + \gamma i = B, \quad \alpha h + \beta j + \gamma c = C,$$
$$\alpha g + \beta i + \gamma j = D, \quad \alpha e + \beta g + \gamma h = E, \quad \alpha d + \beta f + \gamma g = F,$$

we shall have

$$Ax^2 + By^2 + Cz^2 + 2Dyz + 2Exz + 2Fxy = a'^2x'^2 + b'^2y'^2 + c'^2z'^2.$$

The remainder of the solution of this problem (11) will be identical with that part of the solution of the previous problem which follows equation (6), if only we therein substitute A, B, C, &c. for a, β, γ, &c., and a'^2, b'^2, c'^2 for a', b', c'.

The above example will serve as a type of the process to be employed in the reduction of all equations the degree of which is odd, as the previous one serves for those whose degree is even. In each case the reduction will be effected by the symbolical equation

$$\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2} = \frac{d^2}{dx'^2} + \frac{d^2}{dy'^2} + \frac{d^2}{dz'^2},$$

applied as often as may be necessary, in order to reduce the equation to the first or the second degree.

A remarkable difference will be observed to exist in the character of the two solutions which we have exhibited: when the proposed equation is of an even degree, then a', b', c' are given by a cubic equation, and three definite values are obtained for them; but when the proposed equation is of an odd degree, the resulting cubic equation determines a'^2, b'^2, c'^2, whence a', b', c' admit each of a double sign; in fact the signs of the three terms which compose the function $a'x'^3 + b'y'^3 + c'z'^3$ are arbitrary. Now this conclusion is agreeable to the requirements of the case: for consider the particular term $a'x'^3$, and let the linear value of the variable x' which it contains, be

$$x' = \frac{px + qy + rz}{\sqrt{(p^2 + q^2 + r^2)}} \cdots\cdots\cdots\cdots (12);$$

then, if the sign of a' be changed, and if the sign of the above linear value of x' be changed also, the value of the term after substitution will remain unaltered: but any coefficients of the linear value of x' may be changed in sign without departing from the conditions of rectangular trans-

[redacted]

formation, and the radical sign in the denominator of the expression for x' indicates that this change is permissible in the present case.

Hence if the sign of a' be changed, the sign of the linear value of x' must be changed also.

In the following examples, another symmetrical form of the reduced equation will be considered.

Example III.

It is required to ascertain the conditions under which the equation of the fourth degree (1) is reducible to the form

$$a'x'^4 + b'y'^4 + c'z'^4 + 6d'y'^2z'^2 + 6e'x'^2z'^2 + 6f'x'^2y'^2 = 1 \ldots (13),$$

and to assign in such case the value of the coefficients a', b', c', d', e', f', and the linear relations connecting x, y, z with x', y', z'.

Representing as before the first members of the given and the transformed equations by Q and Q', and performing the same process of reduction, we have

$$\begin{aligned} &ax^2 + \beta y^2 + \gamma z^2 + 2\delta yz + 2\epsilon xz + 2\zeta xy \\ &\quad = (a' + e' + f')x'^2 + (b' + d' + f')y'^2 + (c' + e' + d')z'^2 \ldots (14), \end{aligned}$$

α, β, γ, &c. having the same values as in (6). Hence also we derive the same cubic equation (7), whose roots with signs changed, which we will represent by α', β', γ', determine the coefficients of the second members of (14): thus we have

$$\left.\begin{aligned} a' + e' + f' &= \alpha' \\ b' + d' + f' &= \beta' \\ c' + d' + e' &= \gamma' \end{aligned}\right\} \ldots\ldots\ldots\ldots (15),$$

wherein it is to be remembered that α', β', γ' are known quantities. By substitution, (14) thus becomes

$$ax^2 + \beta y^2 + \gamma z^2 + 2\delta yz + 2\epsilon xz + 2\zeta xy = \alpha' x'^2 + \beta' y'^2 + \gamma' z'^2.$$

Between this result and (6) the only difference is that α', β', γ' in the second members stand in the place of a', b', c'; hence, continuing in the track of the former solution, we find

$$p = \zeta\delta - \epsilon(\beta + h), \quad q = \epsilon\zeta - \delta(\alpha + h), \quad r = (\alpha + h)(\beta + h) - \zeta^2,$$

and
$$\frac{px + qy + rz}{\sqrt{(p^2 + q^2 + r^2)}} = x' \text{ or } y' \text{ or } z' \ldots\ldots\ldots (17),$$

according as we give to h the value $-\alpha'$, or $-\beta'$, or $-\gamma'$ in the expressions for p, q, and r: the linear relations are thus completely determined. It remains to determine the values

a', b', c', d', e', f', the coefficients of the transformed equa-n, together with the conditions which render the trans-rmation possible.

If we substitute the values of x', y', z' given by (17) in Q' e first member of (13), and equate coefficients with Q the st member of (1), we shall have a series of equations which ll be linear with respect to the unknown quantities a', b', c', , e', f'. Any three independent equations out of this set, gether with the system (15), will determine the six elements ught. The condition that the transformation is possible that the six quantities thus being formed, the remaining near equations are identically satisfied by their values.

The example thus discussed is, like each of the two former es, the representative of a class. By an analysis similar in l respects to that which we have adopted, it may be deter-ined whether any proposed equation of the n^{th} degree, being even, can be reduced to a form which shall involve ne but even powers of the variables; the coefficients of the near theorems will be functions of each root in succession a cubic equation, and the coefficients of the transformed uation will be determined by a system of linear equations, ree of which will be similar in form to the equations of the stem (15), while the remaining ones will be found by bstitution, and the equating of coefficients in the manner ployed in the example just considered. It is obvious that at example is more general than the first example, the ution of which is necessarily involved in the solution now ven.

I shall not enter in this paper into the discussion of the me system of problems modified by the introduction of lique coordinates. The method of reduction is explained the previous memoir, and is in its main points similar to at employed in the present paper, only necessarily more mplex: and with respect to the whole theory I would serve, that while it is premature to ask whether those new oblems of which it offers a solution are likely to arise in e more extended application of mathematics to the inter-etation of nature or not, it is always desirable to know at objects it is within the resources of analysis to accomplish, d what transcend them: and easy as the present class of estions may now appear to be, they seemed to the author a short time ago to belong for the most part to the latter ision.

. 20, 1851.

Again, it is easy to see from (2), that the equations to the four straight lines joining the corresponding angles of two tetrahedra whose opposite faces intersect in an hyperboloid, are

$$\left.\begin{array}{l} \dfrac{u'}{\lambda} = \dfrac{v'}{\mu} = \dfrac{w'}{\nu} \\[2ex] \dfrac{v'}{\rho} = \dfrac{w'}{\sigma} = \dfrac{t'}{\lambda} \\[2ex] \dfrac{w'}{\tau} = \dfrac{t'}{\mu} = \dfrac{u'}{\rho} \\[2ex] \dfrac{t'}{\nu} = \dfrac{u'}{\sigma} = \dfrac{v'}{\tau} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (7);$$

and it is easily shewn that each of these lines is situated on the hyperboloid whose equation is

$$(\mu\sigma - \nu\rho)(\tau t'u' + \lambda v'w') + (\nu\rho - \lambda\tau)(\sigma t'v' + \mu u'w') + (\lambda\tau - \mu\sigma)(\rho t'w' + \nu u'v') = 0 \ldots (8).$$

Hence

VII. *If the corresponding faces of two tetrahedra intersect in four straight lines belonging to the same system of generators in an hyperboloid of one sheet, then shall the four straight lines which join the corresponding angles lie in another hyperboloid of one sheet, and will belong to the same system of generators.*

It is to be observed that we might, by aid of (3), have denoted the four straight lines joining the corresponding angular points by

$$\frac{u}{\lambda'} = \frac{v}{\mu'} = \frac{w}{\nu'}, \text{ \&c.,}$$

and then the equation to the hyperboloid in which they lie would be denoted by

$$(\mu'\sigma' - \nu'\rho')(\tau' tu + \lambda' vw) + (\nu'\rho' - \lambda'\tau')(\sigma' tv + \mu' uw) + (\lambda'\tau' - \mu'\sigma')(\rho' tw + \nu' uv) = 0 \ldots (9).$$

It thus appears that (8) and (9) denote the very same hyperboloid, and are therefore identical equations.

Conversely,

VIII. *If the straight lines which join the corresponding angles of two tetrahedra lie in an hyperboloid of one sheet and belong to the same system of generators, then shall the corresponding faces intersect in four straight lines belonging to the same system of generators in an hyperboloid of one sheet.**

* The above is Chasles' extension of Poncelet's theorem:—If the four straight lines joining the corr[illegible] two tetrahedra intersect

This is the reciprocal of the last theorem, but I shall give an independent proof.

Let $t = 0$, $u = 0$, $v = 0$, and $w = 0$ denote the faces of one tetrahedron. Since the first mentioned hyperboloid passes through the angular points of this tetrahedron, its equation must be of the form

$$Atu + Btv + Ctw + Duv + Euw + Fvw = 0;$$

or, supposing t, u, v, and w to have been multiplied by the proper constants, we may write the equation thus,

$$a(tu + vw) + b(tv + uw) + c(tw + uv) = 0 \ldots . (10);$$

and I shall now shew that this may be put under the form

$$\left(\frac{m}{n} - \frac{n}{m}\right)(tu+vw) + \left(\frac{p}{m} - \frac{m}{p}\right)(tv+uw) + \left(\frac{n}{p} - \frac{p}{n}\right)(tw+uv) = 0 \ldots (11).$$

---

in a point, the corresponding faces will intersect in four straight lines in one plane.

In this case we may exhibit the equations to the faces of the tetrahedra in neat and symmetrical forms. Let $t = 0$, $u = 0$, $v = 0$, and $w = 0$ denote the faces of one tetrahedron, and $S = 0$, the equation to the plane in which the corresponding faces intersect; then, supposing t, u, v, and w to have been multiplied by the proper constants, we may denote the faces of the other tetrahedron by $t - 2S = 0$, $u - 2S = 0$, $v - 2S = 0$, and $w - 2S = 0$: hence writing $T + S$, $U + S$, $V + S$, and $W + S$ for t, u, v, and w respectively, we see that we may denote the faces of one tetrahedron by

$$T + S = 0, \quad U + S = 0, \quad V + S = 0, \quad W + S = 0 \ldots\ldots (\alpha),$$

and those of the other by

$$T - S = 0, \quad U - S = 0, \quad V - S = 0, \quad W - S = 0 \ldots\ldots (\beta).$$

In this case, the twelve points in which the faces of the tetrahedron (β) intersect the corresponding contiguous edges of (α) lie in a surface of the second degree, whose equation is

$$2S^2 + (T + U + V + W)S + TU + TV + TW + UV + UW + VW = 0 \ldots . (\gamma).$$

Also the equation to the surface of the second degree passing through the twelve points in which the faces of the tetrahedron (α) intersect the corresponding contiguous edges of (β) is

$$2S^2 - (T + U + V + W)S + TU + TV + TW + UV + UW + VW = 0 \ldots . (\delta).$$

Deducting (δ) from (γ), we see that these two surfaces intersect in two conics (real or imaginary) in the planes $S = 0$ and $T + U + V + W = 0$. Hence

The surfaces (γ) *and* (δ) *have two symptotic planes, one of which coincides with the plane in which the corresponding faces intersect.*

So far as I can see, this is not generally true when the faces intersect in a hyperboloid, for the surfaces (1) and (4) do not seem to intersect in two plane curves.

For assume
$$\left.\begin{aligned}\frac{m}{n}-\frac{n}{m}&=ka\\ \frac{p}{m}-\frac{m}{p}&=kb\\ \frac{n}{p}-\frac{p}{n}&=kc\end{aligned}\right\}\dots\dots\dots\dots\dots(12).$$

Now
$$\left(\frac{m}{n}-\frac{n}{m}\right)^2\left(\frac{p}{m}-\frac{m}{p}\right)^2\left(\frac{n}{p}-\frac{p}{n}\right)^2$$
$$=\left(\frac{m}{n}-\frac{n}{m}\right)^4+\left(\frac{p}{m}-\frac{m}{p}\right)^4+\left(\frac{n}{p}-\frac{p}{n}\right)^4$$
$$-2\left(\frac{m}{n}-\frac{n}{m}\right)^2\left(\frac{p}{m}-\frac{m}{p}\right)^2-2\left(\frac{m}{n}-\frac{n}{m}\right)^2\left(\frac{n}{p}-\frac{p}{n}\right)^2$$
$$-2\left(\frac{p}{m}-\frac{m}{p}\right)^2\left(\frac{n}{p}-\frac{p}{n}\right)^2$$

is an identical equation. Hence, (12), we have
$$a^2b^2c^2k^2=a^4+b^4+c^4-2a^2b^2-2a^2c^2-2b^2c^2.$$

This equation determines k, and then the three equations (12) enable us to find the values of the ratios $\frac{m}{n}$, $\frac{p}{m}$, and $\frac{n}{p}$.

Hence we may assume (11) as the equation to the first mentioned hyperboloid.

Now either $\frac{u}{p}=\frac{v}{n}=\frac{w}{m}$ or $pu=nv=mw$ satisfy (11), and hence one of these must denote the straight line drawn from the angle (uvw) of the tetrahedron ($tuvw$) to the corresponding angle of the other tetrahedron. It is manifestly indifferent which system we take, suppose the former, so that the line referred to is denoted by $\frac{u}{p}=\frac{v}{n}=\frac{w}{m}$. In a similar manner it may be shewn that the straight line drawn from the angle (vwt) must be denoted by either $\frac{v}{m}=\frac{w}{n}=\frac{t}{p}$ or $mv=nw=pt$: but the latter system of equations must be rejected, for the equations $\frac{u}{p}=\frac{v}{n}=\frac{w}{m}$, and $mv=nw=pt$, being satisfied by $mv=nw=pt=\frac{mn}{p}u$, the two lines drawn from (uvw) and (vwt) would intersect in a point, and could not therefore belong to the n of generators; hence

the line drawn from (vwt) must be denoted by $\frac{v}{m} = \frac{w}{n} = \frac{t}{p}$. Similarly the equations of the lines drawn from (wtu) and (tuv) may be found; and collecting the whole, we see that the equations of the four lines joining the corresponding angles of the two tetrahedra may be denoted as follows:

$$\left.\begin{array}{l} \frac{u}{p} = \frac{v}{n} = \frac{w}{m} \\[1ex] \frac{v}{m} = \frac{w}{n} = \frac{t}{p} \\[1ex] \frac{w}{p} = \frac{t}{n} = \frac{u}{m} \\[1ex] \frac{t}{m} = \frac{u}{n} = \frac{v}{p} \end{array}\right\} \dots\dots\dots\dots\dots (13).$$

Again, let $t' = 0$, $u' = 0$, $v' = 0$, and $w' = 0$ be the equations to the faces of the second tetrahedron; then supposing t, u', v', and w' to have been multiplied by the proper constants, we may assume

$$\left.\begin{array}{l} t = at' + p_1u' + n_1v' + mw' \\ u = bu' + p_2t' + mv' + n_2w' \\ v = cv' + n_3t' + mu' + p_3w' \\ w = ew' + mt' + n_4u' + p_4v' \end{array}\right\} \dots\dots\dots (14).$$

Hence the equations to the straight line joining the points (uvw) and $(u'v'w')$ are

$$\frac{u}{p_2} = \frac{v}{n_3} = \frac{w}{m};$$

and since these equations must be identical with the first equations (13), we must have $p_2 = p$, and $n_3 = n$. Considering in a similar manner all the four lines, we find that

$$p_1 = p_2 = p_3 = p_4 = p,$$

and $$n_1 = n_2 = n_3 = n_4 = n.$$

Hence the equations to the faces of the first tetrahedron may be denoted by

$$\left.\begin{array}{l} t = at' + pu' + nv' + mw' = 0 \\ u = bu' + pt' + mv' + nw' = 0 \\ v = cv' + nt' + mu' + pw' = 0 \\ w = ew' + mt' + nu' + pv' = 0 \end{array}\right\} \dots\dots (15);$$

and

and consequently the *opposite* faces intersect in

Hence the following theorem:

XIV. *If two tetrahedra be such that each set of three planes drawn through contiguous edges as in (XIII.) intersects in a straight line, then the four straight lines joining the corresponding angular points will lie in an hyperboloid of one sheet, and will belong to the same system of generators. Also the eight straight lines in which the eight sets of planes intersect will lie in this hyperboloid.*

Conversely,

XV. *Draw, as in (XIII.), three planes for every three contiguous edges of each of two tetrahedra, thus forming eight sets of three planes each. If the four straight lines joining the corresponding angular points lie in an hyperboloid of one sheet and belong to the same system of generators, then shall the three planes of each set just drawn intersect in a straight line in the hyperboloid.*

For as we have previously seen, if $(tuvw)$ and $(t'u'v'w')$ denote the two tetrahedra, the connexion between t, u, v, w and t', u', v', w' may be expressed either by (18) or (19); and it may now be easily shewn that the eight sets of planes intersect in the straight lines (23) and (24), which lie in the hyperboloid (8) or (9).

XVI. *The three edges that are in the same face of one of two tetrahedra may be considered to correspond to three faces of the other; let each edge be intersected by the corresponding face, thus forming a set of three points. Obtain in this manner three points on the three edges in every face of one of the tetrahedra, thus forming four sets of three points each. If the three points of each set range in a straight line, then also the three points of each of the analogous sets that can in like manner be found on the edges in every face of the second tetrahedron will range in a straight line.*

This theorem and the two following are the reciprocals of (XIII.), (XIV.), and (XV.) respectively; but I shall give independent proofs.

Let $t = 0$, $u = 0$, $v = 0$, and $w = 0$ denote the faces of the first tetrahedron, and the equations (22) those of the second tetrahedron. Considering the three edges in the face u, the edges (ut), (uv), and (uw) will correspond to the faces t', v', and w' respectively; and hence the points (utt'), (uvv'), and (uww') are to range in a straight line: now (22) these points are denoted by

$$u = t = \mu v + \nu w = 0,$$

$$u = v = \mu t + \tau w = 0,$$

and $$u = w = \nu t + \tau v = 0;$$

hence the equation t ng through the angular

point (tvw) and the first and second of these points, is

$$\frac{t}{\tau} + \frac{v}{\nu} + \frac{w}{\mu} = 0,$$

and the equation to the plane passing through the same angular point and the first and third of these points, is

$$\frac{t}{\tau_{\prime}} + \frac{v}{\nu} + \frac{w}{\mu} = 0.$$

But since the three points are in a straight line, these two equations must denote the same plane, hence $\tau = \tau_{\prime}$.

Considering in like manner each set of edges that are in the planes v and w, we shall get $\sigma = \sigma_{\prime}$, and $\rho = \rho_{\prime}$; and we shall now find that the three points in which the edges that are in the face t are intersected by the corresponding faces u', v', w', are in a straight line.*

Hence (22) coincides with (18), and t, u, v, w are expressed in terms of t', u', v', w' in (19).

Now, considering the edges in the face t', the edges $(t'u')$, $(t'v')$, and $(t'w')$ will correspond to the faces u, v, w respectively, and the points in which the corresponding edges and faces intersect will, (19), be denoted by

$$t' = u' = \rho' v' + \sigma' w' = 0,$$

$$t' = v' = \rho' u' + \tau' w' = 0,$$

and

$$t' = w' = \sigma' u' + \tau' v' = 0,$$

and these points range in the straight line

$$t' = \frac{u'}{\tau'} + \frac{v'}{\sigma'} + \frac{w'}{\rho'} = 0.$$

In the same manner it may be shewn that the three points of each of the three sets of points corresponding to the edges in the faces u', v', and w' range in a straight line. Hence (XVI.) is true.

It is evident from what has just been proved, that the corresponding faces intersect in straight lines in an hyperboloid, the equation of which is either (5) or (6). Also the equations of the four straight lines in which range the four sets of points corresponding to the edges in each face of the tetrahedron $(tuvw)$, are

* This shews that if the three points of each of three sets range in a straight line, the three points of the fourth set will also range in a straight line.

$$\left.\begin{aligned} t &= \frac{u}{\tau} + \frac{v}{\sigma} + \frac{w}{\rho} = 0 \\ u &= \frac{v}{\nu} + \frac{w}{\mu} + \frac{t}{\tau} = 0 \\ v &= \frac{w}{\lambda} + \frac{t}{\sigma} + \frac{u}{\nu} = 0 \\ w &= \frac{t}{\rho} + \frac{u}{\mu} + \frac{v}{\lambda} = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (25);$$

and each of these sets of equations satisfies (5).

Similarly, if we accent all the letters in (25), we shall obtain the equations to the analogous lines for the edges in each face of the tetrahedron $(t'u'v'w')$, and each set of equations, thus modified, satisfies (6). Hence

XVII. *If two tetrahedra be such that the three points of each of the sets of points found, as in (XVI.), on the edges in each face range in a straight line, then the corresponding faces will intersect in four straight lines belonging to the same system of generators in an hyperboloid of one sheet. Also the eight straight lines in which the eight sets of points are situated will lie in this hyperboloid.*

Conversely,

XVIII. *Find, as in (XVI.), three points on the edges in each face of two tetrahedra, thus determining eight sets of three points each. If the corresponding faces of the two tetrahedra intersect in four straight lines belonging to the same system of generators in an hyperboloid of one sheet, then shall the three points of each set just found, range in a straight line situated on the hyperboloid.*

This is evident after what has been said.

To the propositions now proved, we may join theorems (I.), (II.), (III.), and (IV.) established in part II. (*Journal*, new series, vol. V. pp. 58–60), considering the eight planes in (I.) and (III.) to be the faces of two tetrahedra, and the eight points in (II.) and (IV.) to be the angular points of two tetrahedra. By comparing the whole we shall get numerous propositions, but I shall, for the sake of brevity, insert none of them; nor is it necessary to do so, for nearly every thing that has now been proved is included in the following comprehensive summary.

SUMMARY (A).

Two tetrahedra may possess one of the following properties:

1. *The corresponding faces shall intersect in four straight lines belonging to the same system of generators in an hyperboloid of one sheet.*

2. *The four straight lines joining the corresponding angular points shall lie in an hyperboloid of one sheet, and belong to the same system of generators.*

3. *The faces of either tetrahedron shall intersect the corresponding contiguous edges of the other in twelve points in a surface of the second degree.*

4. *The twelve planes drawn through the angular points of either tetrahedron, and the edges in the corresponding faces of the other, shall touch a surface of the second degree.*

5. *The angular points of each tetrahedron shall be the poles relative to a certain surface of the second degree of the faces of the other.*

6. *The faces taken in any order so that however the corresponding faces shall be opposite, shall intersect three and three in order in eight points, such that every surface of the second degree passing through seven of them shall also pass through the eighth* (see theorems I. and III., vol. V. pp. 58–60).

7. *Take the angular points in any order so that however the corresponding angular points shall be opposite, and draw* (*eight*) *planes through every three successive points; these eight planes shall be such that every surface of the second degree touching seven of them shall also touch the eighth* (see theorems II. and IV. *ib.*).

8. *Any three contiguous edges of either tetrahedron will correspond to three angular points of the other, and a plane may be drawn through each edge and the corresponding angular point. Draw in this manner three planes for every three contiguous edges of either tetrahedron, thus forming four sets of three planes each. The three planes of each set shall intersect in a straight line.*

9. *The three edges that are in the same face of either tetrahedron will correspond to three faces of the other; let each edge intersect the corresponding face. Find, in like manner, three points on the edges in every face of either tetrahedron, thus forming four sets of three points each. The three points of each set shall range in a straight line.*

Now I say, that *if the two tetrahedra possess* ANY ONE *of these properties they shall also possess the other* EIGHT.

This enunciation contains 72 (the number of permutations of 9 things taken two together) theorems, from 16 of which the other 56 immediately follow. All these theorems may be presented under different points of view, many of them very interesting; but these modifications, with a few exceptions to be noticed presently, are necessarily excluded here.

It has been virtually shewn in establishing (XIII.), that if through each of the edges *(vw)*, *(uw)*, and *(uv)* of a trihedral

angle (uvw), and the pole of the face opposite that angl a plane be drawn, the three planes so drawn will intersec in the straight line whose equations are $\tau' u = \sigma' v = \rho' w$ Also, in establishing (XVI.) it has been virtually proved tha if each of the sides (tu), (tv), and (tw) of a plane triangle b intersected by the polar plane of the opposite angle of th triangle, the three points so found will range in a straigh line whose equations are $t = \tau^{-1}u + \sigma^{-1}v + \rho^{-1}w = 0$. Henc the following theorems (which are reciprocal):

XIX. *The three planes drawn through the edges of a trihedr angle, and the poles relative to any surface of the second degr of the opposite faces, intersect in a straight line.*

XX. *The polar planes relative to any surface of the second degr of the angles of any plane triangle, will intersect the opposite sides o the triangle in three points in a straight line.*

The theorem (XIX.) may be exhibited in a rather differen form, thus:

If two trihedral angles have the same vertex, and be such that th edges of one of them pass through the poles relative to a surface of th second degree of the faces of the other, then shall the three plan passing through the corresponding edges intersect in a straight line.

Also it may easily be shewn that

If two trihedral angles have the same vertex, and be such that th edges of one of them pass through the poles relative to any surface o the second degree of the faces of the other, then also the edges of th latter will pass through the poles of the faces of the former.

Two such trihedral angles also possess several other pro perties, into the investigation of which, however, I canno here enter, seeing that this paper is already longer than I anticipated. I shall, however, present these theorems unde the form of another summary.

SUMMARY (B).

If two trihedral angles have the same vertex and possess any o of the following properties, they shall also possess the other four.

1. *The corresponding faces shall intersect in three straight lines i one plane.*

2. *The three planes passing through the corresponding edges sha intersect in a straight line.*

3. *The non-corresponding faces shall intersect in six straight lin in a cone of the second degree.*

4. *The non-corresponding faces shall intersect in six straight lin that touch a non-developable surface of the second degree.**

* The six straight lines will touch an *infinite number* of surfaces of th second degree (see *Journal*, new series, vol. IV. p. 43, theorem XVII.).

5. *The edges of each trihedral angle shall pass through the poles relative to a certain surface of the second degree of the faces of the other.*

I come finally to speak of the expressed objects of this memoir.

The forms in which the theorems (II.) and (V.) are here given, are those under which they presented themselves to my mind, Pascal's and Brianchon's theorems being regarded as properties of *two* triangles: but M. Chasles' view is rather different; he regards these plane theorems as properties of *one* triangle, and he enunciates his analogues nearly as follows:

XXI. *The twelve points in which the edges of a tetrahedron intersect a surface of the second degree may be considered as lying three and three on four planes, each of which contains three points situated on edges meeting in the same angle of the tetrahedron; these four planes intersect the faces opposite to these angles in four straight lines belonging to the same system of generators in an hyperboloid of one sheet.*

XXII. *The twelve tangent planes to a surface of the second degree drawn through the edges of a tetrahedron may be considered to intersect three and three in four points, each of which is the intersection of three planes drawn through edges in the same face of the tetrahedron; the four straight lines joining these points to the angles of the tetrahedron opposite to these faces will lie in an hyperboloid of one sheet, and will belong to the same system of generators.**

We may still however take another, and a very interesting view of these theorems.

In Plane Geometry, it is customary to consider Pascal's and Brianchon's theorems as properties of *plane figures* inscribed in, or circumscribed about a conic; and in like manner we may present (II.) and (V.), that is (XXI.) and (XXII.), as properties of *solid figures* inscribed in, or circumscribed about a surface of the second degree.

We may conceive a hexagon to be generated in either of these two ways: 1st, by taking a triangle and cutting off a portion towards each angular point by a straight line; 2nd,

* When the edges of the tetrahedron *touch* the surface, the four lines mentioned in (XXI.) lie in one plane, and the four lines mentioned in (XXII.) pass through the same point. I state this here, though the proof must be reserved, because M. Chasles has enunciated these particular cases in a somewhat defective manner, saying that the four lines in each case lie in an hyperboloid of one sheet. This of course is true in a certain sense, seeing that in the first theorem the hyperboloid degenerates into a plane, and in the latter, into an indeterminate cone passing through the four lines.

by taking three triangles whose *bases* are respectively equ to the sides of a fourth triangle, and applying the bases o the former to the sides equal to them of the latter. In lik manner we may construct two solid figures, as follows:

1. A *dodecangular octahedron* is a solid figure generated by taking a tetrahedron, and cutting off a portion toward each angular point by a plane.

2. An *octangular dodecahedron* is a solid figure generated by taking four tetrahedra whose *bases* are respectively equ to the faces of a fifth tetrahedron, and applying the bases o the former to the faces equal to them of the latter.*

Now a moment's consideration will make it clear that (II. and (V.) are really equivalent to the following theorems:

XXIII. *If a dodecangular octahedron be inscribed in a surface o the second degree, the opposite faces shall intersect in four straig lines belonging to the same system of generators in an hyperboloid o one sheet.*

XXIV. *If an octangular dodecahedron be circumscribed about surface of the second degree, the four straight lines joining the oppos angular points will lie in an hyperboloid of one sheet, and will belo to the same system of generators.*

Conversely (see III. and VI.),

XXV. *If the opposite faces of a dodecangular octahedron inters in four straight lines belonging to the same system of generators in a hyperboloid of one sheet, then shall the solid figure be inscribed in surface of the second degree.*

XXVI. *If the four straight lines joining the opposite angul points of an octangular dodecahedron lie in an hyperboloid of o sheet and belong to the same system of generators, the solid figure wi be circumscribed about a surface of the second degree.*

It is evident that if $t = 0$, $u = 0$, $v = 0$, and $w = 0$ deno four of the faces of a dodecangular octahedron inscribed i a surface of the second degree, the equations to the oth four faces may be denoted by (2).

Also the equations to the faces of an octangular dodec hedron circumscribed about a surface of the second degr may be got as follows. By (2) the equations to the plan passing through the point (uvw) and the straight lines $(t$

* I am aware that the terms "dodecangular octahedron," and "oct gular dodecahedron" are defective, inasmuch as other solid figures m have the same number of angles and faces. However, I want names f these solids (so as to be able to enunciate the propositions), and I cann invent better.

$(t'v')$, and $(t'w')$ respectively, are

$$u' = \lambda t', \quad v' = \mu t', \quad \text{and} \quad w' = \nu t',$$

and these are the equations to three of the faces of the octangular dodecahedron. In a similar manner the equations to the other faces may be got; and collecting the whole and dropping the accents, we find that the equations to the twelve faces may be denoted as follows:

$$\left.\begin{array}{lll} u = \lambda t, & v = \mu t, & w = \nu t \\ t = \lambda u, & v = \rho u, & w = \sigma u \\ t = \mu v, & u = \rho v, & w = \tau v \\ t = \nu w, & u = \sigma w, & v = \tau w \end{array}\right\} \quad \ldots\ldots\ldots (26).$$

Possibly some persons to whom this subject is new may be inclined to ask, Wherein consists the analogy between four straight lines (in space) situated in an hyperboloid, and three points (in a plane) ranging in a straight line, or between the same and three lines (in a plane) intersecting in one point? To such persons (if any) the following view (though not the only one that might be taken) may not perhaps be without its use. To a point in a plane may correspond either a point or a straight line in space; and to a straight line in a plane may correspond either a straight line or a plane in space, so that a straight line in space may be considered to correspond either to a point or a straight line in a plane; consequently a number of straight lines each intersecting the same straight line (or lines) in space may be considered analogous either to straight lines in a plane passing through the same point, or to points ranging in a straight line. Now when we say that four straight lines lie in an hyperboloid and belong to the same system of generators, we evidently affirm neither more nor less than that *every straight line intersecting three of the four lines will also intersect the fourth*, and we may therefore, if we please, substitute the latter for the former in all the preceding theorems in which mention is made of an hyperboloid; and under this form we see that each of the four lines intersects each of one of an indefinite number of straight lines.

It will be observed that in (XXIII.) and (XXIV.) the angles and faces of a solid figure are considered analogous to the angles and sides of a plane one, while in theorems $\left\{\begin{matrix}\text{I.}\\\text{II.}\end{matrix}\right\}$ of Part I. (*Journal*, new series, vol. IV. p. 27), the $\left\{\begin{matrix}\text{edges}\\\text{angles}\end{matrix}\right\}$ and

$\begin{Bmatrix}\text{faces}\\ \text{edges}\end{Bmatrix}$ are considered analogous to the same. There is at least one more combination that might be considered, and if there be any theorems corresponding, then for $\begin{Bmatrix}\text{Pascal's}\\ \text{Brianchon's}\end{Bmatrix}$ theorem, the $\begin{Bmatrix}\text{angles}\\ \text{edges}\end{Bmatrix}$ and $\begin{Bmatrix}\text{edges}\\ \text{faces}\end{Bmatrix}$ of the solid figure would be deemed analogous to the angles and sides of a plane one. I have not been able to discover such theorems, but they may possibly exist.

Again, we may view Pascal's and Brianchon's theorems as properties of a system of straight lines intersecting two and two in order, or of a system of points joined two and two in order by straight lines; and we should have in space an analogous system of planes intersecting two and two or three and three in order; a system of straight lines intersecting two and two in order, or situated two and two in order in planes; or a system of points connected two and two in order by straight lines, or three and three in order by planes; all of which are in fact virtually the same system. Now, in accordance with this view, we have theorems (I.) and (II.) in Part II. (*Journal*, new series, vol. V. p. 58), in which points and planes in space are considered analogous to points and straight lines in a plane; also in theorems $\begin{Bmatrix}\text{XI.}\\ \text{XII.}\end{Bmatrix}$ of Part I. (*Journal*, new series, vol. IV. p. 39), $\begin{Bmatrix}\text{straight lines}\\ \text{points}\end{Bmatrix}$ and $\begin{Bmatrix}\text{planes}\\ \text{straight lines}\end{Bmatrix}$ are considered analogous to the same; and in this case also for $\begin{Bmatrix}\text{Pascal's}\\ \text{Brianchon's}\end{Bmatrix}$ theorem, we have as yet no theorems in which $\begin{Bmatrix}\text{points}\\ \text{straight lines}\end{Bmatrix}$ and $\begin{Bmatrix}\text{straight lines}\\ \text{planes}\end{Bmatrix}$ are considered analogous to points and straight lines.

I may here add the following theorem:

*If the corresponding faces of two tetrahedra intersect in four straight lines in one plane, the twelve straight lines in which the non-corresponding faces intersect will touch a surface of the second degree.**

The truth of this is easily shewn. Let

$$\left.\begin{array}{l} T+S=0,\quad U+S=0,\quad V+S=0,\quad W+S=0\\ \text{and } T-S=0,\quad U-S=0,\quad V-S=0,\quad W-S=0\end{array}\right\}\ldots(2)$$

* Of course we shall get another theorem by transforming the above by the theory of reciprocal polars.

denote the faces of the two tetrahedra (see the foot-note at p. 118); then it is easily shewn that the surface of the second degree, whose equation is

$$4S^2 = T^2 + U^2 + V^2 + W^2 - 2TU - 2TV - 2TW$$
$$- 2UV - 2UW - 2VW \ldots (28),^*$$

touches the twelve straight lines in which the non-corresponding faces of the two tetrahedra intersect.

I introduce this rather elegant theorem here, chiefly to observe that (the converse of it not being true) I have made many attempts to generalize it in such a way that it may become convertible, but without success. For some time I imagined that it would be sufficient to suppose the opposite faces to intersect in an hyperboloid instead of a plane, but I have satisfied myself that the theorem is not then true. I can render the theorem *less* general so as to become convertible, by supposing the corresponding edges to intersect, in which case the faces of the tetrahedra will form those of an octahedron, and the theorem will then coincide with proposition (VI.) in Part I. (*Journal*, new series, vol. IV. p. 34); but I am unable to seize the general and convertible theorem of which that given above is a particular case.

In conclusion, I may mention that the following are such of the preceding theorems as I find in Chasles' "*Aperçu Historique*:" (II.) and (V.) under the forms (XXI.) and (XXII.), (VIII.), (X.), (XII.), (XIX.), (XX.), together with two or three others that are included under the first *five* conditions of Summary (A). Indeed, all the twenty theorems comprised under these five conditions may be considered *implied* in Chasles' "*Aperçu*," but as he has systematically neglected to mention the converse propositions, he only enunciates about one half of them. Of course it is not pretended that any of the twenty theorems comprised in Summary (B) are substantially new; they are here, however, presented in a compact and connected form.

Nov. 15, 1850.

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* It is interesting to observe that the surface (28) touches the surfaces (γ) and (δ) mentioned in the foot-note at p. 118, along the conics (real or imaginary) in which these surfaces are cut by the planes

$$T + U + V + W + 2S = 0, \quad \text{and} \quad T + U + V + W - 2S = 0$$

respectively.

ON CERTAIN DEFINITE INTEGRALS.

By Arthur Cayley.

Suppose that for any positive or negative integral v of r, we have $\psi(x+ra)=U_r\psi x$, U_r being in general a fun of x, and consider the definite integral

$$I=\int_{-\infty}^{\infty}\psi x\Psi x dx\,;$$

Ψx being any other function of x. In case of either of functions ψx, Ψx becoming infinite for any real value α the principal value of the integral is to be taken, *i.e.* I be considered as the limit of

$$\left(\int_{\alpha+\epsilon}^{\infty}+\int_{-\infty}^{\alpha-\epsilon}\right)\psi x.\Psi x\, dx,\quad (\epsilon=0):$$

and similarly, when one of the functions becomes infinit several of such values of x.

We have $\qquad I=\left(\ldots\int_{a}^{(r+1)a}+\ldots\right)\psi x\Psi x\, dx.$

Or changing the variables in the different integrals so make the limits of each a, 0, we have

$$I=\int_0^a\left[\Sigma\psi(x+ra)\Psi(x+ra)\right]dx,$$

Σ extending to all positive or negative integer values of r

$$I=\int_0^a\psi x\left[\Sigma U_r\Psi(x+ra)\right]dx\,.\,.\,.\,.\,.\,.\,.\,.(A$$

which is true, even when the quantity under the int sign becomes infinite for particular values of x, provided integral be replaced by its principal value, *i.e.* provide be considered as the limit of

$$\left(\int_{\alpha+\epsilon}^{a}+\int_{0}^{\alpha-\epsilon}\right)\psi x\left[\Sigma U_r\Psi(x+ra)\right]dx,$$

or

$$\int_{\epsilon}^{a-\epsilon}\psi x\left[\Sigma U_r\Psi(x+ra)\right]dx\,;$$

where α or one of the limiting values a, 0 is the value for which the quantity under the integral sign bec infinite, and ϵ is ultimately evanescent.

In particular, taking for simplicity $a=\pi$, suppose

$$\psi(x+\pi)=\pm\,\psi x,\quad\text{or}\quad\psi(x+r\pi)=(\pm)^r\,\psi x.$$

[redacted]

Then observing the equation

$$\Sigma \frac{(\pm)^r 1}{x + r\pi} = \cot x \quad \text{or} \quad \operatorname{cosec} x,$$

according as the upper or under sign is taken, and assuming $\Psi x = x^{-\mu}$, we have finally

$$\int_{-\infty}^{\infty} \frac{\psi x\, dx}{x^\mu} = \frac{(-)^{\mu-1}}{\Gamma\mu} \int_0^\pi \psi x \left[\left(\frac{d}{dx}\right)^{\mu-1} \cot x\right] dx,$$

$$\int_{-\infty}^{\infty} \frac{\psi x\, dx}{x^\mu} = \frac{(-)^{\mu-1}}{\Gamma\mu} \int_0^\pi \psi x \left[\left(\frac{d}{dx}\right)^{\mu-1} \operatorname{cosec} x\right] dx,$$

the former equation corresponding to the case of $\psi(x+\pi) = \psi x$, the latter to that of $\psi(x+\pi) = -\psi x$.

Suppose $\psi_{,}x = \psi g x$, g being a positive integer. Then

$$\int_{-\infty}^{\infty} \frac{\psi_{,}x\, dx}{x^\mu} = g^{\mu-1} \int_{-\infty}^{\infty} \frac{\psi x\, dx}{x^\mu}.$$

Also if $\psi(x+\pi) = \psi x$, then $\psi_{,}(x+\pi) = \psi_{,}x$; but if $\psi(x+\pi) = -\psi x$, then $\psi_{,}(x+\pi) = \pm\psi_{,}x$, the upper or under sign according as g is even or odd. Combining these equations, we have

$\psi(x+\pi) = \psi x$, g even or odd,

$$\int_{-\infty}^{\infty} \frac{\psi g x\, dx}{x^\mu} = \frac{(-)^{\mu-1}}{\Gamma(\mu)} g^{\mu-1} \int_0^\pi \psi x \left[\left(\frac{d}{dx}\right)^{\mu-1} \cot x\right] dx$$

$$= \frac{(-)^{\mu-1}}{\Gamma\mu} \int_0^\pi \psi g x \left[\left(\frac{d}{dx}\right)^{\mu-1} \cot x\right] dx;$$

$\psi(x+\pi) = -\psi x$, g even,

$$\int_{-\infty}^{\infty} \frac{\psi g x\, dx}{x^\mu} = \frac{(-)^{\mu-1}}{\Gamma\mu} g^{\mu-1} \int_0^\pi \psi x \left[\left(\frac{d}{dx}\right)^{\mu-1} \operatorname{cosec} x\right] dx$$

$$= \frac{(-)^{\mu-1}}{\Gamma\mu} \int_0^\pi \psi g x \left[\left(\frac{d}{dx}\right)^{\mu-1} \operatorname{cosec} x\right] dx;$$

$\psi(x+\pi) = -\psi x$, g odd,

$$\int_{-\infty}^{\infty} \frac{\psi g x\, dx}{x^\mu} = \frac{(-)^{\mu-1}}{\Gamma\mu} g^{\mu-1} \int_0^\pi \psi x \left[\left(\frac{d}{dx}\right)^{\mu-1} \operatorname{cosec} x\right] dx$$

$$= \frac{(-)^{\mu-1}}{\Gamma\mu} \int_0^\pi \psi g x \left[\left(\frac{d}{dx}\right)^{\mu-1} \operatorname{cosec} x\right] dx.$$

In particular $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x\, dx}{x} = \pi,$

$$\int_{-\infty}^{\infty} dx \frac{\sin x}{x^\mu} = \frac{(-)^{\mu-1}}{\Gamma(\mu)} \int_0^\pi \sin x \left[\left(\frac{d}{dx}\right)^{\mu-1} \operatorname{cosec} x\right] dx,$$

$$\int_0^\pi \sin gx \left[\left(\frac{d}{dx}\right)^{\mu-1} \cot x\right] dx = g^{\mu-1} \int_0^\pi \sin x \left[\left(\frac{d}{dx}\right)^{\mu-1} \operatorname{cosec} x \right.$$

g

$$\int_0^\pi \sin gx \left[\left(\frac{d}{dx}\right)^{\mu-1} \operatorname{cosec} x\right] dx = g^{\mu-1} \int_0^\pi \sin x \left[\left(\frac{d}{dx}\right)^{\mu-1} \operatorname{cosec} x \right.$$

g

$$\int_0^\pi \sin gx \cot x \, dx = \pi, \quad g \text{ even},$$

$$\int_0^\pi \sin gx \operatorname{cosec} x \, dx = \pi, \quad g \text{ odd},$$

$$\int_0^\pi \frac{\tan x \, dx}{x} = 0, \text{ \&c.},$$

the number of which might be indefinitely extended.

The same principle applies to multiple integrals o
order: thus for double integrals, if

$$\psi(x + ra, \; y + rb) = U_{r,s} \psi(x, y),$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(x, y) \Psi(x, y) dx \, dy = \int_0^a \int_0^b \psi(x, y) \, \Sigma \, U_{r,s} \Psi(x + ra, \;$$

..........(

In particular, writing w, v for a, b, and assuming

$$\psi(x + rw, \; y + sv) = (\pm)^r (\pm)^s \psi(x, y).$$

Also $\Psi(x, y) = (x + iy)^{-\mu}$, where as usual $i = \sqrt{-1}$,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\psi(x, y) dx \, dy}{(x + iy)^{\mu}} = \frac{(-)^{\mu-1}}{\Gamma\mu} \int_0^w \int_0^v \psi(x, y) \left[\left(\frac{d}{dx}\right)^{\mu-1} \Theta(x + iy)\right]$$

..........(E

where $$\Theta(x + iy) = \Sigma \frac{(\pm)^r (\pm)^s \, 1}{(x + iy + rw + svi)},$$

Σ extending to all positive or negative integer values
and s. Employing the notation of a paper in the *Mathem*
Journal, "On the Inverse Elliptic Functions," (old s
tom. IV. p. 257, we have for the different combinations c
ambiguous sign,

1. $-, \; -, \ldots \quad \Theta(x + iy) = \dfrac{\mathfrak{G}(x + iy)}{\gamma(x + iy)} = \dfrac{1}{\phi(x + iy)}.$

2. $-, \; +, \ldots \quad \Theta(x + iy) = \dfrac{G(x + iy)}{\gamma(x + iy)} = \dfrac{F(x + iy)}{\phi(x + iy)}.$

3. $+, \; -, \ldots \quad \Theta(x + iy) = \dfrac{g(x + iy)}{\gamma(x + iy)} = \dfrac{f(x + iy)}{\phi(x + iy)}.$

4. $+, \; +, \ldots \quad \Theta(x + iy) = \dfrac{\gamma'(x + iy)}{\gamma(x + iy)};$

where ϕ, f, F are in fact the symbols of the inverse elliptic functions (Abel's notation) corresponding very nearly to sin am, cos am, Δam. It is remarkable that the last value of Θ cannot be thus expressed, but only by means of the more complicated transcendant γx, corresponding to the $H(x)$ of M. Jacobi. The four cases correspond obviously to

1. $\psi(x + rw,\ y + sv) = (-)^{r+s}\psi(x, y)$.
2. $\psi(x + rw,\ y + sv) = (-)^{r}\psi(x, y)$.
3. $\psi(x + rw,\ y + sv) = (-)^{s}\psi(x, y)$.
4. $\psi(x + rw,\ y + sv) = \psi(x, y)$.

The above formulæ may be all of them modified, as in the case of single integrals, by means of the obvious equation

$$\iint \frac{\psi(gx, gy)\,dx\,dy}{(x+iy)^{\mu}} = g^{\mu-2}\iint \frac{\psi(x, y)\,dx\,dy}{(x+iy)^{\mu}}, \text{ [limits } \infty, -\infty\text{]}.$$

The most important particular case is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\phi(x+iy)\,dx\,dy}{(x+iy)} = wv,$$

for in almost all the others, *e.g.*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\phi(x+iy)dxdy}{(x+iy)^{\mu}} = \frac{(-)^{\mu-1}}{\Gamma(\mu)}\int_{0}^{w}\int_{0}^{v} \phi(x+iy)\left[\left(\frac{d}{dx}\right)^{\mu-1}\frac{1}{\phi(x+iy)}\right]dxdy,$$

the second integration cannot be effected.

Suppose next $\psi(x, y)$ is one of the functions $\gamma(x+iy)$, $g(x+iy)$, $G(x+iy)$, $\mathfrak{G}(x+iy)$, so that

$$\psi(x + rw,\ y + sv) = (\pm)^{r}(\pm)^{s}U_{r,s}\psi(x, y),$$

where $$U_{r,s} = (-)^{rs}\epsilon^{\beta x(rw-svi)}q_{,}^{-\frac{1}{2}r^2}q^{-\frac{1}{2}s^2},$$

(see memoir quoted). Then, retaining the same value as before of $\Psi(x, y)$, we have still the formula (B), in which

$$\Theta(x+iy) = \Sigma\,\frac{(\pm)^{r}(\pm)^{s}U_{r,s}}{x+iy+rw+svi}.$$

But this summation has not yet been effected; the difficulty consists in the variable factor $\epsilon^{\beta x(rw-svi)}$ in the numerator, nothing being known I believe of the decomposition of functions into series of this form.

On the subject of the preceding paper may be consulted the following memoirs by Raabe, "Ueber die Summation periodischer Reihen," &c., *Crelle*, tom. xv. p. 355, and "Ueber die Summation harmonisch periodischer Reihen," &c. tom. XXIII. p. 105, and *tom.* XXV. *p.* 160. The integrals

he considers, are taken between the limits 0, ∞ (instead of $-\infty$, ∞). His results are consequently more general than those given above, but they might be obtained by an analogous method, instead of the much more complicated one adopted by him: thus if $\phi(x+2\pi)=\phi x$, the integral $\int_0^\infty \phi x \frac{dx}{x}$ reduces itself to

$$\Sigma_0^\infty \int_0^{2\pi} \phi x \frac{dx}{x+2r\pi}$$

$$=\int_0^{2\pi} dx\, \phi x \left[\frac{1}{x}+\Sigma_1^\infty\left(\frac{1}{x+2r\pi}-\frac{1}{2r\pi}\right)\right],$$

provided $\int_0^{2\pi} dx\phi x=0$. The summation in this formula may be effected by means of the function Γ and its differential coefficient, and we have

$$\int_0^\infty \phi x \frac{dx}{x} = -\frac{1}{2\pi}\int_0^{2\pi} \phi x \frac{\Gamma'\left(\dfrac{x}{2\pi}\right)}{\Gamma\left(\dfrac{x}{2\pi}\right)}dx,$$

which is in effect Raabe's formula (10), *Crelle*, tom. xxv. p. 166.

By dividing the integral on the right-hand side of the equation into two others whose limits are 0, π, and 2π respectively, and writing in the second of these $2\pi-x$ instead of x,

$$\int_0^\infty \phi x \frac{dx}{x} = -\frac{1}{2\pi}\int_0^{\pi}\left\{\phi x\frac{\Gamma'\dfrac{x}{2\pi}}{\Gamma\dfrac{x}{2\pi}}+\phi(2\pi-x)\frac{\Gamma'\left(1-\dfrac{x}{2\pi}\right)}{\Gamma\left(1-\dfrac{x}{2\pi}\right)}\right\}dx;$$

or reducing by

$$\frac{\Gamma'\dfrac{x}{2\pi}}{\Gamma\dfrac{x}{2\pi}}=\frac{\Gamma'\left(1-\dfrac{x}{2\pi}\right)}{\Gamma\left(1-\dfrac{x}{2\pi}\right)}-\pi\cot\tfrac{1}{2}x,$$

we have

$$\int_0^\infty \phi x\frac{dx}{x}=\tfrac{1}{2}\int_0^\pi \phi x\cot\tfrac{1}{2}x\,dx-\frac{1}{2\pi}\int_0^\pi [\phi x+\phi(2\pi-x)]\frac{\Gamma'\left(1-\dfrac{x}{2\pi}\right)}{\Gamma\left(1-\dfrac{x}{2\pi}\right)}dx,$$

which corresponds to Raabe's formula (10′). If $\phi(-x)=-\phi x$, so that $\phi x+\phi(2\pi-x)=0$, the last formula is simplified; but then the integral on the first side may be replaced by $\frac{1}{2}\int_{-\infty}^\infty \phi x\frac{dx}{x}$, so that this belongs to the preceding class of *formulæ*.

ON ARBOGAST'S METHOD OF DERIVATIONS.

By W. F. Donkin.

The following paper contains a view of the first principles and most important processes of Arbogast's method of development. It was intended at first merely to give an investigation of a general principle from which rules may be obtained in *all* cases for avoiding the production of superfluous terms. But it was found that this could hardly be made intelligible without entering so much into the general subject, that it was worth while to make it available as an introduction to the method for those readers who may have been hitherto unacquainted with it.

1. The true nature of the elementary operation employed in Arbogast's method, appears to have been first observed by Professor De Morgan, who stated it in a paper published in this *Journal* (vol. II. p. 244, new series; vol. V. p. 244), namely, that derivation is *differentiation accompanied by integration.* This view I have adopted, and founded on it a demonstration of the rules for developing a function of a single series involving only one variable, which does not differ essentially (though it does in form) from that of Mr. De Morgan. I believe however that it will be found easier. In what follows, I have taken the course pursued by Arbogast himself, namely, to examine *first* the process to be followed in developing a function of several series involving the same variable, and thence to deduce the development of a function of a single series proceeding by powers and products of two variables. The rules which I have given for avoiding, in these two last cases, the production of superfluous terms, are I think simpler than those of Arbogast, and more easy of application. At all events, they have the advantage of being merely particular adaptations of one very simple and general rule from which they may in any case be immediately derived (see Art. 15). Whether such rules be or be not a saving of trouble in any particular problem, must be decided by the circumstances of the case; but it will in general depend chiefly upon the notation chosen. In the most common cases, the advantage of employing them is unquestionable. For example, in performing the development of $\dfrac{a_0 + a_1x + a_2x^2 + \dots}{b_0 + b_1x + b_2x^2 + \dots}$, the application of the rule is perfectly easy, and avoids the pro
of nine superfluous terms out of twenty-one, which
duced by operating on *every letter in the* coefficient

2. I proceed now to consider the simplest and most important problem to which the method is applied. The reader is requested particularly to observe that throughout this paper the word *index* will signify *subscript index*, as in a_0, a_1, D_m, &c., and the word *exponent* will be used for the number or letter expressing a *power*. Thus in a_n^m, n is the *index*, and m the *exponent*.

Given $$u = \phi(a_0 + a_1x + a_2x^2 + \ldots),$$

it is required to develope u in a series of the form

$$a_0 + a_1x + a_2x^2 + \ldots$$

Let differentiation with respect to any coefficient a_{m-1}, followed (or preceded) by integration (from 0) with respect to the *next* coefficient a_m, be denoted by the symbol D_m, so that we have

$$D_m = \frac{d}{da_{m-1}} \cdot \left(\frac{d}{da_m}\right)^{-1}.$$

The laws of combination of such operations as D_m are very simple. In the first place, it is plain that $D_m^n.v = 0$, unless the subject v contain a_{m-1} with an exponent at least equal to n, or otherwise so involved, that the n differentiations shall not produce zero; and secondly, that whenever the operation D_m does not produce zero, it will introduce a_m, or modify the way in which the subject contains that coefficient already: thus if the subject do not contain a_m, the operation D_m^n will introduce a_m^n. Lastly, either $D_n.D_m$ is equivalent to $D_m.D_n$, or the result of one of these combinations is zero.

The only cases with which we shall be concerned at present, are comprised in the form

$$D_m^{\alpha}.D_{m'}^{\alpha'}.D_{m''}^{\alpha''}\ldots\phi(a_0)\ldots\ldots\ldots(1),$$

where $m, m', m'', \ldots$ form a *decreasing series*. And here it is plain that since the operation $D_{m'}^{\alpha'}$ introduces $a_{m'}$ for the first time, and with the exponent α', the next operation D_m^{α}, which involves α differentiations with respect to a_{m-1}, will produce zero, first, if $m - 1 > m'$, and secondly, if $\alpha > \alpha'$. And applying the same reasoning to the other indices and exponents, we see that every such term as (1) is zero, unless it be of the form

$$D^{\alpha}_m.D^{\beta}_{m-1}.D^{\gamma}_{m-2}\ldots D_2^{\lambda}.D_1^{\mu}.\phi(a_0),$$

in which no one of t[illegible] $\beta, \gamma, \ldots$ is greater than the succeedi[illegible], the indices must

form an unbroken progression from m to 1, and the exponent of D_i must not be greater than that of D_{i-1}.

3. The development of $\phi(a_0 + a_1x)$ by Taylor's theorem is equivalent (as is easily seen) to

$$\left\{1 + x\frac{d}{da_0}\cdot\left(\frac{d}{da_1}\right)^{-1} + x^2\left(\frac{d}{da_0}\right)^2\cdot\left(\frac{d}{da_1}\right)^{-2} + \ldots\right\}\phi(a_0),^*$$

or
$$(1 - xD_1)^{-1}.\phi(a_0).$$

And since $\phi(a_0 + a_1x)$ is changed into $\phi(a_0 + a_1x + a_2x^2)$, by writing $a_1 + a_2x$ instead of a_1, we have in like manner

$$\phi(a_0 + a_1x + a_2x^2) = (1 - xD_2)^{-1}(1 - xD_1)^{-1}\phi(a_0);$$

and, continuing the same reasoning,

$$\phi(a_0 + a_1x + a_2x^2 + \ldots + a_nx^n)$$
$$= (1 - xD_n)^{-1}(1 - xD_{n-1})^{-1}\ldots(1 - xD_1)^{-1}\phi(a_0)\ldots(2),$$

and this is enough to give the coefficient of x^n in the required development, since that coefficient cannot contain a with any higher index than n.

Now if we develope the operating factors in (2), and then collect the coefficient of x^n, we shall get the sum of all terms such as
$$D^{\alpha}_{n}.D^{\beta}_{n-1}\ldots D_1^{\mu}.\phi(a_0),$$
in which $\alpha + \beta + \ldots + \mu = n$: but of these we have seen that every one will vanish except those in which $\alpha, \beta, \gamma, \ldots \mu$ form a progression in which no term is greater than the next succeeding one. Moreover α can only be 0 or 1, because a_n cannot enter with an exponent greater than 1.†

By means of this expression it is easy to construct the coefficient of x^n independently. But our object at present is to "derive" the coefficient of x^{n+1} from that of x^n, supposed given. We suppose given, namely, all such combinations as
$$D^{\alpha}_{n}.D^{\beta}_{n-1}\ldots D_1^{\mu}\ldots\ldots\ldots\ldots(3),$$
and we want all such combinations as
$$D^{\alpha'}_{n+1}.D_n^{\beta'}\ldots D_1^{\nu'}\ldots\ldots\ldots\ldots(4),$$
where
$$\alpha + \beta + \ldots + \mu = n,$$
and
$$\alpha' + \beta' + \ldots + \nu' = n + 1.$$

Now recollecting that α and α' can only be 0 or 1, we see

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* This remark is due to Mr. De Morgan.

† If we apply Taylor's theorem to $\phi(a_0 + a_1x + \ldots + a_nx^n)$, considering all the terms after a_0 as an increment, we see that the only term containing a_n and x^n will be $\phi'(a_0).a_nx^n$.

that every term such as (4) can be got from some term such as (3), either by adding 1 to the exponent of some one of the symbols of operation already contained in it, or else by prefixing the symbol with the *next higher* subscript index. But it is also obvious that it will never be necessary to increase the exponent of any symbol already contained in the term, *except the last.* For suppose the last symbol in the term (*i.e.* the one which has the highest index) is D_m^i, and suppose another symbol in the term is D^k_{m-p}. There will be another term differing from this only in containing D_m^{i-1} and D_{m-p}^{k+1}; and this gives the same combination by adding 1 to the exponent of D_m, as is given by the former on adding 1 to that of D_{m-p}.

Thus it appears that in order to produce all such terms as (4), we have only to take each term, such as (3), and perform upon it separately *two* operations, viz.:

(1). *Add* 1 *to the exponent of the symbol which has the highest index;*

(2). *Prefix the symbol with the next higher index.*

The latter process will always produce an effective term; but the former will produce a term whose effect is zero, whenever the exponents belonging to the two highest indices are equal. For since D^i_{m-1} introduces a^i_{m-1} by i integrations, and D^i_m subtracts i from the exponent of a_{m-1} by i differentiations, it is plain that

$$D_m^i . D^i_{m-1} \ldots D_1^\mu . \phi(a_0) \ldots\ldots\ldots\ldots (5)$$

does not contain a_{m-1} at all; so that if the exponent of D_m be increased, which implies another differentiation with respect to a_{m-1}, the result is zero. In fact, this is merely a repetition of what we observed before (Art. 2) as to the progression of exponents. The point to be now observed is, that such a term as (5) will contain a_m but not a_{m-1}.

4. Thus the coefficient of x^{n+1} may be derived from that of x^n by the following rule:

Take each term in the coefficient of x^n, *and,* (1) *repeat the last operation which can be traced in it;* (2) *perform upon it the operation next after the last.*

The first of these two processes produces zero whenever the highest index contained in the term differs from the next lower by more than 1.

The best form of the rule for practice is—

If i *be the highest index in the term, operate upon* a_i *only, unless* a_{i-1} *be also contained in the term, and then operate upon* a_{i-1} *also.*

"*Operating*" upon a_r here (and in all that follows), means performing the operation $D_{r,r+1}$; that is, differentiating with respect to a_r, and integrating with respect to a_{r+1}.

I think it is best *always* to keep this in mind; but the same result is produced if we differentiate with respect to an imaginary independent variable t, and suppose $\frac{da_r}{dt} = a_{r+1}$, provided that whenever an exponent is increased by the operation, we divide by the new exponent.

If instead of using the above rule of the "last and last but one," we operated upon every one of the coefficients a_0, a_1, a_2, ... contained in the term, it appears from what has preceded, that we should only produce repetitions of some of the terms given by using the rule, and all terms so repeated would have to be rejected.

If we used a progression of *letters* instead of *indices*, the rule would be—

"Operate upon the *last letter* in the term, and also on the *last but one* if it be the next preceding letter in the alphabet; otherwise on the last only."

5. If we denote by D the whole operation by which the coefficient of x^{n+1} is derived from that of x^n, we have, since the first term must be $\phi(a_0)$,

$$\phi(a_0 + a_1x + \ldots.) = \phi(a_0) + D\phi(a_0)\,x + D^2\phi(a_0)\,x^2 + \ldots.,$$

and applying the rules of derivation explained in the last article,

$$D\phi(a_0) = \phi'(a_0)\,a_1,$$

$$D^2\phi(a_0) = \phi''(a_0)\,\frac{a_1^2}{2} + \phi'(a_0)\,a_2,$$

$$D^3\phi(a_0) = \phi'''(a_0)\,\frac{a_1^3}{2.3} + \phi''(a_0)\,a_1a_2 + \phi'(a_0)\,a_3,$$

$$D^4\phi(a_0) = \phi''''(a_0)\frac{a_1^4}{2.3.4} + \phi'''(a_0)\frac{a_1^2a_2}{2} + \phi''(a_0)\left(\frac{a_2^2}{2} + a_1a_3\right) + \phi'(a_0)\,a_4,$$

&c.

6. It is easy also to find a general expression for $D^n\phi(a_0)$, which enables us to calculate beforehand, and tabulate all those terms which are outside the functional symbols. For if we put for shortness $\phi_n(a_0)$ instead of $\frac{1}{1.2\ldots n}\,\phi^{(n)}(a_0)$, we

have by Taylor's theorem

$$\phi(a_0 + a_1x + a_2x^2 + \ldots\ldots)$$

$$= \phi(a_0) + \phi_1(a_0)\,x\,(a_1 + a_2x + \ldots) + \phi_2(a_0)\,x^2(a_1 + a_2x + \ldots)^2 + \ldots;$$

and since $(a_1 + a_2x + \ldots)^m = a_1^m + Da_1^m.x + D^2a_1^m.x^2 + \ldots,$

the whole coefficient of x^n is easily found to be

$$\phi_1(a_0)D^{n-1}a_1 + \phi_2(a_0)D^{n-2}a_1^2 + \ldots + \phi_{n-1}(a_0)Da_1^{n-1} + \phi_n(a_0)a_1^n \ldots (6),$$

which is equivalent to $D^n\phi(a_0)$; and thus it is only requisite to have a table of derivatives of the successive powers of a_1, in order to obtain at once the n^{th} derivative of $\phi(a_0)$ by the help of n differentiations. Such a table is given by Mr. De Morgan (*Diff. Cal.* p. 331), adapted to a progression of *letters*. But the student will find it worth while to calculate for himself a sufficient number of terms to obtain complete familiarity with the process, both for a progression of indices and of letters. I give as an example for verification the 4^{th} derivative of a_1^7, omitting the lower ones,

$$D^4a_1^7 = 35a_1^3a_2^4 + 105a_1^4a_2^2a_3 + 21a_1^5a_3^2 + 42a_1^5a_2a_4 + 7a_1^6a_5.$$

7. It may be remarked here that every term in $D^n\phi(a_0)$ being of the form

$$D^{\alpha}_{m}.D^{\beta}_{m-1}.\ldots D_2^{\lambda}.D_1^{\mu}.\phi(a_0),$$

if we actually perform the operations indicated, we get

$$\phi^{(\mu)}(a_0)\,\frac{a_1^{\mu-\lambda}}{1.2\ldots(\mu-\lambda)}\cdot\frac{a_2^{\lambda-\kappa}}{1.2\ldots(\lambda-\kappa)}\cdot\ldots\cdot\frac{a_{m-1}^{\beta-\alpha}}{1.2\ldots(\beta-\alpha)}\cdot\frac{a_m^{\alpha}}{1.2\ldots\alpha}\ldots(7);$$

in which we see that (considering that part which is outside the functional symbol) the sum of the *exponents* (or the whole number of factors) is μ, and the sum of the *indices* (reckoning the index of every single factor) is

$$\mu - \lambda + 2(\lambda - \kappa) + 3(\kappa - \iota) + \ldots\ldots + (m-1)(\beta - \alpha) + m\alpha$$

$$= \alpha + \beta + \ldots\ldots + \lambda + \mu = n;$$

and hence the rule for finding independently the coefficient of $\phi^{\mu}(a_0)$ in $D^n\phi(a_0)$ would be:

Find every way in which n may be resolved into the sum of μ numbers (excluding 0). Each way gives a term such as (7), in which there is a factor a_i corresponding to each number (i) of the μ components of n. The numerical coefficient is found by introducing a divisor $1.2.3\ldots r$, for every exponent r.

For example, to find the coefficient of $\phi'''(a_0)$ in $D^7\phi(a_0)$, we have

$$7 = 1 + 1 + 5 = 1 + 2 + 4 = [illegible] + 3 = 2 + 2 + 3,$$

nd the coefficient required is therefore

$$\frac{a_1^2 a_5}{1.2} + a_1 a_2 a_4 + \frac{a_1 a_3^2}{1.2} + \frac{a_2^2 a_3}{1.2}.$$

Now the expression (6) Art. 6, shews that this ought to be $\frac{1}{1.2.3} D^4 a_1^3$, so that we ought to have

$$D^4 a_1^3 = 3a_1^2 a_5 + 6a_1 a_2 a_4 + 3a_1 a_3^2 + 3a_2^2 a_3,$$

hich is easily verified. In general, $D^m a_1^r$ is the coefficient $\phi_r(a_0)$ in $D^{m+r}\phi(a_0)$; so that finding $D^m a_1^r$ independently olves finding all the ways in which $m + r$ can be made of r numbers, excluding 0. The reader may compare s with an equivalent process deduced in another way by r. De Morgan, (*Diff. Cal.* p. 336). Observe that $m + r$ may e resolved into the sum of r numbers *excluding* 0, by first solving m into r numbers *including* 0, and then adding 1 to very number.

8. I return from this digression to the general subject. he more complex problems are mostly (as we shall see) luded in the following:

To develope $\phi(u, v, w, \ldots.)$ where

$$u = a_0 + a_1 x + a_2 x^2 + \ldots.,$$
$$v = b_0 + b_1 x + b_2 x^2 + \ldots.,$$
$$w = c_0 + c_1 x + c_2 x^2 + \ldots.,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

he process to be employed for any number of series .... will be apparent from an examination of the case hich there are three. We have then to develope $v, w)$.

et $A_m = \frac{d}{da_{m-1}}\left(\frac{d}{da_m}\right)^{-1}$, $B_m = \frac{d}{db_{m-1}}\left(\frac{d}{db_m}\right)^{-1}$, &c.;

, putting ϕ for $\phi(a_0, b_0, c_0)$, and reasoning exactly as e, we have

$$\phi(u, v, w) = \phi + D\phi.x + D^2\phi.x^2 + \ldots.,$$

e $D^n\phi$ is the coefficient of x^n in the development of

$$(1 - xC_n)^{-1}(1 - xC_{n-1})^{-1}\ldots.(1 - xC_1)^{-1},$$
$$\times (1 - xB_n)^{-1}(1 - xB_{n-1})^{-1}\ldots.(1 - xB_1)^{-1},$$
$$\times (1 - xA_n)^{-1}(1 - xA_{n-1})^{-1}\ldots.(1 - xA_1)^{-1}\phi.$$

is coefficient is the sum of such terms as

$$C_n^{\gamma}.C^{\gamma'}_{n-1}\ldots.B_n^{\beta}.B^{\beta'}_{n-1}\ldots.A_n^{\alpha}.A^{\alpha'}_{n-1}\ldots.\phi\ldots.(8),$$

e $\quad \alpha + \alpha' + \ldots.\beta + \beta' + \ldots. + \gamma + \gamma' + \ldots. = n.$

[redacted]

The order in which the A-operations are written with reference to the B or C-operations is indifferent; but we must keep to *one* order, which shall be that of the alphabet, so that an A-operation cannot succeed a B or C-operation, and so on.

Now, in deriving the coefficient of x^{n+1} from that of x^n, exactly the same reasoning will apply as in the simple case (Art. 4). We have only to take each term such as (8) and get from it all the terms which it will yield by the following rule: (1) *Add* 1 *to the exponent of the last operation, and* (2) *Prefix the symbol of each operation which can immediately succeed the last.*

With respect to increasing the exponent of the last operation, the same considerations apply as before. Suppose the last operation has introduced (or increased the exponent of) b_m, then the repetition of it gives zero, unless the term also contain b_{m-1}.

With respect to the *next* operation, we are to remember that a B-operation can be succeeded by another B-operation (of the next higher order), or by a C or E-operation, &c., but not by an A-operation.

The *general* rule then in operating on any term, is to take the last letter in alphabetical order, which has an index greater than 0, and operate on that and on the *succeeding letters* only, observing *also* the rule of the "last and last but one," whenever we have to operate on a letter which occurs with different indices.

Thus, in forming the derivative of $a_0b_2b_3{}^2c_0$, the only *letters* to be operated on are b and c. And in operating on b, we must operate both on b_3 and b_2. And in forming that of $a_1b_3c_2c_4$, we must operate only on c_4.

The most convenient form of the rule in practice is: *Operate first on the last* letter (*in alphabetical order*), *and if it occur with an index greater than* 0, *on that* letter *only*: *if it occur with the index* 0 *alone, then operate also on the next preceding* letter, *and so on till you have operated on a letter with an index greater than* 0; *then operate on no more* letters. *Observe the rule of the "last and last but one" with respect to* indices of the same letter.

Thus in forming the derivative of $a_3b_1{}^2b_2c_0$, we should operate on c_0, b_2, b_1. But in forming that of $a_3b_1{}^2b_3c_0e_0$, only upon e_0, c_0, b_3. When *all* the letters in a term occur with the index 0 only, then of course all must be operated upon.

In applying these rules directly to particular functions, it is to be observed that the [illegible]ry single "operation"

to be considered as a separate term. Fractions, for instance, must not be reduced to a common denominator and added so as to make one term out of several.

9. I give as an example (which is a good one for practice) the three first derivatives of $\frac{b_0c_0}{a_0}$, which are the coefficients of x, x^2, x^3, in the development of

$$(b_0 + b_1x + \ldots .)(c_0 + c_1x + \ldots .)(a_0 + a_1x + \ldots .)^{-1}.$$

Put $\phi = \frac{b_0c_0}{a_0}$, and we have

$$D\phi = \frac{b_0c_1}{a_0} + \frac{b_1c_0}{a_0} - \frac{a_1b_0c_0}{a_0^2},$$

$$D^2\phi = \frac{b_0c_2}{a_0} + \frac{b_1c_1}{a_0} + \frac{b_2c_0}{a_0} - \frac{a_1b_0c_1}{a_0^2} - \frac{a_1b_1c_0}{a_0^2} - \frac{a_2b_0c_0}{a_0^2} + \frac{a_1^2b_0c_0}{a_0^3},$$

$$D^3\phi = \frac{b_0c_3}{a_0} + \frac{b_1c_2}{a_0} + \frac{b_2c_1}{a_0} + \frac{b_3c_0}{a_0} - \frac{a_1b_0c_2}{a_0^2} - \frac{a_1b_1c_1}{a_0^2} - \frac{a_1b_2c_0}{a_0^2}$$

$$- \frac{a_2b_0c_1}{a_0^2} - \frac{a_2b_1c_0}{a_0^2} - \frac{a_3b_0c_0}{a_0^3} + \frac{a_1^2b_0c_1}{a_0^3} + \frac{a_1^2b_1c_0}{a_0^3} + \frac{2a_1a_2b_0c_0}{a_0^3} - \frac{a_1^3b_0c_0}{a_0^4}.$$

After a very little practice the process involves no difficulty whatever. It may be observed that if in forming $D^3\phi$, we had operated on every letter in $D^2\phi$, we should have had to reject ten superfluous terms. The student will also find it a good exercise to find a few terms of the *general* development in the following manner. Take for instance $\phi(u, v)$: put ϕ for $\phi(a_0, b_0)$, and let A, B signify $\frac{d}{da_0}$, $\frac{d}{db_0}$, and we have

$$D\phi = B\phi.b_1 + A\phi.a_1,$$

$$D^2\phi = B\phi.b_2 + B^2\phi.\frac{b_1^2}{2} + AB\phi.a_1b_1 + A^2\phi.\frac{a_1^2}{2} + A\phi.a_2,$$

&c.

I leave to the reader the adaptation of the rule to the case in which the notation employed is

$$u = a_0 + b_0x + c_0x^2 + \ldots ., \quad v = a_1 + b_1x + c_1x^2 + \ldots ., \text{ \&c.}$$

(which is rather less convenient), and proceed to consider the development of a function of one series involving two variables.

10. Let
$$\begin{aligned} u = a_0 &+ b_0 x + c_0 x^2 + e_0 x^3 + \ldots, \\ &+ b_1 y + c_1 xy + e_1 x^2 y + \ldots, \\ &+ c_2 y^2 + e_2 xy^2 + \ldots, \\ &+ e_3 y^3 + \ldots. \end{aligned}$$

It is required to develope $\phi(u)$ in a series of the [illegible] form. This is easily done as follows:

Develope $\phi(a_0 + b_0 x + c_0 x^2 + \ldots)$ into
$$\phi(a_0) + D_x \phi(a_0).x + D_x^2 \phi(a_0).x^2 + \ldots *$$
by the first method (Arts. 4–6), remembering that the pro-gression is now one of *letters*.

Every term in $D_x^n \phi(a_0)$ will be (neglecting numerical coefficients) of the form
$$\phi^{(m)}(a_0) b_0^\beta c_0^\gamma e_0^\epsilon \ldots;$$
and if we now put
$$b_0 + b_1 t \text{ instead of } b_0,$$
$$c_0 + c_1 t + c_2 t^2 \text{ instead of } c_0,$$
$$e_0 + e_1 t + e_2 t^2 + e_3 t^3 \text{ instead of } e_0, \text{ \&c.},$$
and, after developing, write $\frac{y}{x}$ for t, we shall obviously get the required series, in which all the terms of the n^{th} degree will come from $D_x^n \phi(a_0) x^n$.

The preceding theory (Arts. 8, 9) enables us to effect this at once; we have only to perform *index-derivations* upon such terms as $b_0^\beta c_0^\gamma e_0^\epsilon \ldots$, according to the rules just explained, and we get the coefficients of the successive powers of t. Thus the complete development involves two processes: one of *horizontal* or *letter-derivation*, and the other of *vertical* or *index-derivation*.

11. For example, the coefficient of x^3, got by horizontal derivation, is
$$\phi_3.b_0^3 + \phi_2.2b_0 c_0 + \phi_1.e_0, \quad \left\{\phi_3 = \frac{1}{1.2.3}\phi'''(a_0), \text{ \&c.}\right\},$$
and the coefficients of x^2y, xy^2, y^3 are the successive vertical derivatives of this. In performing the derivations, remember that a has no *index-derivative*, so that ϕ_1, ϕ_2, &c. will merely

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* [illegible] to distinguish this horizontal or x-derivation, as it may [illegible] the vertical or t-derivation which occurs immediately after-[illegible] what Mr. De Morgan denotes by D:, the colon being [illegible]

constant multipliers; b_0 has only one derivative b_1, so that $= 0$, &c.; c_0 has only two, and so on. Hence applying the les of (8), we get for the three coefficients in question,

$$\phi_3.3b_0^2b_1 + \phi_2.(2b_0c_1 + 2b_1c_0) + \phi_1.e_1,$$
$$\phi_3.3b_0b_1^2 + \phi_2.(2b_0c_2 + 2b_1c_1) + \phi_1.e_2,$$
$$\phi_3.b_1^3 + \phi_2.2b_1c_2 + \phi_1.e_3.$$

The process hardly requires explanation, after what has ecceded, but I will notice two terms by way of example. c_1 only gives b_0c_2, because since the *last letter* c occurs with index 1, we must not operate upon b; b_0c_2 gives no term, cause for a similar reason b is not to be operated on, and is 0. I should recommend the reader to proceed to the rms of the 4th and 5th degrees, and compare them with ose given by Mr. De Morgan (*Journal*, vol. v. p. 254). he two notations may be made to coincide by omitting r. De Morgan's upper accents, and changing the lower es into indices.

12. The preceding is the easiest way of performing the velopment. But we might also get the coefficient of y^{n+1} from that of x^my^n, by what may be called a *diagonal y-derivation.* In order to make the principles of Art. 8 ily applicable to this process, a change of notation is quired. Let

$$\begin{aligned} u = a_0 + a_1x + a_2x^2 + a_3x^3 &+ \ldots, \\ + a_0'y + a_1'xy + a_2'x^2y &+ \ldots, \\ + a_0''y^2 + a_1''xy^2 &+ \ldots, \\ + a_0'''y^3 &+ \ldots, \end{aligned}$$

that the coefficient of x^my^{n+1} is an *accent-derivative* from t of x^my^n; the exponents of y being indicated by accents, d those of x by indices.

If we now develope $\phi(a_0 + a_1x + a_2x^2 + \ldots)$, as in Arts. and 5, the coefficient of x^m will consist of terms such as $(a_0)a_1^\alpha a_2^\beta \ldots$; and if we then put throughout

$$a_0 + a_0'y + a_0''y^2 + \ldots \text{ for } a_0,$$
$$a_1 + a_1'y + a_1''y^2 + \ldots \text{ for } a_1,$$
$$a_2 + a_2'y + a_2''y^2 + \ldots \text{ for } a_2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

apply the principles of Art. 8, we shall evidently g ment required; the coefficient of x^my^n being t *ivative from that of* x^m.

The reader may, if he prefer it, substi... progression for that of indices, retaini... above.

The reason why this process is less co... of Art. 10, is, that the letter under the fu... now liable to be operated upon.

13. As an example of the above, taking... the last article, the coefficient of x^3 is (Art...

$$\phi'(a_0)a_3 + \phi''(a_0)a_1a_2 + \phi'''(a_0)\frac{a_1^3}{2.3}$$

and taking the *accent* or *y-derivatives* of this, ... coefficients of

$$x^3y, \quad \phi'(a_0)a_3' + \phi''(a_0)(a_0'a_3 + a_1a_2' + a_1'a_2)$$

$$+ \phi'''(a_0)\left(a_0'a_1a_2 + \frac{a_1^2a_1'}{2}\right) + \ldots$$

$$x^3y^2, \quad \phi'(a_0)a_3'' + \phi''(a_0)(a_0'a_3' + a_0''a_3 + a_1a_2'' + \ldots$$

$$+ \phi'''(a_0)\left(\frac{a_0'^2a_3}{2} + a_0'a_1a_3' + a_0'a_1'a_2 + a_0''a_1a_2 + \ldots\right.$$

$$+ \phi^{iv}(a_0)\left(\frac{a_0'^2a_1a_2}{2} + \frac{a_0'a_1^2a_1'}{\ldots} \ldots\right.$$

...rent notations. But if we put

$$\begin{aligned} u = a_0 + b_0 x + c_0 x^2 &+ e_0 x^3 + \ldots, \\ + a_1 y + b_1 xy &+ c_1 x^2 y + \ldots, \\ + a_2 y^2 &+ b_2 xy^2 + \ldots, \\ &+ a_3 y^3 + \ldots, \end{aligned}$$

...diagonal or *y-derivation* would be *index-derivation*, and ...rule for performing it would be precisely that of Art. 8, ...that the process would only differ from that of Art. 11 in ...respect, that *every letter* has now an unlimited number ...index-derivatives.

Whichever notation be adopted, it is obvious that we may ...the process, and *first* performing a series of *y-deri-*...*tions* on $\phi(a_0)$, afterwards perform *x-derivations* on the ...efficients of the powers of y. With the notation of this ...ticle the rule for *x-derivations* (which are *letter-derivations*) will be expressed shortly, *Operate first on the letter which has the* highest index, *and if it be any other than* a, *on letters with that index only*, &c. *Observe the rule of the "last and last but one" with respect to letters with the same index.*

Thus, the coefficient of y^2 being (Art. 5),

$$\phi'(a_0)a_2 + \phi''(a_0)\frac{a_1^2}{2},$$

we get for the coefficients of

$$xy^2, \quad \phi'(a_0)b_2 + \phi''(a_0)(b_0 a_2 + a_1 b_1) + \phi'''(a_0)\frac{a_1^2 b_0}{2},$$

$$x^2y^2, \quad \phi'(a_0)c_2 + \phi''(a_0)\left(b_0 b_2 + c_0 a_2 + a_1 c_1 + \frac{b_1^2}{2}\right)$$

$$+ \phi'''(a_0)\left(\frac{b_0^2 a_2}{2} + a_1 b_0 b_1 + \frac{a_1^2 c_0}{2}\right) + \phi^{iv}(a_0)\frac{a_1^2 b_0^2}{2.2},$$

which I leave without further remark.

It is essential to observe that we cannot use this rule for ...*-derivation* with the notation of Art. 10. Instead of the ...rocess there adopted, we might have begun by vertical or ...dex-derivatives from $\phi(a_0)$, and have performed horizontal ... letter-derivations upon them; not putting $a_1 = 0$, $b_2 = 0$, ...c. till *after* the process was complete. But we cannot per-...rm horizontal derivations by this rule (except from $\phi(a_0)$), ...*ter* putting $a_1 = 0$, &c., without losing all the terms which ...rivation would have *saved from vanishing*.

15. The reader will perceive without difficulty that rea-...ning similar to that of Art. 8 *may* be applied to every

possible case, and that the *general* rule will be the one there given, namely:

Repeat the last operation, and perform every operation which can immediately succeed it.

The order of operations must be fixed on beforehand, so far as it is arbitrary.

16. For instance, let us take the development of $\phi(u, v)$, where u is the same as in Art. 10, and v is a similar series in which the coefficients are *Greek letters*.

The order of operations on *different letters* being arbitrary, let us assume, as before, the alphabetical order in each alphabet separately, with the additional condition that an operation on an *English letter* shall never follow one on a *Greek letter*. Then the process of Art. 10 extended to this case will be as follows:

First develope

$$\phi(a_1 + b_1 x + c_1 x^2 + \ldots, \quad \alpha_1 + \beta_1 x + \gamma_1 x^2 + \ldots),$$

by letter-derivations on $\phi(a_1, \alpha_1)$ according to the principles of Art. 8. The rule will be, "Operate first on Greek letters, and if any other than α occur, on them only; if α only occur, operate also on English letters. Observe the rule of the 'last and last but one' in each alphabet."

Next, taking the coefficient of x^m in this development, the coefficient of $x^m y^n$ will be got by n vertical or index-derivations performed upon it (as in Art. 10), for which the rule will be:

"Operate on the last Greek letter, and if it occur with any index but 0, on that *letter* only, &c. If all the Greek letters have the index 0 alone, then operate also on the English letters in the same way, until you have operated on some letter with a higher index than 0. Observe the rule of the 'last and last but one' with respect to indices of the *same letter*."

uv and $\dfrac{v}{u}$ may be taken as simple examples. As they present no kind of difficulty, it would be useless to occupy more space by giving the details here.

17. The elementary operation in all cases is differentiation with respect to one coefficient, with integration with respect to its derivative. But in some cases it is necessary to observe (what is true in *all*) that the *integration* affects those terms only which are *outside the functional symbol*, whilst the *differentiation* affects the terms both within and without.

The following problem will illustrate this last remark, as ll as the general rule.

It is required to develope $\phi(u, v)$, where

$$u = a_0 + a_1 x + a_2 x^2 + \ldots.,$$
$$v = a_m + a_{m+1} x + a_{m+2} x^2 + \ldots.,$$

e coefficients in v belonging to the same progression as se in u.

Here the operations on coefficients in v may either precede follow operations on those in u; but as respects the efficients of either series separately, the order of operations st be that of indices. Let us then assume the order of dices to be *always* that of operations; and observing that operation on a_m can follow immediately an operation on y lower coefficient, we see that the rule of Art. 15 applied this case may be expressed as follows:

"In forming the successive derivatives of $\phi(a_0, a_m)$ operate cording to the usual rules for a single series of one variable ith the following modifications:—Whenever a term contains , or any higher coefficient, *outside the functional symbol*, perate in *all* respects according to the usual rules. When does not contain such coefficients outside the functional ymbol, operate first without reference to a_m, according to e usual rules, and then operate on a_m *also*. Remember lways to treat a_m *within* the function, as a constant so far integration is concerned."

In a particular example, it is more convenient to use a ogression of letters than of indices.

Suppose for instance

$$u = a + bx + cx^2 + ex^3 + \ldots.,$$
$$v = c + ex + fx^2 + gx^3 + \ldots..$$

Let ϕ stand for $\phi(a, c)$, and A, C respectively for $\dfrac{d}{da}$ and $\dfrac{d}{dc}$, and let A and C operate upon ϕ *only*.

Then the above rule gives the following:

$$D.\phi = A\phi.b + C\phi.e,$$

$$D^2.\phi = A\phi.c + A^2\phi.\frac{b^2}{2} + AC\phi.be + C\phi.f + C^2\phi.\frac{e^2}{2},$$

$$D^3.\phi = A\phi.e + AC\phi.ce + A^2\phi.bc + A^3\phi.\frac{b^3}{2.3} + A^2C\phi.\frac{b^2e}{2} + AC\phi.bf$$
$$+ AC^2\phi.\frac{be^2}{2} + C\phi.g + C^2\phi.cf + C^3\phi.\frac{e^3}{2.3}.$$

It is worth while to verify this, so far as it goes, in a simple case such as $\frac{u}{v}$, or uv; but I leave the process to the reader.

It is to be observed, that if in a problem of this kind we operated directly upon a *particular function*, it would be necessary to adopt some device for distinguishing between those quantities which are, and those which are not subject to *integration*. It is therefore much more convenient to perform the *general* development first, as above, and particularise the form of the function afterwards.

18. In more complicated cases, such as the development of a function of *several* series whose coefficients belong to the same progression, it is best to apply the general rule of Art. 15 *directly*, without attempting to give it a special form adapted to the particular problem. The process in general involves little more trouble than the writing down a great number of terms. Such complicated problems will hardly occur in practice, but they are useful as exercises. I believe also that there would be no better way of attaining the clearest possible ideas of the whole subject, than the direct application of the same *general rule* to all the problems which have been here treated by *special forms* of it. Special rules are always used most safely, as well as most advantageously, when we know how to do without them.

Oxford, Jan. 14, 1850.

ON THE MODE OF USING THE SIGNS + AND − IN PLANE GEOMETRY.

By Professor De Morgan.

The theory of signs in the application of common algebra to plane geometry, has not, I believe, been made complete. In my *Differential Calculus*, pp. 341–345, I suggested some addition to the usual conventions, for the purpose of forming a general mode of measuring angles, and demonstrating the common differential formulæ without subdivision of cases. I have at different times endeavoured to complete this theory in such manner as to give a system under which geometrical and trigonometrical investigations might be conducted by universal rules, without the necessity of considering any

ecific diagram. In this I have not succeeded until now. he following summary of the plan I propose is submitted the consideration of those who have felt the unsoundness the method of relying upon one case of a diagram, leaving gebra to take care of the others.

1. The *signature of translation* of a line is the mode of ving sign to the two directions of translation upon it. Any ne may have either of two distinct signatures. A line with ne signature must be distinguished from what would be lled in geometry *the same line*, with the other signature; consider these as *different* lines, making with each other an gle of 180°; and I call each the *inversion* of the other. And by a given line, I mean a line given in position and gnature. The signature of a given line is to be the same all points in it.

2. Two points A and B on the same line, give the distances AB and BA of different signs. And on one and the me line, we have $AB + BA = 0$, $AB + BC + CA = 0$, &c. Also $AB + BC = AC$, $AB - CB = AC$, $BC - BA = AC$.

3. The signatures of parallel lines are not necessarily e same. And all coordinates are to be measured *on* their xes, not parallel to them. Thus, if P project on the axes nto M and N, the coordinates of P (O being the origin) are OM and ON, not NP and MP. Of the two, OM, NP, the signs may be different.

4. Of the two modes of revolution about a point, one is selected as positive, the other as negative. Every point has then what we may call its *signature of rotation.* Inversion of the signature of translation of a line is to be accompanied by inversion of the signature of rotation of every point upon it.

5. The angle made by the line P with Q (which I designate by $P^\circ Q$) is to be distinguished from the angle made by Q with P, in the same manner as AB is distinguished from BA, by difference of sign. So that $P^\circ Q + Q^\circ P = 0$: but as we do not distinguish θ from $\theta \pm 2\pi$, $\theta \pm 4\pi$, &c., we are o read 0 in the last equation, and others, as "0 or some alue of $\pm 2m\pi$."

6. The least of the *positive* values of $P^\circ Q$ is the angle aced out by a line revolving from the positive side of Q the positive side of P in the *positive* direction of revoluti nd for the word *positive* in italics, we may read *negati*

both places together; while in both cases we may, if we like, read "negative" in the other two cases in which "positive" is written.

7. The origin and axes (X and Y) being given, we prescribe that $Y^\circ X = \frac{1}{2}\pi$, by which we make the positive direction of revolution about the origin to be the usual one. This is the only signature of rotation which cannot be altered without a fundamental change of system.

8. A line R which passes through the origin, has its signature of translation determined by the meaning of $R^\circ X$. The positive side of R is that which bounds the positive value of $R^\circ X$.

9. A line which does not pass through the origin has its signature thus determined. The radius of any point in the line is the line drawn from the origin to that point; and the positive direction is that in which a *positive* radius revolves *positively* about the origin, or a *negative* radius *negatively*, while the end of the radius moves along the line. Thus the signature of every such line depends upon the sign of the radii drawn to its points from the origin.

10. If two points of P, say A, B, be projected upon Q into a, b; then the definition of the cosine of $P^\circ Q$ is $ab : AB$ or $ba : BA$. Take a line at *one* right angle to Q, and let a', b' be the projections of A, B upon it; then the *sine* of $P^\circ Q$ is $a'b' : AB$ or $b'a' : BA$. But if the second line make *three* right angles with Q, then $-a'b' : AB$ or $-b'a' : BA$ is the *sine* of $P^\circ Q$.

The following theorems will now be immediately proved:—
The sum of all the angles of any polygon is 0 or a substitute. The angle made by any line with Y is $\frac{1}{2}\pi$ less than that made with X, $(P^\circ X + X^\circ Y + Y^\circ P = 0)$. If a semicircle of positive revolution commence from the positive side of its bounding diameter, it is on the same side of that diameter as the origin or not, according as the line drawn from the origin to the centre is positive or negative. The positive angle made with a line drawn through the origin never exceeds π.

The only points, I think, in which this system is repugnant to commonly received notions are these two. First, in that no line has a signature essential to its position; next, in that parallel lines have not always the same signature. I need not say anything on the first point, which has been virtually given up by all who admit the polar equation of a curve to

ultaneously with the rectilinear one. The revolving not negative when it coincides with the negative he axis of x, because it is then a line inclined at the to that axis. The second point deserves further ation.

$'$ be a positive point (a point on the positive side) of of y, and through it draw a parallel to the axis of x. ON is positive, this parallel is of opposite signature hough all algebraists use it as a line on which the e is that of X. But continuity requires that the s should be different. a point set out from the d move along the negat ide of the axis of x to ite distance. It is then ebraically speaking, on llel: let it then move on the parallel to N. uity be preserved, *its ra is still negative*, and the s drawn, not through N the axis of y, but on the of that axis. And the nature of the parallel is same as that of x, as ally taken. But if we that the radius of the nt moving on the parallel ve, there is discontinuity at the algebraic junction its parallel. The radius is there inverted, and with allel also. The radius line making an instantaneous lution, brings round, as it were, the parallel to x rough N' (N' being situated similarly to N on the side of Y), and makes it change places with that N. With continuity of radius there is continuity tion, the two parallels being taken as an infinitely oval. In the ordinary mode of giving signature allel, there is discontinuity in the revolution. But be noted that when two parallels are of the same , their infinitely distant intersections must be con- s on the same *ends* of the parallel; if of different s, on opposite ends.

refer to the pages cited from my Differential Calculus of the manner in which universal demonstration is use of this system, in cases of differential formulæ. w give without diagram, a complete demonstration amental trigonometrical formula.

it must be shewn that the cosine and sine of this e in all cases the same as the ordinary ones. This s as to *values*, but by no means so obvious as to Ve must shew then that the cosine of this system ommon cosine have an initial agreement, and after- ange sign together. When the identity of the established, that of the sines follows, for in both in $\theta = \cos(\theta - \frac{1}{2}\pi)$.

[redacted]

$$ab.cd + ac.db + ad.bc = 0 \ldots\ldots\ldots\ldots (3),$$

$$\frac{\sin a^2\beta \sin \gamma^2\delta}{ab} + \frac{\sin a^2\gamma \sin \delta^2\beta}{ac} + \frac{\sin a^2\delta \sin \beta^2\gamma}{ad} = 0 \ldots (4),$$

$$\cot a^2\beta . db.cd + \cot a^2\gamma . ac.db + \cot a^2\delta . ad.bc = 0 \ldots (5);$$

from these it is not difficult to deduce the known formulæ, of which the following is one,

$$\frac{ac}{bc} : \frac{ad}{bd} = \frac{\sin a^2\gamma}{\sin \beta^2\gamma} : \frac{\sin a^2\delta}{\sin \beta^2\delta} \ldots\ldots\ldots\ldots (6).$$

It is obvious from (1) that when the ratios

$$\lambda(\mu\nu' - \mu'\nu) : \mu(\nu\lambda' - \nu'\lambda) : \nu(\lambda\mu' - \lambda'\mu)$$

are constant, the ratios

$$x(y - z) : y(z - x) : z(x - y)$$

are so also; hence the relations established above hold good; (1) for all pencils whose rays pass through the points $a, b, c, d, \ldots$; (2) for all transversals which meet a given fixed pencil.

The theory of four harmonic points is given by putting any of the ratios $=1$; for instance, the formulæ when written in the following different ways give rise to a variety of theorems,

$$\left.\begin{array}{lll} \mu : \nu = (\lambda\mu' - \lambda'\mu) : (\nu\lambda' - \nu'\lambda), & & 2\mu\nu\lambda' - \lambda(\mu\nu' + \mu'\nu) = 0 \\ \dfrac{y}{z} = \dfrac{x - y}{z - x}, & 2yz - x(y + z) = 0, & \left(y - \dfrac{x}{2}\right)\left(z - \dfrac{x}{2}\right) = \left(\dfrac{x}{2}\right)^2 \end{array}\right\} \ldots (7).$$

If two transversals be projective to one another, they are projective to the same pencil; hence taking the point of intersection of the two transversals as the origin, and calling the distances of the points $a, b, c, d, \ldots$ $a_1, b_1, c_1, d_1, \ldots$ from the origin $w, x, y, z, \ldots$ $w_1, x_1, y_1, z_1, \ldots$ respectively, and writing for convenience

$$X = \frac{1}{x - w}, \quad Y = \frac{1}{y - w}, \quad Z = \frac{1}{z - w},$$

with similar expressions for X_1, Y_1, Z_1, we have

$$\begin{vmatrix} \lambda & \lambda & \lambda' \\ \mu & \mu & \mu' \\ \nu & \nu & \nu' \end{vmatrix} = 0, \quad \begin{vmatrix} \lambda X & \lambda & \lambda' \\ \mu Y & \mu & \mu' \\ \nu Z & \nu & \nu' \end{vmatrix} = 0, \quad \begin{vmatrix} \lambda X_1 & \lambda & \lambda' \\ \mu Y_1 & \mu & \mu' \\ \nu Z_1 & \nu & \nu' \end{vmatrix} = 0,$$

and consequently

$$\begin{vmatrix} X & X_1 & 1 \\ Y & Y_1 & 1 \\ Z & Z_1 & 1 \end{vmatrix} = 0 \ldots\ldots\ldots\ldots (8),$$

[wh]ich is consequently the condition for the projectivity of [tw]o transversals. Similarly may be found the condition for [th]e projectivity of two pencils, viz.

$$\begin{vmatrix} \cot a^2\beta & \cot a_1{}^2\beta_1 & 1 \\ \cot a^2\gamma & \cot a_1{}^2\gamma_1 & 1 \\ \cot a^2\delta & \cot a_1{}^2\delta_1 & 1 \end{vmatrix} = 0 \ldots\ldots\ldots (9).$$

If two corresponding points (a, a_1) meet at the point of [in]tersection of the two transversals w, w_1 vanish; and if $\xi, \eta, \zeta, \ldots$ be the values of $X, Y, Z, \ldots$ in this case (8) [b]ecomes

$$\begin{vmatrix} \xi & \xi_1 & 1 \\ \eta & \eta_1 & 1 \\ \zeta & \zeta_1 & 1 \end{vmatrix} = 0 \ldots\ldots\ldots\ldots (10).$$

If then the rays β, γ; γ, a; $a, \beta \ldots$ successively fall [p]arallel to the two transversals, the points b, c_1; c, a_1; a, b_1 [w]ill be infinitely distant; and if x, y be the values of η, ζ; ζ, ξ; ξ, η in these cases, there will result successively,

$$\begin{vmatrix} \xi & \xi_1 & 1 \\ 0 & x & 1 \\ y & 0 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} x & 0 & 1 \\ \eta & \eta_1 & 1 \\ 0 & y & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} 0 & y & 1 \\ x & 0 & 1 \\ \zeta & \zeta_1 & 1 \end{vmatrix} = 0,$$

[w]hich are equations to straight lines passing through the [p]oints a, a_1, B; b, b_1, B; $c, c_1, B, \ldots$ respectively, as may [b]e seen by writing them in the following form,

$$\frac{\xi}{x} + \frac{\xi_1}{y} = 1, \quad \frac{\eta}{x} + \frac{\eta_1}{y} = 1, \quad \frac{\zeta}{z} + \frac{\zeta_1}{z} = 1,$$

[a]nd the elimination of xy from these will reproduce (10); [h]ence (10) is the condition of perspectivity of A, A_1.

If in two projective pencils two rays a, a_1 coincide, (9) [b]ecomes

$$\begin{vmatrix} \cot a^2\beta & \cot a^2\beta_1 & 1 \\ \cot a^2\gamma & \cot a^2\gamma_1 & 1 \\ \cot a^2\delta & \cot a^2\delta_1 & 1 \end{vmatrix} = 0 \ldots\ldots\ldots (11);$$

[an]d if the transversal be taken for the axis of x, and a line [pe]rpendicular to it for the axis of y, and if h be the distance [A]B_1, the above equation may be thus written,

$$\begin{vmatrix} x : y & x - h : y & 1 \\ x' : y' & x' - h : y' & 1 \\ x'' : y'' & x'' - h : y'' & 1 \end{vmatrix} = 0, \text{ or } \begin{vmatrix} x & y & 1 \\ x' & y' & 1 \\ x'' & y'' & 1 \end{vmatrix} = 0 \ldots$$

(x, y; x', y'; x'', y'', being the coordinates of the intersec
of β, β_1; γ, γ_1; δ, δ_1, respectively). This is the cond
that the three points lie in a straight line; hence (11) or
is the condition of the perspectivity of B, B_1.

In any quadrilateral each pair of diagonals is interse
by a pencil formed by the two sides and the other diago
hence for each pair of diameters

$$\begin{vmatrix} X_1 & X_2 & 1 \\ Y_1 & Y_2 & 1 \\ Z_1 & Z_2 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} X_2 & X & 1 \\ Y_2 & Y & 1 \\ Z_2 & Z & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} X & X_1 & 1 \\ Y & Y_1 & 1 \\ Z & Z_1 & 1 \end{vmatrix}$$

which may always be satisfied by the same value
$X, Y, Z, \ldots$ in each; hence in the system

$$X + Y + Z = 3X, \quad \text{or} = 3Y, \quad \text{or} = 3Z,$$
$$X_1 + Y_1 + Z_1 = 3X_1, \quad \text{or} = 3Y_1, \quad \text{or} = 3Z_1,$$
$$X_2 + Y_2 + Z_2 = 3X_2, \quad \text{or} = 3Y_2, \quad \text{or} = 3Z_2,$$

whenever one equation of any vertical row is satisfied,
other two of the same vertical row are so also; that is to
if the intersections of any diameter with the two sides
one other diameter be given, the fourth harmonic point
on the other diameter. Hence the theorems—

In a complete quadrilateral the points in which the three diagonals intersect one another are harmonically conjugate to the corresponding angular points.

In a complete quadrila the rays which join the in sections of the opposite are harmonically conjuga those sides.

If three mutually projective transversals pass throu
point, and three corresponding points be united at the p
of intersection, the transversals will be situated two and
perspectively, so that

$$\begin{vmatrix} \xi_1 & \xi_2 & 1 \\ \eta_1 & \eta_2 & 1 \\ \zeta_1 & \zeta_2 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} \xi_2 & \xi & 1 \\ \eta_2 & \eta & 1 \\ \zeta_2 & \zeta & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} \xi & \xi_1 & 1 \\ \eta & \eta_1 & 1 \\ \zeta & \zeta_1 & 1 \end{vmatrix} =$$

which equations may always be satisfied by the same va
of ξ, η, ζ; ξ_1, η_1, ζ_1; ξ_2, η_2, ζ_2, in each; so that the rays
pass two and two through the various points of the tr
versals: and if ξ, ξ_1, ξ_2 refer to the intersections of the
passing through B_1B_2, B_2B, BB_1, with the three transve
respectively, the above equations will express the condi
that B, B_1, B_2 lie in a straight line.

ee mutually projective pencils have their centres ight line (κ), which is a common ray to the three then

$$\begin{vmatrix} \text{ċ}^2\alpha_1 & \cot\kappa^2\alpha_2 & 1 \\ \text{ċ}^2\beta_1 & \cot\kappa^2\beta_2 & 1 \\ \text{ċ}^2\gamma_1 & \cot\kappa^2\gamma_2 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} \cot\kappa^2\alpha_2 & \cot\kappa^2\alpha & 1 \\ \cot\kappa^2\beta_2 & \cot\kappa^2\beta & 1 \\ \cot\kappa^2\gamma_2 & \cot\kappa^2\gamma & 1 \end{vmatrix} = 0,$$

$$\begin{vmatrix} \cot\kappa^2\alpha & \cot\kappa^2\alpha_1 & 1 \\ \cot\kappa^2\beta & \cot\kappa^2\beta_1 & 1 \\ \cot\kappa^2\gamma & \cot\kappa^2\gamma_1 & 1 \end{vmatrix} = 0,$$

ıy be satisfied with the same positions for α, β, γ; ; α_2, β_2, γ_2 in each; so that the rays will pass two :hrough the various points of the transversals: hence, ı transversals intersect, two corresponding points are : the point of intersection; and consequently the ısversal will intersect at the same point.

the theorems—

e mutually projective als A, A_1, A_2, be so hat three correspond- ts are united at the their mutual inter- they will be situated vely, and the centres ɔencils B, B_1, B_2 will traight line.

three straight lines A_2, which join the f any two triangles b_2 in a given order, a point, the three ı which the opposite ersect one another in ɜ order will be in a line.

angles of a variable aa_1a_2 move on three raight lines which a point e, and if two the same (aa_1, aa_2) ıt fixed points B_1B_2, side a_1a_2 will always ugh a fixed point B,

If three mutually projective pencils be so situated that three corresponding rays fall upon one another, their centres consequently lying in that straight line, they will be perspectively and the three transversals will meet in a point.

If the three points B, B_1, B_2 in which the sides of any two triangles aa_1a_2, bb_1b_2 in a given order intersect in a straight line, the three straight lines AA_1A_2, which join the opposite angles in the same order two and two, will meet in a point.

If three sides of a variable triangle aa_1a_2 turn about three fixed points BB_1B_2 which turn in a straight line d, and if two angles of the same (a, a_1) move along two fixed straight lines AA_1, the third angle a_2 will move on a third

which lies in a straight line with the two former ones B_1B_2.

fixed straight line A_2 whi passes through the interse tion of AA_1.

On Geometrical Involution.

The condition that six points, whose distances from given point are

$$x, \quad x', \quad y, \quad y', \quad z, \quad z'$$

respectively, are in involution, may be thus expressed,

$$\begin{vmatrix} 0 & x'-z & x-z' \\ x'-y & 0 & y-z' \\ x-y' & y'-z & 0 \end{vmatrix} = 0, \text{ or } \begin{vmatrix} 0 & y'-x & y-x' \\ y'-z & 0 & z-x' \\ y-z' & z'-x & 0 \end{vmatrix} = $$

$$\text{or } \begin{vmatrix} 0 & z'-y & z-y' \\ z'-x & 0 & x-y' \\ z-x' & x'-y & 0 \end{vmatrix} = 0,$$

which are easily transformed to

$$\begin{vmatrix} 0 & x'-z' & x-z \\ x'-y & 0 & y-z \\ x-y' & y'-z' & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} 0 & y'-x' & y-x \\ y'-z & 0 & z-z \\ y-z & z'-x & 0 \end{vmatrix} = $$

$$\begin{vmatrix} 0 & z'-y' & z-y \\ z'-x & 0 & x-y \\ z-x' & x'-y' & 0 \end{vmatrix} = 0.$$

These express the relations given by M. Chasles in History of Geometry, and do not involve any of the labori algebraical processes to which he alludes. They also aff a simple demonstration of another theorem relating to volution and anharmonic ratio.

Consider the determinant

$$\begin{vmatrix} (x-w)(x'-w') & x-w & x'-w' \\ (y-w)(y'-w') & y-w & y'-w' \\ (z-w)(z'-w') & z-w & z'-w' \end{vmatrix} = 0,$$

which expresses that the anharmonic ratio of the four po whose distances from a fixed point are w, x, y, z, is equi lent to that of the four points whose distances are w', x', y' Let these 8 points be represented by

$$0, \quad A, \quad B, \quad C, \quad 0', \quad A', \quad B', \quad C' \quad .$$

respectively.

hen if $\quad$ 0 coincides with A', B', C' successively,

nd $\quad$ $0'$ . . . . A, B, C . . . .,

nd 0 be taken as the origin; then there will result successively

$$0 = w = x', \quad \text{or} = y', \quad \text{or} = z',$$
$$w' = x, \quad \text{or} = y, \quad \text{or} = z;$$

nd (1) becomes successively

$$\begin{vmatrix} 0 & 1 & -1 \\ yy' & y & y'-x \\ zz' & z & z'-x \end{vmatrix} = 0, \quad \begin{vmatrix} xx' & x & x'-y \\ 0 & 1 & -1 \\ zz' & z & z'-y \end{vmatrix} = 0, \quad \begin{vmatrix} xx' & x & x'-z \\ yy' & y & y'-z \\ 0 & 1 & -1 \end{vmatrix} = 0,$$

hich may be written thus

$$\begin{vmatrix} 0 & y'-x & y \\ y'-z & 0 & z \\ y-z' & z'-x & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} 0 & z'-y & z \\ z'-x & 0 & x \\ z-x' & x'-y & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} 0 & x'-z & x \\ x'-y & 0 & y \\ x-y' & y'-z & 0 \end{vmatrix} = 0,$$

hence the systems of six points so formed are severally in nvolution.

ON MARRIOTTE'S LAW OF FLUID ELASTICITY.

By Henry Wilbraham.

In a memoir by Lieutenant Hunt in the *Philosophical Magazine* for July 1850, extracted from Silliman's *American Magazine*, the author tries to prove that Marriotte's Law may e accounted for, not only by the hypothesis of the mutual epulsions of the particles of the elastic fluid varying inversely as their mutual distances, which is the law stated y Newton (*Princip.* Lib. II. Prop. XXIII.) to be necessary nd sufficient, but also by any other law of repulsive force mong the particles, provided that the medium be homogeneous. I will try to shew that, though Newton's result incorrect, Lieut. Hunt's is not less so; and further, that ot only the law of the inverse distance, but also any other w in which the mutual force depends solely on the distance etween the two particles, conducts to a result at variance ith Marriotte's law.

The error in Newton's reasoning is this—that wherein he ccounts for the mutual effect of *the same number* of particles the greater and smaller cubes severally, he accounts in ger cube for the effect *of particles* situate at a distance

the assumed plane at which distance the effect of any particles in the case of the smaller cube is not accounted for. He supposes, in short, the *number of particles affected by any single particle*, not *the distance at which the effect is sensible*, to be the constant, while the density varies. He thus assumes a law of force between any two particles, not varying merely as a function of the distance between them, but involving another variable element in the density of the surrounding medium. In the scholium to the proposition Newton states, that the proposition is only applicable to forces whose action is confined to the particles nearest or very near to the centres of force. If he means that the action of the force is confined to the particles *nearest* to the attracting particle, at whatever distance therefrom these *nearest* particles be, Newton's reasoning is correct; but the system assumed is not a law of force depending merely on the distance. If he means that the law of force must be discontinuous, such that the force varying as a certain function of the distance within some definite sphere of action, vanish for particles exterior to such sphere, the objection above taken is as applicable as if the law be supposed continuous.

The erroneous assumption made by Lieut. Hunt in the above-mentioned memoir is this,—he assumes that the whole action of the fluid on one side of any plane upon that on the other side is correctly measured by the resultant action (calculated in a direction perpendicular to the plane) exerted on this plane by all the forces on one side of it, observing that this resultant is balanced by an equal and opposite resultant of all the forces on the other side; and he proceeds to reason as if the imaginary plane were a material lamina capable of being acted on by force.

The correct representation of the pressure is the sum of the resultant action (calculated perpendicularly to the plane) of all the particles on the one side of the plane on every particle on the other side. If now the whole space be divided into very small parallelepipeds, each of which, when the density of the medium is ρ, contains m particles, and r be the distance between two of them, one on either side of the plane, the sum of the forces of all the particles in the one parallelepiped on all other particles in the other, is generally $m^2\phi(r)$, ($\phi(r)$ being the law of force between two particles,) and consequently the whole resultant of all the forces on one side of the plane on all the particles on the other, is correctly measured by $m^2\phi(r)$. If the density alter and become ρ', (the magnitude and position of each p maining constant),

m will become $\frac{m\rho'}{\rho}$; and so the whole resultant action will be measured by $\left(\frac{m\rho'}{\rho}\right)^2\phi(r)$. Hence the pressure at a plane of constant extent varies as the square of the density.

This reasoning, however, is not quite or in all cases correct; for when the two parallelepipeds are adjacent or very near one another, the forces exercised by the particles in the one on those in the other is not correctly represented by $m^2\phi(r)$; this expression being true only when the difference of distance from any point in one parallelepiped of the several particles contained in the other may be neglected in comparison with the distance between the two parallelepipeds. But the result is evidently true whenever the law of force is such that the mutual action of the particles very near the plane, on each side thereof, is inconsiderable in comparison with that of the particles further from it. This is the case certainly when $\phi(r) = \frac{\mu}{r^n}$, n being not greater than 3; for then, as is well known, if a particle be in contact with an infinitely extended mass of matter, the action of any definite part of the mass upon the particle is infinitely small as compared with the whole action of the body upon it.

If $\phi(r) = \frac{\mu}{r^n}$, where n is large—we shall see presently that it is when n is not less than 4—we may well conclude that the law given in the scholium to Newton's proposition (viz. that the pressure is as $\rho^{\frac{1}{3}(n+2)}$) will be true; for as in that case the most effective force must be that exercised among adjacent or nearly adjacent particles, the cubes used in Newton's proof may be supposed of such size that the effect of particles included in the larger, but not included in the smaller (the neglect of whose effect was the cause of my objection as above), may be negligible.

The following investigation will, I think, shew the pressure for any given value of n, where $\phi(r) = \frac{\mu}{r^n}$.

Consider the total resultant action (calculated in a direction perpendicular to the plane) of the whole mass situated on the one side of a plane of infinite magnitude upon a single particle at a distance z on the other side of the plane; the mass being bounded by a second plane parallel to and at a distance h behind the first one.

This resultant action we may find to be

$$\int_z^{z+h}\int_0^{\frac{1}{2}\pi} 2\pi\rho x^2 \sec\theta \tan\theta\ \phi(x\sec\theta)\, d\theta\, dx$$

(ρ being the density of the mass). This is equal to

$$2\pi\rho\int_z^{z+h} x\int_x^{x\sec\frac{1}{2}\pi} \phi(y)\,dy\,dx = 2\pi\rho\mu\int_z^{z+h} x\int_x^{x\sec\frac{1}{2}\pi} y^{-n}\,dy\,dx,$$

which being integrated is

$$\frac{-2\pi\mu\rho}{(n-1)(n-3)}\left\{(z+h)^{-(n-3)} - z^{-(n-3)}\right\}.$$

Suppose now, that instead of a single particle there is a line perpendicular to the plane along which line the particles are uniformly distributed so that there are p of them in a unit of length, and therefore that they are at distances $\frac{1}{p}$ from one another. The whole effect of the mass on this line (supposing the nearest particle to the plane to be at a distance from the plane smaller than, but comparable with $\frac{1}{p}$ and the furthest at a distance k) will be

$$\frac{-2\pi\mu\rho p}{(n-1)(n-3)}\int_\delta^k\left\{(z+h)^{-(n-3)} - z^{-(n-3)}\right\}dz,$$

where δ is some length smaller than $\frac{1}{p}$.

This integral is

$$\frac{2\pi\mu\rho p}{(n-1)(n-3)(n-4)}\left\{(k+h)^{-(n-4)} - (h+\delta)^{-(n-4)} - k^{-(n-4)} + \delta^{-(n-4)}\right\}.$$

If, instead of considering the effect on this single line of particles, we consider that on a column of the same length as the line, and whose base is a unit of surface, this last expression must be multiplied by p^2, or by some quantity varying as p^2. Where then the law of force is $\phi(r) = \frac{\mu}{r^n}$, the pressure on a given surface is measured by

$$\rho p^3\left\{(k+h)^{-(n-4)} - (h+\delta)^{-(n-4)} - k^{-(n-4)} + \delta^{-(n-4)}\right\}.$$

If n be greater than 4, the last term in the bracket is the only one which need be regarded, being very much greater than the others. The expression then becomes $\rho p^3\delta^{-(n-4)}$. As the density is the same on either side of the plane, p^3, which measures the den ne side, must vary as ρ:

To find the numerator of Mr. Boole's fraction, we must form the symbolical operator

$$\begin{Bmatrix} l^2 \dfrac{d}{da} + m^2 \dfrac{d}{db} + n^2 \dfrac{d}{dc} + p^2 \dfrac{d}{dd} \\ + 2lm \dfrac{d}{de} + 2np \dfrac{d}{d\epsilon} + 2lm \dfrac{d}{dg} + 2mp \dfrac{d}{d\gamma} + 2lp \dfrac{d}{dh} + 2mn \dfrac{d}{d\eta} \end{Bmatrix}$$

$$\times \begin{Bmatrix} l'^2 \dfrac{d}{da} + m'^2 \dfrac{d}{db} + n'^2 \dfrac{d}{dc} + p'^2 \dfrac{d}{dd} \\ + 2l'm' \dfrac{d}{de} + 2n'p' \dfrac{d}{d\epsilon} + 2l'n' \dfrac{d}{dg} + 2m'p' \dfrac{d}{d\gamma} + 2l'p' \dfrac{d}{dh} + 2m'n' \dfrac{d}{d\eta} \end{Bmatrix}$$

and after expanding the determinant here under written,

$$\begin{matrix} a & e & g & \eta \\ e & b & h & \gamma \\ g & h & c & \epsilon \\ \eta & \gamma & \epsilon & d, \end{matrix}$$

perform the operations above indicated upon the result so obtained.

These are the operations and processes which, on Professor Boole's authority, we are to accept "*as without doubt far more convenient*" than the one simple process of forming, and when necessary, calculating the extended determinant above given. Here for the present I leave the case between Mr. Boole and myself to the judgment of the readers of this Journal.

In the April number of the *Philosophical Magazine*, I have shewn that the extended determinant serves, not only to represent the full and complete determinant of the reduced quadratic function, but likewise all the minor determinants thereof; the last *set* of which will be evidently no other than the coefficients themselves. For instance, in the example above given, if we wish to find the coefficient of x^2 after (z) and (t) have been eliminated, we have only to strike out the line and column $e\ b\ h\ g\ m\ m'$ from the extended determinant; if we wish to find the coefficient of y^2, we must strike out the line and column $a\ e\ g\ \eta\ l\ l'$; to find the coefficient of xy, we must strike out the line $a\ e\ g\ \eta\ l\ l'$ and the column $e\ b\ h\ \gamma\ m\ m'$, or *vice versâ*.

In each of these cases the determinant so obtained is the numerator of the equivalent fraction; the denominator remaining always the same function of the coefficients of transformation as in the original theorem.

Again, if there be taken only one linear equation, and if any one of x y z be supposed to be eliminated; and if the reduced quadratic function be called

$$Lx^2 + My^2 + Nz^2 + 2Pyz + 2Qxz + 2Rxy,$$

the same extended determinant as before given will serve, when stripped of its outer border, consisting of the line and column (n), to produce the various equivalent fractions thus form the square

$$\begin{matrix} L & R & Q \\ R & M & P \\ Q & P & N \end{matrix}$$

The numerator of the fraction equivalent to $\frac{L\ R}{R\ M}$, i.e. to $LM - R^2$, may be found by striking out from the form of the extended determinant the line and column $\eta\ \gamma\ \epsilon\ \delta\ p$; that corresponding to $\frac{L\ Q}{R\ P}$, i.e. $LP - RQ$, will be found by striking out the line $g\ h\ c\ \epsilon\ n$ and the column $\eta\ \gamma\ \epsilon\ d\ p$, or *vice versâ*; and so forth for all the first minor determinants; and similarly the second minors, i.e. L, M, N, P, Q, R, may be obtained by striking out in each case a correspondent pair of lines and pair of columns. Thus, to find the numerator of L the same pair of lines and columns, viz. $(g\ h\ c\ \epsilon\ n)$, $(\eta\ \gamma\ \epsilon\ d\ p)$, must be elided. To find the numerator of R, the pair of lines $(g\ h\ c\ \epsilon\ n)$, $(\eta\ \gamma\ \epsilon\ d\ p)$, and the pair of columns $(e\ b\ h\ \gamma\ m)$, $(\eta\ \gamma\ \epsilon\ d\ p)$, or *vice versâ*, will have to be elided; and so forth for the remaining second minors. I may conclude with observing that the theorem contested by Mr. Boole is an immediate corollary from the general Theory of Relative Determinants alluded to in the "Sketch" inserted in the present number of the *Journal*.

ON THE METHOD OF VANISHING GROUPS.

By James Cockle.

[*Continued from* p. 181, Vol. III. N. S.]

XI. By the Method of Vanishing Groups is meant that species of Indeterminate Analysis discussed in my two papers "On certain Algebraic Functions," published respectively

at pp. 267–273 of vol. II., and at pp. 179–181 of vol. III. of the present series of this *Journal*. To avoid repetition I shall treat those papers as if they were incorporated with the present one, and number the paragraphs accordingly.

The spirit of the method consists in the reducing an algebraic function to the form of a sum of like powers, grouping the powers two and two together, and making each group vanish. The analysis may, however, be made to take a different and, sometimes perhaps, a slightly more general form. Thus, if we denote the result of paragraph I. by

$$f^2(m) = h_1^2 + h_2^2 + . + h_r^2 + . + h_m^2 = \Sigma_m(h^2),$$

and make

$$p_r = h_r + \sqrt{(-1)}\, h_{r+1}, \text{ and } p_{r+1} = h_r - \sqrt{(-1)}\, h_{r+1},$$

we shall have (r being considered as an odd number),

$$f^2(m) = p_1 p_2 + p_3 p_4 + . + p_r p_{r+1} + . + \tfrac{1}{2}\{1-(-1)^m\}\, h_m^2:$$

and thus, when m is even, we may write

$$f^2(m) = \Sigma_{\frac{1}{2}m}(p_1 p_2),$$

and $f^2(m)$ may be made to vanish by the $\frac{1}{2}m$ relations

$$p_1 = 0, \quad p_3 = 0, \ldots \; p_r = 0, \ldots$$

or by any of the corresponding ones; the only restriction being, that either p_r or p_{r+1} must vanish. So that, when exhibited under the latter aspect, our process might not improperly be termed the *method of vanishing products.*

XII. Under this generalized aspect the higher results of the method of vanishing groups may be viewed. For, let $\rho_1, \rho_2, \ldots \rho_n$ represent the n roots of

$$1 + \rho^n = 0,$$

then will

$$h_1^n + h_2^n = (h_1 - \rho_1 h_2)(h_1 - \rho_2 h_2) \ldots (h_1 - \rho_n h_2);$$

and if (r, as before, being odd) we make

$$h_r + \rho_t h_{r+1} = H_{r,t},$$

we may in general consider the method of vanishing groups (or products) as the method by which we are enabled to determine for what values of t, and under what circumstances, the relation

$$f^n(t) = (H_{1,1} \times H_{1,2} \times . \times H_{1,n}) + (H_{3,1} \times H_{3,2} \times . \times H_{3,n})$$
$$+ . + (H_{r,1} \times H_{r,2} \times . \times H_{r,n}) = \Sigma_r(H_{1,1} \times H_{1,2} \times . \times H_{1,n})$$

can be satisfied. In such cases the equations

$$H_{1,1} = 0, \quad H_{3,1} = 0, \ldots \; H_{r,1} = 0,$$

(or any corresponding ones) would cause $f^n(t)$ to vanish. If we exhibit $f^n(t)$ under the above form at once, without the intermediate grouping of the h^n's, we avoid the obstacles sometimes thrown in the way of the direct method of vanishing groups by the disappearance of the squares, or other powers, of the indeterminate quantities involved in the expressions given for discussion.

XIII. Without now dwelling further upon this part of the subject, I shall observe that, of the enormous masses of results with which the method furnishes us in almost infinite variety, one portion alone is capable of practical application, the rest possesses an interest purely speculative. Yet the latter and the greater portion must not be considered as useless to science. By its aid we are, as will be hereafter seen, able to obtain a satisfactory indication of the possibility of solving problems the actual solution of which would, from intolerable complexity and incalculable length, be utterly unattainable. But the importance of ascertaining the solvibility of a problem must not be measured by the utility which would attach to its solution if actually written out. The possibility of solving the general equation of the fifth degree is a question the weight and interest of which would, perhaps, be best made apparent by mentioning the names of those algebraists who have bent their attention to the subject.

By way of example, let it be required to determine what must be the value of u_m, in order that the function $f^4(u_m)$ should be capable of being reduced to the form of a sum of two fourth powers. By means of paragraph III. we see that $u_0 = 2$, and consequently that

$$u_1 = 3.2^{1+4} - 1 = 3.2^5 - 1 = 95,$$
$$u_2 = 3.2^{1+190} - 1 = 3.2^{191} - 1,$$

an enormous number. And, since in the present instance $m = 2$, we see that the mere physical labour, required by the nature of the question, is such as to render the actual performance of the operations involved in it impossible for any human being to accomplish. But it is not the less certain that the reduction in question is *algebraically* possible; and on this algebraic possibility we may build any arguments that we please, or that the exigencies of any other problem may require.

XIV. It is no doubt true that the Method of Vanishing Groups may be so modified, in many cases, as to abridge considerably the operations required by it in its unmodified

form. But upon this discussion I shall not here enter, as other points claim our attention first. And among these the foremost that presents itself is that of notation, which, important in every department of mathematics, is of vital consequence here, dealing as we have to do with operations that, from their very magnitude and extent, defy all power of conception, other than of the most general nature. By a notation, that I hope will be found suitable to the purpose, I hope to exhibit clearly the results of operations which it would baffle the genius of the most subtle and weary the industry of the most untiring to perform; and which, from sheer physical necessity, must have their existence in the imagination only: and I hope to exhibit them in such a way as to render them subjects of distinct and definite contemplation.

XV. By γ I shall denote the state of a function after the performance of the operation by which a power

$$(h^2,\ \mathrm{h}^3,\ \mathrm{H}^4,\ \mathfrak{h}^6,\ \&c.)$$

is isolated from the rest of the function. I shall use an index to represent the degree of the power last isolated, and a suffix to indicate the number of times the operation has been performed. And I may premise that, as γ relates to the form and not to the value of the expression, we have the general relation $\gamma(u) = u$, whatever be the index or suffix of γ.

XVI. If we represent by $f^2(m)$ the homogeneous function of the second degree alluded to at the commencement of paragraph I., we may avail ourselves of this notation to express that paragraph as follows:

$$\begin{aligned} f^2(m) &= \gamma_1^2 f^2(m) = h_1^2 + \phi(\xi', \xi'', .\,.\,, \xi^{(m)}) \\ &= \gamma_2^2 f^2(m) = h_1^2 + h_2^2 + \psi(\xi''', \xi^{v}, .\,.\,, \xi^{(m)}) \\ &= \&c. \qquad = \&c. \end{aligned}$$

$$\gamma_m^2 f^2(m) = h_1^2 + h_2^2 + .\, + h_r^2 + .\, + h_m^2;$$

and in general we have

$$\gamma_r^2 f^2(m) = h_1^2 + h_2^2 + .\, + h_r^2 + f^2(m-r).$$

XVII. In the present paper I shall proceed no further with the development of the Method of Vanishing Groups than to shew its application to a theorem, interesting on account of its resemblance to one given by Mr. Sylvester in a preceding page (15, Note) of this volume. By the aid of paragraphs II. and XI. we see that

$$\gamma^3 f^3(9) = \mathrm{h}_1^3 + (p_1 p_2 + p_3 p_4 + p_5 p_6 + p_7 p_8)\mathrm{K}'\Xi' + f^3(8) \ldots (14).$$

Make $p_1 = 0,\ p_3 = 0,\ p_6 = 0,\ p_7 = 0;$

and by means of these four linear equations eliminate four of the unknowns from (14), then that equation becomes

$$\gamma^3 f^3(5) = h_1^3 + f^3(4);$$

hence $$\gamma_2^3 f^3(5) = \mathrm{h}_1^3 + \mathrm{h}_2^3 + f^2(3)\xi + f^3(3).$$

Make $f^2(3)$ and $f^3(3)$ vanish simultaneously in the ordinary way. This may be done by means of an equation of the sixth degree. Then, by means of Ξ' and ξ, we may satisfy

$$\mathrm{h}_1^3 + \mathrm{h}_2^3 = 0 \ldots\ldots\ldots\ldots\ldots (15),$$

and also any given equation of the n^{th} degree originally involving the same nine quantities as $f^3(9)$, but from which we eliminate the quantities in the same manner as they are eliminated from the preceding equations in this paragraph. No elevation of degree will be introduced by (15); hence we see that an indeterminate system of two homogeneous equations of *nine* variables, one of the third and the other of the n^{th} degree, may be completely resolved by means of two equations, one of the sixth and the other of the n^{th} degree.*

2, *Pump Court, Temple, April* 3, 1851.

[*To be Continued.*]

Postscript.—The reader will be pleased to make the following corrections in one of my previous Papers.

Vol. II., N. S., p. 268, line 8, *for* 1 *read* −1.
" " " 269, " 9 from the bottom, *for* h *read* h.
" " " 270, " 18, *for* $+(1)^{\frac{1}{4}}$ *read* $-(1)^{\frac{1}{4}}$.

ON THE LAWS OF THE ELASTICITY OF SOLID BODIES.

By W. J. Macquorn Rankine, C.E., F.R.S.E., F.R.S.S.A., &c.

[*Supplementary Paper to Section III., Article* 17.]

In the portion above referred to of my paper on the Elasticity of Solids, published in the *Cambridge and Dublin Mathematical Journal* for February, 1851, the theorem is

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* Had Mr. Sylvester's equations been homogeneous they would have involved *five* variables; had mine not been so they would have involved *eight*. Whether the number of quantities which I have above employed may not be diminished, will form a subject of future inquiry.

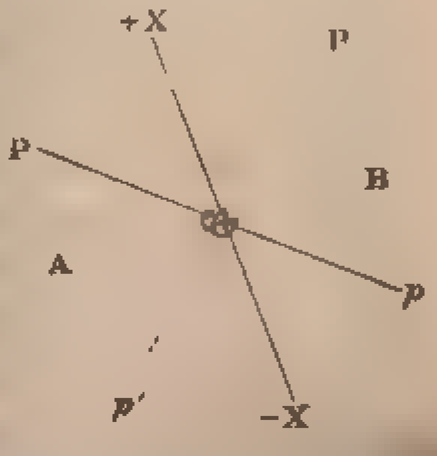
laid down, that in a given plane in an elastic solid consisting entirely of atoms acting on each other by attractions and repulsions between their centres, the coefficients of rigidity and of lateral elasticity are equal.

The proof of this proposition depends on the principle that *the elastic force in such a solid, called into play by a strain, in which the relative displacements of the atoms are very small as compared with their distances apart, is sensibly the resultant of the variations of force due to the variations of distance only, the variations of relative direction producing no appreciable effect.* This principle being granted, it is easily shewn that the portion of that resultant for each pair of atoms is the same for a given amount of strain in a given plane, whether lateral or transverse with respect to the plane on which elastic pressure is estimated.

In the paper referred to, I assumed this principle without demonstration. The editor of this *Journal*, however, has since shewn me, that my having done so may be considered as causing a defect in the chain of reasoning. I shall now therefore proceed to prove it.

Let it be possible for a solid to exist in an unstrained condition, consisting entirely of atomic centres of force acting on each other along the lines joining them, with forces which are functions of the lengths of these lines. Then must the pressure, estimated in any direction, on any portion of any plane in that solid be null. That pressure is the resultant, in the direction assumed, of the mutual actions of all the atoms whose lines of junction pass through the given portion of the given plane.

Let the given portion be indefinitely small, and let it be called ω, being situated in the arbitrarily-assumed plane $P\omega p$, which divides the solid into two portions A and B. Let $-X\omega+X$ be an arbitrary axis, along which pressure is to be estimated. The pressure exerted by the portion A upon the infinitesimal area ω of the portion B, is the resultant reduced to the direction $-X\omega+X$, of all the forces exerted by the atoms in A on the atoms in B, in lines passing through ω; and the body being unstrained, this *resultant must be null.*



Assume a new position $P'\omega p'$ for the plane of separation, making an equal angle $P'\omega + X = + X\omega P$ on the opposite side of the axis to the original position. The same letters applying to the two portions of the solid, the pressure of A on the area ω of B along $- X\omega X$ must still be null.

The two planes divide the solid into two pairs of opposite wedges. The action of A on B along X through ω in the original position of the plane, may be divided into two parts, viz.—

The resultant of the actions of the atoms in the wedge $P\omega p$ on those in the opposite wedge $P'\omega p$;

The resultant of the actions of the atoms in the wedge $p\omega p$ on those in the wedge $P\omega P'$.

In the new position of the plane, the pressure on ω is made up as follows:

The resultant of the actions of the atoms in the wedge $P\omega p'$ on those in the wedge $P'\omega p$, which is the same as in the original position of the plane;

The resultant of the actions of the atoms in the wedge $P\omega P'$ on those in the wedge $p'\omega p$, being identical in amount but *opposite in direction* to that of the atoms in $p'\omega p$ on those in $P\omega P'$, which formed part of the pressure in the original position of the plane.

Now the pressures in the two positions of the plane of separation cannot both be null, unless the resultant of the mutual actions of the atoms in each pair of opposite wedges is separately null; for we see that the action of a pair of wedges can be reversed in direction without affecting the nullity of the total resultant. The position of the pair of opposite wedges is arbitrary; so also is their angular magnitude, which may be indefinitely small.

Therefore no mere change of angular position of a pair of opposite elementary wedges can produce a pressure.

Every strain in which the relative displacements of the particles are small as compared with their relative distances, may be reduced to angular displacements of pairs of opposite elementary wedges, and variations of the mutual distances of the particles contained in them. The angular displacements can produce no pressure of themselves; the variations of distance are therefore the sole cause of that portion of the pressure, which is of the same order of small quantities with the strain: being the principle to be proved.

The combination of the angular displacements with the variations of distance wil [illegible] essures of the second

and higher orders of small quantities as compared with the strain; but for the small strains to which the present inquiry is limited, those are inappreciable and may be neglected.

London, February, 1851.

MATHEMATICAL NOTES.

I.—*Construction by the Ruler alone to determine the ninth Point of Intersection of two Curves of the third Degree.*

By A. S. Hart, Trinity College, Dublin.

In the last number of the *Mathematical Journal*, Mr. Weddle has mentioned the analysis from which I derived a construction to determine the ninth point of intersection of two curves of the third degree; but he had not seen my construction, which I now send to complete the solution of the problem. It is obvious that the point is immediately determined by the intersection of conics; but as there is only one point, it should be determined by right lines. Mr. Weddle's construction involves the determination of the intersections of a right line and conic, which is unnecessary in a question of this nature; and Euclid's constructions are equally inadmissible, as they involve the description of circles: I must therefore premise some elementary propositions.

Prop. 1. To find the fourth harmonical to three diverging lines.

From any point on the second draw two lines cutting the first and third; the lines which join their intersections will meet on the fourth harmonical.

Prop. 2. Given three diverging lines to find a fourth, so that the anharmonic ratio of the pencil may be given.

Let OA, OB, OC, OD, be the given anharmonic ratio, and let PA, Pb, Pc, be the given lines: through A draw two right lines $ABCD$ and Abc: join Bb, Cc, and join their intersection and D; this line will cut Abc on the fourth line of the required pencil.

Prop. 3. To find an anharmonic pencil whose ratio is the product of the ratios of two given pencils.

Let the given pencils be cut by two lines at A, B, C, D, A, b, c, d: join Bd, bD, and from their intersection draw lines to C and c: these lines will form the required pencil.

Prop. 4. Given five points A, B, C, D, E, on a conic, t find the polar of any given point P.

Let the fourth harmonics to AB, P, CD; AC, P, BD, meet at Q; and let the fourth harmonics to AB, P, CE, AC, P, BE, meet at R: QR is the required polar.

Prop. 5. Given five points A, B, C, D, E, to find a sixth P, such that the anharmonic ratios $P(ABCD)$, $P(ABCE)$ may be given.

Let AE, BD meet at M; and on BD find a fourth point Q such that $BMDQ$ may be in the first given ratio: also le AD, CE meet at N; and on CE find R so that $CENR$ ma be in the second given ratio; join QR meeting AE at K an AD at L, BK and CL will meet at the required point P.

Lemma. If a right line PQ meet a conic at A, B, and th polars of P and Q at p, q, then $\dfrac{PA.PB}{QA.QB} = \dfrac{Pp.Pq}{Qp.Qq}$.

Prop. 6. To find the ninth point of intersection of tw curves of the third degree.

Find the polars of 7 and 8 with respect to each of th three conics 12345, 12346, 12356, and let them cut 78 the points A, a, B, b, C, c, (A, B, C being on the pola of 7 and a, b, c on the polars of 8). Then find (by Prop 3 and 5) the point 9, such that the anharmonic ratio o 9(6758) may be equal to the product of the anharmoni ratios $(7A\,8B)$, $(7a\,8b)$, and that the anharmonic ratio 9(6748) may be equal to the product of $(7A\,8C)$, $(7a\,8c)$: the poin thus found is the ninth point of intersection.

II.—*On Clairaut's Theorem.*

By Samuel Haughton.

In the fourth volume of the *Cambridge and Dublin Mathematical Journal*, p. 202, Mr. Stokes has proved Clairaut's Theorem as a consequence of two observed facts; viz. 1st, That gravity is perpendicular to the surface of the Earth; 2nd, That the figure of the Earth is a spheroid of small ellipticity.

The following proof of the same theorem was suggested to me by a theorem published by Professor M'Cullagh in the *Dublin University Calendar* for 1834, p. 268.

It leads directly to the physical meaning of the constants employed, and in this respect se as simple as can be

desired; the rest of the proof is substantially the same as that given by Mr. Stokes.

The potential of any attracting mass upon a distant point is

$$V = \frac{M}{R} + \frac{A + B + C - 3I}{2R^3} \ldots\ldots\ldots\ldots (1).$$

A, B, C, are the moments of inertia with respect to the principal axes, which are taken as axes of coordinates; the centre of gravity being the origin. I is the moment of inertia with respect to the axis passing through the centre of gravity and the distant point, M is the mass, and R the distance of point from origin.

Equation (1) has been shewn by Prof. M'Cullagh to be the potential for all points of the surface of an ellipsoid, whose principal sections have small ellipticities, and which is composed of ellipsoidal couches, whose densities and small ellipticities vary according to any law.

This is evident from the consideration that the attraction of the ellipsoid may be replaced by the attraction of a very small confocal ellipsoid; with respect to which the points of the surface of the first ellipsoid may be considered as very distant. We are entitled to assume, that in the Earth A and B are q. p. equal; but

$$I = A\cos^2\lambda + B\cos^2\mu + C\cos^2\nu.$$

Hence $\quad A + B + C - 3I = (C - A)(1 - 3\sin^2\lambda) \ldots . (2),$

λ denoting the latitude, or angle made by line joining origin with attracted point, and plane of x, y.

We may now assume the potential of the Earth upon any external point to be

$$V = \frac{M}{R} + \frac{C - A}{2R^3}(1 - 3\sin^2\lambda) + U \ldots\ldots (3),$$

the indeterminate function U justifying this assumption.

Let V_0, U_0; and V_∞, U_∞ denote the values of V, U at the surface of the Earth, and at an infinite distance respectively; U_∞ is equal zero, and we shall shew that, if the two facts from which we set out be granted, U_0 is also zero.

But if $U_0 = 0$, and $U_\infty = 0$, then $U = 0$ for all external points. (C. F. Gauss, "On General Propositions regarding Forces acting inversely as the square of the distance." Taylor's *Scientific Memoirs*, Vol. III. p. 183.)

To prove that $U_e = 0$.

1st. *Gravity is perpendicular to the surface.*

This is expressed by the equation

$$\left(\frac{dV}{dx} + \omega^2 x\right) dx + \left(\frac{dV}{dy} + \omega^2 y\right) dy + \frac{dV}{dz}\, dz = 0,$$

ω denoting the angular velocity: or, integrating,

$$V_e + \tfrac{1}{2}\omega^2 (x^2 + y^2) + \text{const.} = 0 \text{ .... } (4).$$

2nd. *The surface of the Earth is spheroidal and of small ellipticity.*

This is expressed by the equation

$$r = a(1 - e \sin \lambda) \text{ .... } (5),$$

r denoting the radius vector of the surface and a the major axis; e the ellipticity and λ the latitude.

Substituting from (4) and (5) in equation (3), we find

$$\frac{M}{a}(1 + e \sin^2\lambda) + \frac{C - A}{2a^3}(1 - 3 \sin^2\lambda) + \frac{\omega^2 a^2}{2}(1 - \sin^2\lambda)$$
$$+ \text{const.} + U_e = 0.$$

This equation will be satisfied, if

$$\left.\begin{aligned} U_e &= 0, \\ \frac{M}{a} + \frac{C - A}{2a^3} + \frac{\omega^2 a^2}{2} + \text{const.} &= 0, \\ \frac{Me}{a} - \frac{3}{2} \cdot \frac{C - A}{a^3} - \frac{\omega^2 a^2}{2} &= 0. \end{aligned}\right\} \text{ .... } (6),$$

The second of these equations is satisfied by means of the constant, and the first will be true subject to whatever condition is imposed by the third, which determines the difference of polar and equatoreal inertia as a function of the figure and rotation.

If q denote the ratio of centrifugal force to gravity, we have

$$\omega^2 a^2 = q\,\frac{M}{a}.$$

Hence the third of equations (6) becomes

$$C - A = \frac{Ma^2}{3}(2e - q) \text{ .... } (7).$$

Substituting this [illegible] U, as we are

entitled to do by Gauss's Theorem, we obtain finally for the potential of the Earth on any external point,

$$V = \frac{M}{R} + \frac{Ma}{6R^3}(2e - q)(1 - 3\sin^2\lambda) \dots\dots (8).$$

Differentiating this with respect to R, substituting r for R, and adding the term $\omega^2 r(1 - \sin^2\lambda)$ arising from centrifugal force, we find ($-g$ denoting gravity at any latitude)

$$g = \frac{M}{a^2}\left\{(1 + e - \tfrac{3}{2}q) + (\tfrac{5}{2}q - e)\sin^2\lambda\right\} \dots\dots (9).$$

Let G_p and G_e denote polar and equatoreal gravity; then

$$G_p = \frac{M}{a^2}\{1 + q\}, \quad G_e = \frac{M}{a^2}(1 + e - \tfrac{3}{2}q).$$

Hence $$m = \frac{G_p - G_e}{G_p} = \tfrac{5}{2}q - e,$$

or $$e + m = \tfrac{5}{2}q \dots\dots\dots\dots (10),$$

which is Clairaut's Theorem.

Trinity College, Dublin, April 4, 1851.

III.—*Laws of the Elasticity of Solid Bodies.*

Note respecting Mr. Clerk Maxwell's paper "On the Equilibrium of Elastic Solids." (*Trans Roy. Soc. Edinb.*, Vol. xx. Part I.)

By W. J. Macquorn Rankine.

I have already referred to the researches of Mr. Clerk Maxwell; of the general nature of which only I was aware at the time of the publication of my paper on this subject in the *Cambridge and Dublin Mathematical Journal* for February 1851.

Since then I have had an opportunity of reading Mr. Maxwell's paper, so as to compare his notation with my own.

Mr. Maxwell's investigations relate to such solids only as are equally elastic in all directions. He expresses their elasticity by means of two coefficients, μ and m, having the following properties:

$$\mu = -\frac{1}{3}\cdot\frac{P_1 + P_2 + P_3}{\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}},$$

$$m = -\frac{P_1 - P_2}{\frac{d\xi}{dx} - \frac{d\eta}{dy}} = -\frac{P_2 - P_3}{\frac{d\eta}{dy} - \frac{d\zeta}{dz}} = -\frac{P_3 - P_1}{\frac{d\zeta}{dz} - \frac{d\xi}{dx}}.$$

From which it is clear, that those coefficients have [illegible] following values in the notation of the paper which I h[illegible] published:

$$\mu = \frac{1}{\delta} = \tfrac{5}{3}C + J = \text{reciprocal of the cubic compressibility},$$

$$m = 2C = \text{twice the rigidity};$$

consequently $\qquad C = \tfrac{1}{2}m, \quad J = \mu - \tfrac{5}{6}m.$

The particular problems solved by Mr. Clerk Maxwell are of a very interesting character, especially those relative to the optical changes produced in transparent bodies by straining them.

London, February, 1851.

IV.—*Note on Mr. Cockle's Solution of a Cubic Equation.*

By A. Cayley.

The final result of Mr. Cockle's elegant method for the solution of a cubic equation, as applied to the equation

$$ax^3 + 3bx^2 + 3cx + d = 0,$$

is

$$x = \frac{\{(ad - bc) - \sqrt{(-M)}\}\sqrt[3]{P} + 2(ac - b^2)c}{\{ad - bc - \sqrt{(-M)}\}a - 2(ac - b^2)\sqrt[3]{P}};$$

where

$$P = \tfrac{1}{2}[(3abc - 2b^3 - a^2d) + a\sqrt{(-M)}],$$

$$M = 6abcd - 4ac^3 - 4b^3d + 3b^2c^2 - a^2d^2.$$

SKETCH OF A MEMOIR ON ELIMINATION, TRANSFORMATION, AND CANONICAL FORMS.

By J. J. Sylvester, M.A., F.R.S.

There exists a peculiar system of analytical logic, founded upon the properties of zero, whereby, from dependencies of equations, transition may be made to the relations of functional forms, and *vice versâ*: this I call the logic of characteristics.

The resultant of a given system of homogeneous equations of as many variables, is the function whose nullity implies and is implied by the possibility of their co-existence, *i.e.* is the characteristic of suc[illegible] inasmuch as any

[redacted]

numerical product of any power of a characteristic is itself an equivalent characteristic, in order to give definiteness to the notion of a resultant, it must further be restricted to signify the characteristic taken in the *lowest form* of which it *in general* admits.

The following very general and important proposition for the change of the independent variables in the process of elimination, is an immediate consequence of the doctrine of characteristics.

Let there be two sets of homogeneous forms of function;

$$\text{the } 1^{\text{st}},\ \phi_1,\ \phi_2 \ldots\ldots \phi_n,$$
$$\text{the } 2^{\text{nd}},\ \psi_1,\ \psi_2 \ldots\ldots \psi_n.$$

Let the results of applying these forms to any sets of (n) variables be called

$$(\phi_1),\ (\phi_2) \ldots\ldots (\phi_n),$$
$$(\psi_1),\ (\psi_2) \ldots\ldots (\psi_n);$$

then will the resultant (in respect to those variables) of

$$\phi_1\{(\psi_1),\ (\psi_2) \ldots\ldots (\psi_n)\},$$
$$\phi_2\{(\psi_1),\ (\psi_2) \ldots\ldots (\psi_n)\},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\phi_n\{(\psi_1),\ (\psi_2) \ldots\ldots (\psi_n)\},$$

be the product of powers (assignable by the law of homogeneity) of the separate resultants of the two systems,

$$\{(\phi_1),\ (\phi_2) \ldots\ldots (\phi_n)\},$$
$$\{(\psi_1),\ (\psi_2) \ldots\ldots (\psi_n)\}.$$

By means of the doctrine of characteristics the following general problem may be resolved.

Given any number of functions of as many letters, and an inferior number of functions of the same inferior number of letters, obtained by combining, *inter se*, in a known manner, the given functions, to determine the factor by which, the resultant of the reduced system being divided, the resultant of the original system may be obtained.

If in the theorem for the change of the independent variables both sets of forms of functions be taken linear, we obtain the common rule for the multiplication of determinants: if we take one set linear and the other not, we deduce two rules, viz. That the resultant of a given set of functional forms of a given set of variables, enters as a factor into the resultant,

1st, of linear functions of the given functions of the given variables;

2nd, of the given functions of linear functions of the given variables:

the extraneous factor in each case being a power of wh[...] may be conveniently termed the *modulus of transformati[on]*, *i. e.* the resultant of the imported linear forms of functions.

From the second of these rules we obtain the law first stated I believe for functions beyond the second degree by Mr. Boole, to wit, that the determinant of any homogeneous algebraical function (meaning thereby the resultant of its first partial differential coefficients) is unaltered by any linear transformations of the variables, except so far as regards the introduction of a power of the modulus of transformation. This is also abundantly apparent from the fact, that the nullity of such determinant implies an immutable, *i.e.* a fixed and inherent, property of a certain corresponding geometrical locus.

There exist (as is now well known) other functions besides the determinant, called by their discoverer (Mr. Cayley) hyperdeterminants, gifted with a similar property of immutability. I have discovered a process for finding hyperdeterminants of functions of any degree of any number of letters, by means of a process of Compound Permutation. All Mr. Cayley's forms for functions of two letters may be obtained in this manner by the aid of one of the two processes (to wit, that one which will hereafter be called the derivational process,) for passing from immutable constants to immutable forms. Such constants and forms, derived from given forms, may be best termed adjunctive; a term slightly varied from that employed by M. Hermite in a more restricted sense.

The two processes alluded to may be termed respectively appositional and derivational. The appositional is founded upon the properties of the binary function $x\xi + y\eta + z\zeta + \ldots$; in which, whether we substitute linear functions of x, y, z, &c., or linear functions of ξ, η, ζ, &c., in place of x, y, z, &c., or ξ, η, ζ, &c., the result is the same.

Consequently, if we apply the form ϕ to $\xi, \eta, \ldots \zeta$, and take any constant (in respect to $\xi, \eta, \ldots \zeta$) adjunctive to

$$\phi(\xi, \eta, \ldots \zeta) + (x\xi + y\eta + \ldots + z\zeta + kt^{n-1})\, t,$$

calling this quantity $\psi(x, y \ldots . z, t)$, the form ψ is evidently adjunctive to the form ϕ: and if we expand so as to obtain

$$\psi(x, y \ldots z, t) = \psi_1(x, y \ldots z)\, t^{\alpha} + \psi_2(x, y \ldots z)\, t^{\beta} + \text{\&c.},$$

it is evident ψ_1, ψ_2, &c. will be each separately adjunctive to ϕ. These forms, when ψ is obtained by finding the determinant in respect to $\xi, \eta \ldots \zeta$ of S, are, in fact, identical with Hermite's "formes

The derivational mode of generating forms from constants depends upon the property of the operative symbol

$$\chi = \xi \frac{d}{dx} + \eta \frac{d}{dy} + \ldots + z \frac{d}{dz},$$

applied to (ϕ) a function of $x, y \ldots z$; viz. that if in (ϕ), in place of these letters, we write linear functions thereof, to wit $x', y' \ldots . z'$, we may write

$$\chi = \xi' \frac{d}{dx'} + \eta' \frac{d}{dy'} + \ldots . + \zeta' \frac{d}{dz'},$$

where $\xi', \eta', \ldots . \zeta'$ will be the same functions of $\xi, \eta, \ldots . \zeta$ that $x', y', \ldots . z'$ are of $x, y, \ldots . z$.

Suppose now, in the first place, that in regard to $\xi, \eta, \ldots \zeta$, $\psi(x, y, \ldots . z)$ is adjunctive to $\chi^r . \phi(x, y, \ldots . z)$; then is the form ψ adjunctive to the form ϕ, for on changing $x, y, \ldots . z$ to $x', y', \ldots . z'$,

$$\left(\xi \frac{d}{dx} + \eta \frac{d}{dy} + \ldots . + \zeta \frac{d}{dz}\right)^r \phi(x', y', \ldots . z')$$

becomes $\left(\xi' \dfrac{d}{dx'} + \eta' \dfrac{d}{dy'} + \ldots . + \zeta' \dfrac{d}{dz'}\right) \phi(x', y', \ldots . z')$;

and consequently $\psi(x, y, \ldots . z)$ becomes $\psi(x', y', \ldots . z')$, multiplied by a power of the modulus of transformation, the modulus of that transformation, be it well observed, whereby $x', y', \ldots . z'$ would be replaced by $x, y, \ldots . z$, and not as in the appositional mode of that converse transformation according to which $x, y, \ldots . z$ would be replaced by $x', y', \ldots . z'$. It is on account of this converseness of the modes of transformation that the appositional and derivational modes of generating forms cannot except for a certain class of *restricted* linear transformations be combined in a single process. More generally, if instead of a single function $\chi^r . \phi(x, y, \ldots z)$, we take as many such with different indices to χ as there are variables, and form either the resultant in respect to $\xi, \eta, \ldots . \zeta$, or any other immutable constant in regard to those variables, (presuming in extension of the hyperdeterminant theory and as no doubt is the case, that such exist,) every such resultant or other constant will give a form of function of $x, y, \ldots . z$ adjunctive to the given form (ϕ).

It may be shewn that every such resultant so formed will contain ϕ as a factor.

Again, in the former more available determinant mode

of generation, if we take the determinant in respect to $\xi, \eta, \ldots . \zeta$, it may be shewn that all the adjunctive functions so obtained will be algebraical derivees of the partial differential coefficients of ϕ in respect to $x, y, \ldots . z$; that is to say, if these be respectively zero, all such adjunctive functions so derived, as last aforesaid, will be zero, or in other words, each such adjunctive is a syzygetic function of the partial differential coefficients of the primitive function.

To Mr. Boole is due the high praise of discovering and announcing, under a somewhat different and more qualified form and mode of statement, this marvel-working process of derivational generation of adjunctive forms. I was led back to it, in ignorance of what Mr. Boole had done, by the necessity which I felt to exist of combining Hesse's so-called functional determinant, under a common point of view with the common constant determinant of a function; under pressure of which sense of necessity, it was not long before I perceived that they formed the two ends of a chain of which Hesse's end exists for all homogeneous functions, but the other only when such functions are algebraical.

In fact, if we give to (r) every value from (2) upwards, the successive determinants in respect to $\xi, \eta, \ldots . \zeta$ of

$$\left(\xi \frac{d}{dx} + \eta \frac{d}{dy} + \ldots . \zeta \frac{d}{dz}\right)^r . \phi(x, y, z),$$

will produce the chain in question, which, when ϕ is algebraical and of n dimensions, comes to a natural termination when $r = n - 1$. The last member of and the number of terms in this chain are identical with the last member of and the number of terms in Sturm's auxiliary functions, when the variables are reduced to two. There is some reason to anticipate that this chain of functions may be made available in superseding Sturm's chain of auxiliaries; and if so, then the fatal hindrance to progress, arising from the unsymmetrical nature of the latter, is overcome, and we shall be able to pass from Sturm's theorem, which relates to the theory of Kenothemes, or Point-systems, to certain corresponding but much higher theories for lines, surfaces, and n-themes generally.

The restriction of space allowed to me in the present number of the *Journal* will permit me only to allude in the briefest terms to the theory of Relative Determinants, which, as it will be seen, plays an important part in the effectuation of the reductions of the higher algebraical functions to their simplest forms. Nor can the effect of the processes to be indicated be correctly appreciated without [illegible] dge of the circumstances

der which the resultant of a *given* system of equations can k in degree below the resultant of the *general* type of such stem. Abstracting from the case when the equations sepaely, or in combination, subdivide into factors, this lowering degree, as may be shewn by the doctrine of characteristics, n only happen in one of two ways. Either the particular ultant obtained is a rational root of the general resultant, the general resultant becomes zero for the case supposed, d the particular resultant is of a distinct character from the neral resultant, being in fact the characteristic of the posbility not of the given system of equations being merely le to coexist (for that is already supposed), but of their ing able to coexist for a certain system of values *other than* given system or given systems. Such a resultant may be rmed a Sub-resultant; the lowest resultant in the former e may be termed a Reduced-resultant. The theory of Subsultants is one altogether remaining to be constructed, and well worthy equally of the attention of geometers and of alysts.

As to the theory of Relative Determinants, the object this theory is to obtain the determinant resulting from iminating as many variables as can be eliminated, chosen pleasure from a set of variables greater in number han the equations containing them; and the mode of efctuating this object is through the method of the indeterinate multiplier. To avoid the discussion of the theory of ub-resultants and other particularities, I shall content myself ith giving the rule applicable to the case (the only one which as yet a practical application has offered itself to e in the course of my present inquiries) when all but one the functions is linear.

If $U, L_1 L_2 \dots L_m$ be the first an n^c and the others linear nctions of (n) variables, and it be desired to find the deterinant of the resultant arising from the elimination of any) out of the (n) variables, the following is the rule:

Find the determinant, *i.e.* the resultant of the partial differntial coefficients in respect to the given variables, and of

$$\lambda_1\lambda_2 \dots \lambda_m \text{ of } U + L_1 . \lambda_1 + L_2 . \lambda_2 + \dots + L_m . \lambda_m .$$

This resultant, in its lowest form, will be always a rational $-1)^{\text{th}}$ root of the resultant of the homogeneous system of uations to which the system above given can be referr its type; and this reduced resultant divided by a po terminable by the law of homogeneity) of the resultar $L_2 \dots L_m$, when all but the selected variables are n

zero, will be the resultant determinant required.* As regards what has been said concerning the reducibility of the general typical resultant in the case before us, this is a consequence of, and may be brought into connection with, the following theorem, which is easily demonstrable by the theory of characteristics. If $Q_1 Q_2 \ldots . Q_m$ be (m) homogeneous functions of (m) variables of the same degree, (r) of which enter in each equation only as simple powers uncombined with any of the other variables, then the degree of the reduced resultant is equal to the number of the equations multiplied by the $(m - r - 1)^{\text{th}}$ power of the units of number on the degree of each, subject to the obvious exception that when r is m, (there being in fact but *one* step from $r = m - 2$ to $r = m$,) instead of r, $(r - 1)$ must be employed in the above formula. As an example of a sub-resultant as distinguished from a reduced-resultant, I instance the case of three quadratics U, V, W, functions of x, y, z, in each of which no squared power of z is supposed to enter: it may easily be shewn by my dialytic method that instead of six equations, between which to eliminate x^2, y^2, z^2, xy, xz, yz, we shall have only 5, the three original ones and two instead of three auxiliaries between which to eliminate x^2, y^2, xy, xz, yz, the *apparent* resultant is accordingly of the 9th instead of the 12th degree. But this is not the true characteristic of the possibility of the coexistence of the given systems, which in fact is zero, as is evidenced by the fact that they always *do* coexist, since they are always satisfiable by only *two* relations between the variables, to wit $x = 0$, $y = 0$. The apparent resultant is then something different, and what has been termed by the above a Sub-resultant.

I take this opportunity of entering my simple protest against the appropriation of my method of finding the resultant of any set of three equations of degrees equal or differing only by a unit, one from those of the other two, by Dr. Hesse, so far as regards quadratic functions, without acknowledgment, four years after the publication of my memoir in the *Philosophical Magazine:* the fundamental idea of Dr. Hesse's partial method is identical with that of my general one. Still more unjustifiable is the subsequent use of the *dialytic* principle, by the same author, equally without acknowledgment, and in cases where there is no

* The same method applies not only to the Final or Constant Determinant, but likewise to all the Functional Determinants in the chain above described, extending un- [illegible] the Hessian, or as it ought to be termed, the first Bool

dure to give even a plausible
ledgment. It is capable
ten on the matter was
nigsberg, at each

re peculiar
de of reducing
d most symme-
rmite well proposes
Every quadratic func-
always be linearly trans-
functions of the same, and
of ways; but in every other
mited number of ways, where-
m will admit of being transmuted
h the sole exception of a cubic func-
uch transmutation will never be possible,
condition, or certain conditions, be satisfied
constants of the forms proposed for transmu-
number of such conditions is the number of
entering into the canonical form, and is of course
the number of terms in the general form of the
diminished by the square of the number of letters.
there is one parameter in the canonical form for the
quadratic function of two and the cubic function of three
tters, and no parameter in the cubic function of two letters. Hitherto no canonical forms have been studied beyond the cases above cited, but I have succeded, as will presently be shewn, in obtaining methods for reducing to their canonical forms functions with *two* and *four* parameters respectively. Owing to what has been remarked above, the theory of quadratic functions is a theory apart. Simultaneous transformation gives definiteness to that theory, but has no existence for any useful purpose for functions of the higher degrees. Where the theory of simultaneous transformation ends, that of canonical forms properly begins; and in what follows, the case of quadratic forms is to be understood as entirely excluded. Such exclusion being understood, there is no difficulty in assigning the canonical, *i.e.* the simplest and most symmetrical general form to which every function of two letters admits of being reduced by linear transformations. If the degree be odd, say $2m+1$, the canonical form will be $u_1^{2m+1} + u_2^{2m+1} + \ldots + u_{m+1}^{2m+1}$; if the degree be even, say $2m$, the canonical form will be

$$u_1^{2m} + u_2^{2m} + \ldots + u_m^{2m} + K(u_1 u_2 \ldots u_m)^2,$$

all the u's being linear functions of the two given variables. It is easy to extend an analogous mode of representation to functions of any number of letters. From the above we see that for cubic, biquadratic, and quintic functions of two letters, the canonical forms will be respectively

$$u^3 + v^3, \quad u^4 + v^4 + Ku^2v^2,$$
$$u^5 + v^5 + w^5,$$

with a linear relation in the last-named case between u, v, w.

First as to the reduction of any 4^c function to Cayley's form

$$u^4 + v^4 + Ku^2v^2.$$

This may be effected in a great variety of ways, of which the following is not the simplest as regards the calculations required, but the most obvious. Let the modulus of transformation, whereby the given biquadratic function, say $F(x, y)$, becomes transmuted into its canonical form, be called M; let the determinant of F be called D, and the determinant of the determinant in respect to

$$\xi \text{ and } \eta \text{ of} \left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right)^2 F(x, y),$$

which latter, for brevity's sake, may be termed the Hessian of F, (although in stricter justice the Boolian would be the more proper designation) be called D_2. Then, by examining the canonical form itself (which is as it were the very *palpitating heart* of the function laid bare to inspection), we shall obtain without difficulty the two equations

$$(1 - 9m^2)^2 = M^{12} D_1 \frac{1}{4^6},$$

$$m^2 (1 - 9m^2)^2 (m^2 - 1)^2 = M^{24} D_2 \frac{1}{12^{12} 4^6}.$$

Eliminating the unknown quantity M, we obtain

$$\frac{m^2 (m^2 - 1)^2}{(1 - 9m^2)^2} = c, \quad \text{or} \quad \frac{m^3 - m}{1 - 9m^2} = c^{\frac{1}{2}},$$

where (c) is a known quantity.

This *cubic* equation for finding m is of a peculiar form; it being easy to shew *à priori*, by going back to the canonical form, that its three roots are m, $\theta(m)$, $\theta^2(m)$, where

$$\theta(m) = \frac{m - 1}{3m + 1},$$

θ being a periodical form of function such that $\theta^3(m) = m$.

This it is which accounts for the simple expression for (m), that may be obtained by solving the cubic above given. A

tter practical mode is to take, instead of the determinant f the given function and its Hessian, the two hyperdeterinants and eliminate as before: a cubic equation having recisely the same properties, and in fact virtually identical ith the former, will result. (m) and consequently M being ound, there is no difficulty whatever, calling the given nction F and its Hessian $H(F)$, to form linear functions f the two, as

$$\left.\begin{array}{l}\phi(m).F+\psi(m).H(F)\\ \phi_1(m).F+\psi_1(m).H(F)\end{array}\right\},$$

hich shall be equal to, *i.e.* identical with, $(u^3+v^3)^2$ and u^3v^3, hence u and v are completely determined.

Another and interesting mode of solution is to take, besides e given function F and its Hessian, either the *second* essian or the post-Hessian of the given function, by the ost-Hessian understanding the determinant in respect to

$$\xi \text{ and } \eta \text{ of} \left(\xi\frac{d}{dx}+\eta.\frac{d}{dy}\right)^2.F:$$

ny three of the four functions will be linearly related, and it may be shewn that, calling either the second Hessian (*i.e.* he Hessian of the Hessian) or the post-Hessian H', we shall ave

$$H'.F+a.H.(F)+b(F)=0,$$

here (a) and (b) will be *rational* and *integer* functions of the coefficients of (F), and numerical multiples of two quantities R and S, such that the determinant of F will be equal R^3+S^2; nd this, be it observed, without any previous knowledge of he existence of these hyperdeterminants R and S.

If now we go to Hesse's form for a cubic function of hree letters, we shall find that precisely similar modes of nvestigation apply step for step. Calling the function F and its Hessian $H(F)$, and the post-Hessian or second Hessian t choice $H'.F$, we shall find

$$H'.F+m.S.H(F)+n.R^2.(F)=0,$$

here m and n are numerical quantities and R^3+S^2 equal he determinant of F. It is interesting to contrast this equaon with the one previously mentioned as applicable to the functions of two letters, viz.

$$H'.(F)+m.RH(F)+n.S(F)=0.$$

In both instances there is no difficulty in assigning the lations between the original R and S, and the R and S ny adjunctive form. All Arnohold's results may be thus d and further extended without the slightest difficulty.

As regards the equation for finding the parameter in Hesse's canonical form for the cubic of three letters, this will be of the 4^{th} degree in respect to the cube of the parameter, and the roots will be functionally representable as

$$x;\quad \theta(x);\quad \phi(x);\quad \psi(x),$$

where

$$\theta^2(x) = \phi^2(x) = \psi^2(x) = x;$$
$$\theta\phi(x) = \phi\theta(x) = \psi(x),$$
$$\phi\psi(x) = \psi\phi(x) = \theta(x),$$
$$\psi\theta(x) = \theta\psi(x) = \phi(x);$$

owing to which property the equation is soluble under the peculiar form observed by Aronhold.

I pass on now to a brief account of the method, or rather of a method (for I doubt not of being able to discover others more practical), of reducing a function of the 5^{th} degree of two letters (say of x and y) to its canonical form $u^5 + v^5 + w^5$, subject to the linear relation $au + bv + cw = 0$, where the ratios $a : b : c$, and the linear relations between u, v, w and the two given variables are the objects of research. Here I have found great aid from the method of Relative Determinants; and I may notice that the successful application of more compendious methods to the question would be greatly facilitated were there in existence a theory of Relative Hyperdeterminants, which is still all to form, but which I little doubt, with the blessing of God, to be able to accomplish. It may some little facilitate the comprehension of what follows, if c be considered as representing unity.

Calling as before the given quintic function F, the modulus of transformation M, the Hessian and post-Hessian of F, H and H', and its ordinary or constant determinant D, we shall find

$$a^2.v^3.w^3 + b^2.w^3.u^3 + c^2.u^3.v^3 = M^2.H,$$

and

$$P_1.P_2.P_3.P_4 = M^6.H', \text{ where}$$

$$P_1 = a^{\frac{3}{4}}vw + b^{\frac{3}{4}}wu + c^{\frac{3}{4}}uv,$$

$$P_2 = a^{\frac{3}{4}}vw - b^{\frac{3}{4}}wu - c^{\frac{3}{4}}uv,$$

$$P_3 = -a^{\frac{3}{4}}vw + b^{\frac{3}{4}}wu - c^{\frac{3}{4}}uv,$$

$$P_4 = -a^{\frac{3}{4}}vw - b^{\frac{3}{4}}wu + c^{\frac{3}{4}}uv;$$

also $D = M^{20}$ multiplied by the product of the sixteen values of

$$a^{\frac{5}{4}} + b^{\frac{5}{4}}.(1)^{\frac{1}{4}} + c^{\frac{5}{4}}.(1)^{\frac{1}{4}}.$$

From the above equations it may be shewn that H', (a

known function of the 8^{th} degree of the given variables x, y) must be capable of being thrown under the form

$$L\{(x - a_1.y)(x - a_2y) \times (x - a_3.y)(x - a_4y)$$
$$\times (x - a_5.y)(x - a_6y) \times (x - a_7.y)(x - a_8y)\},$$

where
$$(a_1 - a_2)^2 \times (a_3 - a_4)^2 \times (a_5 - a_6)^2 \times (a_7 - a_8)^2$$
$$= \frac{D}{L^2} = K,$$

so that K is a known quantity.* Accordingly the said equation of the 8^{th} degree, considered as an algebraical equation in $\frac{x}{y}$, may by known methods be found by means of equations not exceeding the 4^{th} or even the 3^{rd} degree: in fact, to do this it is only necessary to form the equation to the squares of the differences of the roots of $\frac{x}{y}$ in the equation $H' \div y^8 = 0$, which new equation will be of the 28^{th} degree. If we then form two other equations of the 378^{th} degree, one having its roots equal to $\sqrt{K}$ multiplied by the binary products of the twenty-eight roots of the equation last named, the other to $\sqrt{K}$ multiplied by the reciprocal of such binary products, the left-hand members of these two equations expressed under the usual form will have a factor in common, which may be found by the process of common measure and will be of the 6^{th} degree, but whose roots consisting of three pairs of reciprocals may be found by the solution of cubics only.

In this way, by means of cubics and quadratics,

$$(a_1 - a_2)^2, \quad (a_3 - a_4)^2, \quad (a_5 - a_6)^2, \quad (a_7 - a_8)^2,$$

can be found, which being known,

$$a_1a_2, \quad a_3a_4, \quad a_5a_6, \quad a_7a_8,$$

can be determined in pairs by means of quadratics from the equation $H' \div y^8 = 0$. This being supposed to be done, we have

$$P_1 = f.L_1,$$
$$P_2 = g.L_2,$$
$$P_3 = l.L_3,$$
$$P_4 = k.L_4,$$

where L_1, L_2, L_3, L_4, are *known* quadratic functions of

* Or in other words, the post-Hessian determinant of a given function in two letters of the second degree, may be divided into four quadratic factors in such a way that the product of the determinants of these several factors shall be equal to the determinant of the given function.

x and y. To determine the ratios of f, g, l, k, we have three equations* obtained from the identity

$$fL_1 + gL_2 + hL_3 + kL_4 \, (= P_1 + P_2 + P_3 + P_4) = 0;$$

$f : g : h : k$, being known $fL_1 : gL_2 : hL_3 : kL_4$ are known ratios.

But

$$P_1 + P_2 = 2a^{\frac{3}{2}}.vw,$$
$$P_1 + P_3 = 2b^{\frac{3}{2}}.wu,$$
$$P_1 + P_4 = 2c^{\frac{3}{2}}.uv.$$

Hence

$$a^{\frac{3}{2}}.vw = \lambda.P,$$
$$b^{\frac{3}{2}}.wu = \lambda.Q,$$
$$c^{\frac{3}{2}}.uv = \lambda.R,$$

where P, Q, R are known quadratic functions of x, y.

Hence $a : b : c$ may be found by means of the identical equation

$$a^2.v^3u^3 + b^2.u^3w^3 + c^2.v^3u^3 = H(F),$$

whereby the ratios $a^{-\frac{5}{2}} : b^{-\frac{5}{2}} : c^{-\frac{5}{2}}$ can be obtained without any further extraction of roots, shewing that there is but one single true system of ratios $a^5 : b^5 : c^5$ applicable to the problem; $a : b : c$ being thus found, λ is easily determined, and thus finally u, v, w are found in terms of x and y.†

I have little doubt that a more expeditious mode of solution than the foregoing‡ will be afforded by an examination of the properties and relations of the *quadratic and cubic forms*, adjunctive to the general quintic functions, and indeed to every $(4n + 1)^{ic}$ function of two letters hereinbefore adverted to.

Sufficient space does not remain for detailing the steps whereby the general cubic function of *four* letters may, by aid of equations *not transcending the fifth degree*, be reduced to its canonical form $u^5 + v^5 + w^5 + p^5 + q^5$, wherein u, v, w, p, q are connected by a linear equation

$$au + bv + cw + dp + eq = 0;$$

the four ratios of whose coefficients $a : b : c : d : e$ give the

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* For we must have the coefficients of x^2 of xy and y^2 in

$$fL_1 + gL_2 + hL_3 + kL_4,$$

all of them zero.

† The problem thus solved may be stated as consisting in reducing the general function $ax^6 + bx^4y + cx^3y^2 + dx^2y^3 + exy^4 + fy^5$ to the form

$$(lx + my)^5 + (l'x + m'y)^5 + (l''x + m''y)^5.$$

‡ The coefficients in the reducing recurrent equation of the 6th degree in the process above detailed may rise to be of 541632 dimensions in respect to the original coefficients in F.

[redacted]

:cessary number $\frac{4.5.6}{1.2.3} - 4^2$ parameters furnished by the eneral rule. Suffice it for the present to say, that the anartical mode of solution depends upon a circumstance capable f the following geometrical statement: "Every surface of ne 4th degree represented by a function which is the Hessian o any given cubic function whatever of four letters, has lying ipon it ten straight lines meeting three and three in ten »oints, and these ten points are the only points which enjoy he following property in respect to the surface of the 3rd legree denoted by equating to zero the cubical function in question, to wit, that the cone drawn from any one of them as vertex to envelop the surface, will meet it not in a continuous double curve of the 6th degree, but in two curves each of the 3rd degree, lying in *planes* which intersect in the ten lines respectively above named; so that to each of the ten points corresponds one of the ten lines: these ten points and lines are the intersections taken respectively three with three, and two with two, of *a single and unique system* of five principal planes appurtenant to every surface of the 3rd degree, and these planes are no other than those denoted by

$$u = 0, \quad v = 0, \quad w = 0, \quad p = 0, \quad q = 0.$$

I have found also by the theory of Sub-resultants, that the analogy between lines and surfaces of the 3rd degree, in regard to the existence of double and conical points, is preserved in this wise: that in the same way as a double point on a curve of the 3rd degree commands the existence of a double point on its Hessian, so does a conical point in a surface of the 3rd degree command over and above the 10 necessary, and so to speak natural conical points, at least one extra, that is to say an 11th conical point on *its* Hessian. And here for the present I must quit my brief and imperfect notice of this subject, composed amidst the interruptions and distractions of an official and professional life.

Observation.—It may be somewhat interesting and instructive to my readers, to have a table of the successive scalar* determinants of a quintic function of two letters presented to them at a single glance. Preserving the notation of page 197, we have the following expressions:

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* By which I mean the determinants in respect to

$$\xi, \eta \text{ of} \left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right)^r . F(xy).$$

The given function $= u^6 + v^6 + w^6$,

its Hessian $= M^2 (a^3v^3w^3 + b^3w^3u^3 + c^3u^3v^3)$,

its post-Hessian $= M^6 \times$ the product of the *four* fo

$$a^{\frac12}vw + b^{\frac12}(1)^{\frac14}wu + c^{\frac12}(1)^{\frac14}uv;$$

its præter-post-Hessian $= M^{12} \times$ the product of the *nine* fo

$$a^{\frac13}v^{\frac13}w^{\frac13} + b^{\frac13}(1)^{\frac13}w^{\frac13}u^{\frac13} + c^{\frac13}(1)^{\frac13}u^{\frac13}v^{\frac13},$$

and the final determinant $= M^{20} \times$ the product of the
forms of

$$a^{\frac23} + (1)^{\frac13}b^{\frac23} + (1)^{\frac13}c^{\frac23}.$$

The success of the method applied depends (as
shewn) upon the fact of a certain function of the roots
post-Hessian (which is an octavic function of the va
being known, which fact *hinges* upon the circumstance

$$(M^6)^2 \times (M^2)^4 = M^{20}.$$

P.S.—I have much pleasure in subjoining the c
hyperdeterminant of the 12[th] degree function of two l
worked out upon the principle of Compound Perm
hinted at in the foregoing pages, for which I am ind
to the kindness and skill of my friend Mr. Spottiswoode
The function being called

$$a.x^{12} + 12bx^{11}.y + \frac{12.11}{2}.cx^{10}.y^2 + \&c.\ldots + ly^{12},$$

the following is its cubical hyperdeterminant:

$$agm - bahl + 15aik + 10aj^2 - 6bfm,$$
$$- 24bhk + 30bgl + 20bij - 24cfl + 114cgk,$$
$$- 145ci^2 + 50chj + 15cem + 20cgi + 20ch^2,$$
$$- 400dgj + 280dhi + 20del + 50dfe + 10d^2k,$$
$$+ 385egi - 135e^2k - 290eh^2 + 705fgh,$$
$$- 330f^2i - 50g^3.$$

Mr. Spottiswoode will I hope publish the work in
the next number of the *Journal*, in which I shall also
how the hyperdeterminants of the cubical function of
letters, Aronhold's S and T, may be similarly obtained.

April, 1851.

ON DUPLICATE SURFACES OF THE SECOND ORDER.

By John Y. Rutledge.

1. The following results have originated with some formulæ published in a preceding number of the *Journal*; they are therefore not to be regarded as a collection of isolated theorems, possessing, as the case may be, a greater or less degree of interest, but as the natural development of a method established in the articles to which I have referred. For the convenience of my readers, I shall briefly state the fundamental theorems, and in all that follows, restrict myself to the consideration of *central* surfaces of the second order.

Suppose that any central surface of the second order, whose semi-major axis is a, is given, and that in space any point (x_0, y_0, z_0) has been selected, the axes of coordinates being the principal axes of the given surface. Let the point be now defined by the intersection of three confocal surfaces of the second order whose semi-major axes are (ρ, μ, ν), and at this point conceive the three normals to the three confocal surfaces drawn, and let them be considered as the rectangular axes of a new system of coordinates (ξ, η, ζ). The following equation is then true (vol. v. p. 112),

$$\left(\frac{\xi_0^2}{\rho^2-a^2}+\frac{\eta_0^2}{\mu^2-a^2}+\frac{\zeta_0^2}{\nu^2-a^2}-1\right)\left(\frac{x_0^2}{a^2}+\frac{y_0^2}{a^2-b^2}+\frac{z_0^2}{a^2-c^2}-1\right)=1\ldots(1);$$

where (ξ_0, η_0, ζ_0) are the coordinates of the centre of the original given surface, referred to the three normals as axes of coordinates, and b and c are the known constants employed in the theory of elliptic coordinates. We also have (vol. v. p. 113)

$$\frac{\xi_0^2}{\rho^2-a^2}+\frac{\eta_0^2}{\mu^2-a^2}+\frac{\zeta_0^2}{\nu^2-a^2}=1+\frac{a^2(a^2-b^2)(a^2-c^2)}{(\rho^2-a^2)(\mu^2-a^2)(\nu^2-a^2)}\ldots(2),$$

and $$\frac{x_0^2}{a^2}+\frac{y_0^2}{a^2-b^2}+\frac{z_0^2}{a^2-c^2}=1+\frac{(\rho^2-a^2)(\mu^2-a^2)(\nu^2-a^2)}{a^2(a^2-b^2)(a^2-c^2)}\ldots(3).$$

With the point (x_0, y_0, z_0) as vertex, let a cone be described enveloping the given central surface a, the equation of its plane of contact, in coordinates (ξ, η, ζ), is (vol. v. p. 112)

$$\frac{\xi\xi_0}{\rho^2-a^2}+\frac{\eta\eta_0}{\mu^2-a^2}+\frac{\zeta\zeta_0}{\nu^2-a^2}=1\ \ldots\ldots\ldots(4),$$

while in the ordinary rectangular coordinates its known equation is

$$\frac{xx_0}{a^2}+\frac{yy_0}{a^2-b^2}+\frac{zz_0}{a^2-c^2}=1\ \ldots\ldots\ldots\ldots(5).$$

The preceding expressions are the basis of the investigations which follow.

2. In any point (x', y', z') conceive three central confocal surfaces of the second order to intersect, whose equations are

$$\left.\begin{aligned} \frac{x^2}{\alpha^2} + \frac{y^2}{\alpha^2 - b^2} + \frac{z^2}{\alpha^2 - c^2} = 1 \\ \frac{x^2}{\beta^2} + \frac{y^2}{\beta^2 - b^2} + \frac{z^2}{\beta^2 - c^2} = 1 \\ \frac{x^2}{\gamma^2} + \frac{y^2}{\gamma^2 - b^2} + \frac{z^2}{\gamma^2 - c^2} = 1 \end{aligned}\right\} \dots\dots\dots\dots (6).$$

The common centre of the system being the point (ξ_0, η_0, ζ_0). With the point (x_0, y_0, z_0) as centre, let us now construct three confocal surfaces intersecting in a point (ξ', η', ζ'), whose equations are

$$\left.\begin{aligned} \frac{\xi^2}{\rho^2 - \alpha^2} + \frac{\eta^2}{\mu^2 - \alpha^2} + \frac{\zeta^2}{\nu^2 - \alpha^2} = 1 \\ \frac{\xi^2}{\rho^2 - \beta^2} + \frac{\eta^2}{\mu^2 - \beta^2} + \frac{\zeta^2}{\nu^2 - \beta^2} = 1 \\ \frac{\xi^2}{\rho^2 - \gamma^2} + \frac{\eta^2}{\mu^2 - \gamma^2} + \frac{\zeta^2}{\nu^2 - \gamma^2} = 1 \end{aligned}\right\} \dots\dots\dots (7).$$

Any two systems, such as (6) and (7), constitute *a duplicate system of confocal surfaces of the second order*, and we shall see that the relations existing between them are most curious and interesting. The two centres (ξ_0, η_0, ζ_0) and (x_0, y_0, z_0) we shall call *duplicate centres*, and any two points related as are the points (x', y', z') and (ξ', η', ζ'), we shall call *duplicate points*.

Now let the right line which connects the duplicate centres (ξ_0, η_0, ζ_0) and (x_0, y_0, z_0) be represented by the letter (R), and let λ denote the central radius vector drawn from the centre (ξ_0, η_0, ζ_0) of the system (6) to the point of intersection (x', y', z'), while (λ') denotes the central radius vector drawn from the centre (x_0, y_0, z_0) of the system (7) to the point of intersection (ξ', η', ζ').

Any two radii vectores related as are the radii λ and λ', we shall call *duplicate radii vectores*. It is, however, well known that

$$\lambda^2 = \alpha^2 + \beta^2 + \gamma^2 - b^2 - c^2:$$

similarly $\quad \lambda'^2 = \rho^2 + \mu^2 + \nu^2 - \alpha^2 - \beta^2 - \gamma^2;$

we therefore obviously have

$$R^2 = \lambda^2 + \lambda'^2 \quad \ldots\ldots\ldots\ldots\ldots (8).$$

This relation is of course perfectly general between any two duplicate systems of confocal surfaces, whose centres respectively are the points (ξ_0, η_0, ζ_0) and (x_0, y_0, z_0). We consequently have the following theorem. *The square of the right line which joins the duplicate centres, is always equal to the sum of the squares of any two duplicate radii vectores.*

It is manifest that the asymptotic cones of the system (7) are the circumscribing cones of the corresponding surfaces of the duplicate system (6) (*Journal*, vol. v. p. 84), and *vice versâ* the asymptotic cones of the system (6) are the circumscribing cones of the corresponding surfaces of the duplicate system (7). From the equations (4) and (5) we see that *the planes of contact are identical* of the asymptotic cone of the surface, whose semi-major axis is $\sqrt{(\rho^2 - a^2)}$, enveloping the duplicate surface a, and of the asymptotic cone of the surface a, enveloping the duplicate surface $\sqrt{(\rho^2 - a^2)}$. If in the surface a a central section be drawn parallel to the tangent plane at the point (a, β, γ), the squares of its principal semi-axes are

$$a^2 - \beta^2, \text{ and } a^2 - \gamma^2.$$

Now, if in the duplicate surface $\sqrt{(\rho^2 - a^2)}$ a central section be drawn parallel to the tangent plane at the point

$$\{\sqrt{(\rho^2 - a^2)}, \quad \sqrt{(\rho^2 - \beta^2)}, \quad \sqrt{(\rho^2 - \gamma^2)}\},$$

the squares of its principal semi-axes are

$$\beta^2 - a^2, \text{ and } \gamma^2 - a^2;$$

consequently the areas of the two sections are the same, except as to sign. We have now demonstrated the following theorems:

Given any two duplicate surfaces of the second order, the asymptotic cone of the one is an enveloping cone of the other.

The enveloping cones have the same plane of contact.

The areas of the central sections parallel to the tangent planes at any two duplicate points, are equal, except as to sign.

From the point (x_0, y_0, z_0) let fall a perpendicular upon any plane of contact (4), and, as is well known, the perpendicular at the point in which it cuts the plane will be normal to a surface of the second order which touches the plane of contact at that point, and is confocal with the surface a. Similarly, from the duplicate centre (ξ_0, η_0, ζ_0) let fall a per-

duplicate surfaces whose centres are the duplicate points (ξ_0, η_0, ζ_0) and (x_0, y_0, z_0), let us consider the particular duplicate surfaces whose semi-major axes respectively are ρ, μ, ν of the one system, and ρ, $\sqrt{(\rho^2 - b^2)}$, $\sqrt{(\rho^2 - c^2)}$ of the other. If with respect to these surfaces we seek the different particular cases of the general theorems which have been demonstrated throughout the previous portion of the present article, we shall see that they are in fact equivalent to a well-known theorem of Professor Chasles'. Suppose that we are given any ellipsoid (E) and any fixed plane (P). *Through any point in the fixed plane, describe three surfaces confocal with the given surface* (E), *and at this point draw their respective normals; upon each normal measure a portion equal to the semi-major axis of its surface, and construct a new ellipsoid with these right lines for its semi-major axes and with its centre at the assumed point in the plane* (P). This ellipsoid will possess the following properties:

(1). *It will pass through the centre of the ellipsoid* (E), *and will be tangent to the plane normal to the semi-major axis of* (E).

(2). *Its central section parallel to the principal plane just mentioned, will always have a constant area whatever be the position of the assumed point in the plane* (P). Similarly, if upon the normals drawn at the assumed point in the plane (P), we measure portions respectively equal to $\sqrt{(\rho^2 - b^2)}$, $\sqrt{(\mu^2 - b^2)}$, $\sqrt{(\nu^2 - b^2)}$ and $\sqrt{(\rho^2 - c^2)}$, $\sqrt{(\mu^2 - c^2)}$, $\sqrt{(\nu^2 - c^2)}$, and construct surfaces with the assumed point as centre, it is evident that analogous properties will hold with respect to the constructed surfaces. We are now enabled to perceive that Prof. Chasles' theorem, beautiful and important as it is, *is in all respects but a very particular case of a theorem far more general.* This fact, without at all entering into the details, we have in a previous communication remarked (vol. v. p. 83).

The further consideration of many interesting particular cases we shall omit, as the reader can without difficulty infer them from the general theorems demonstrated in the present article.

4. We shall now resume the consideration of the duplicate systems (6) and (7). From the mere inspection of the equations it is manifest, *that to a line of curvature upon any one surface corresponds a line of curvature upon the duplicate surface.* Also, *to a set of corresponding points upon any*

3. We have seen (vol. v. p. 118) that

$$\frac{ll'}{D^2} = \frac{(\rho^2 - a^2)(\mu^2 - a^2)(\nu^2 - a^2)}{a^2(a^2 - b^2)(a^2 - c^2)} \ldots\ldots\ldots (13),$$

where l, l' denote the intercepts from the point (x_0, y_0, z_0) made by the surface a upon the right line which joins the duplicate centres, and D denotes the coincident semi-diameter of the surface a. Hence we easily see that, if R indicate the length of the right line connecting the duplicate centres,

$$\frac{R^2}{D^2} = 1 + \frac{(\rho^2 - a^2)(\mu^2 - a^2)(\nu^2 - a^2)}{a^2(a^2 - b^2)(a^2 - c^2)} \ldots\ldots (14).$$

Similarly if Δ denote the semi-diameter of the *duplicate* of the surface a, coincident with the right line R, it is easy to see that, attending to the equation (8),

$$\frac{R^2}{\Delta^2} = 1 + \frac{a^2(a^2 - b^2)(a^2 - c^2)}{(\rho^2 - a^2)(\mu^2 - a^2)(\nu^2 - a^2)} \ldots\ldots (15).$$

If we now write

$$\frac{R^2}{\Delta^2} = \kappa^2, \quad \frac{R^2}{D^2} = \kappa'^2,$$

from the equations (2) and (3) we shall obtain

$$\frac{\xi_0^2}{\kappa^2(\rho^2 - a^2)} + \frac{\eta_0^2}{\kappa^2(\mu^2 - a^2)} + \frac{\zeta_0^2}{\kappa^2(\nu^2 - a^2)} = 1 \ldots (16),$$

and

$$\frac{x_0^2}{\kappa'^2 a^2} + \frac{y_0^2}{\kappa'^2(a^2 - b^2)} + \frac{z_0^2}{\kappa'^2(a^2 - c^2)} = 1 \ldots\ldots (17),$$

the equations of two surfaces concentric and similar with the duplicate surfaces $\sqrt{(\rho^2 - a^2)}$ and a. It is evident that they pass through the duplicate centres, and that at these centres their tangent planes are parallel to each other and the common plane of contact (4). If we next conceive the duplicate surfaces $\sqrt{(\rho^2 - a^2)}$ and a to vary, and in each case construct surfaces analogous to (16) and (17), *it is obvious that they will envelope the cones* (9) *and* (10), which are the reciprocals of the similar cones described by the normals drawn from the duplicate centres to the given systems of duplicate confocal surfaces. We shall next consider what variations of the duplicate surfaces determine among the surfaces (16) and (17) the *confocal* systems which intersect in the duplicate centres. Conceive any three confocal surfaces (α, β, γ), whose centre is the point (ξ_0, η_0, ζ_0), to intersect upon the right line R which joins the duplicate centres. It

hence we may infer an important and interesting theorem. We shall for the sake of shortness call such sides as L and L' of any two duplicate cones, *duplicate sides*, *i.e.* when any side L' (suppose) touches the two confocal surfaces, duplicate of the two confocal surfaces touched by the right line L. We shall call P and P' *duplicate perpendiculars*, *i.e.* where P is a perpendicular from the vertex of the one cone upon the tangent plane to the other containing a duplicate side, and *vice versâ*.

Any two semi-diameters, such as D and D', of two duplicate surfaces which are parallel to any two duplicate sides, such as L and L', of their duplicate asymptotic cones, we shall also call *duplicate semi-diameters*. The value which we have found for the square of $P'L'$ is the same, except as to sign, as the square of the PD of that particular geodesic line upon the surface a, which the right line L touches; for we have (vol. v. p. 121)

$$PD = \frac{a\sqrt{(a^2-b^2)}\sqrt{(a^2-c^2)}}{\sqrt{(a^2-a'^2)}}.$$

Hence we obtain the curious relations

$$P'L' = \sqrt{-1}\, PD \ldots\ldots\ldots\ldots (20),$$

and

$$PL = \sqrt{-1}\, P'D' \ldots\ldots\ldots\ldots (21).$$

We now perceive the truth of the following theorem: *Given any two duplicate surfaces of the second order* a *and* $\sqrt{(\rho^2-a^2)}$, *and the lengths* L *and* L' *of any two duplicate sides of their asymptotic cones. Let* P *and* P', D *and* D' *represent the corresponding duplicate perpendiculars and semi-diameters, we shall then always have the square of the* P'L' *of the one equal, except as to sign, to the square of the* PD *of the other.* We also perceive that, *if the* PD *of the one surface be real, the corresponding* P'L' *of the duplicate surface will be imaginary and the contrary.* Now since along the geodesic line PD is constant, and since the successive tangents form a developable surface, which circumscribes, in the present instance, the surface a', it is easy to see that if upon this developable surface any point (x_0, y_0, z_0) be taken as the duplicate of the common centre of the confocal surfaces a and a', and we construct the duplicates of the latter surfaces, *we shall, in every instance, have the value of the corresponding* P'L' *constant.* Similarly, if we have PL constant (21) for all duplicate centres (x_0, y_0, z_0), it is obvious that the $P'D'$ of all the related geodesic every instance upon

[the] surface duplicate of the surface a, will be constant. Let [us] next suppose that PL being constant, the duplicate centre [(x_0], y_0, z_0) moves on the surface of the ellipsoid ρ, we shall [de]monstrate that it will trace upon the surface of the ellipsoid [a] *curve of the second degree.* Since for the locus under con[s]ideration PL is constant, and since the confocal surfaces a and a' are supposed fixed, it follows that the value (13) is constant for the same locus, viz.

$$\frac{U'}{D^2} = \frac{(\rho^2 - a^2)(\mu^2 - a^2)(\nu^2 - a^2)}{a^2(a^2 - b^2)(a^2 - c^2)} = K.$$

From this equation, if we eliminate μ, ν by means of the known equations

$$\mu^2 + \nu^2 = x_0^{\,2} + y_0^{\,2} + z_0^{\,2} + b^2 + c^2 - \rho^2,$$

$$\mu\nu = \frac{bcx_0}{\rho},$$

we shall obtain the equation of a surface of the second order, whose intersection with the given ellipsoid ρ will determine the locus in question. Attending however to the equation (8), *we shall, without difficulty, find the equation of the locus to be*

$$\left.\begin{array}{l} \dfrac{x^2}{a^2\rho^2} + \dfrac{y^2}{(a^2 - b^2)(\rho^2 - b^2)} + \dfrac{z^2}{(a^2 - c^2)(\rho^2 - c^2)} \\ \quad = \dfrac{K}{\rho^2 - a^2}\left(\dfrac{x^2}{\rho^2} + \dfrac{y^2}{\rho^2 - b^2} + \dfrac{z^2}{\rho^2 - c^2}\right) \end{array}\right\} \ldots (22).$$

When the constant K vanishes, then a must equal μ or ν, *and we shall have as particular cases of the curve the known lines of curvature upon the ellipsoid.* The geometrical properties of the curve are, however, in general indicated by the equations (13) and (21).

5. Referring to the first of the equations (18), we shall have

$$\cos\iota = \frac{\xi_0\sqrt{(\rho^2 - \beta^2)}\sqrt{(\rho^2 - \gamma^2)}}{\rho\sqrt{(\rho^2 - b^2)}\sqrt{(\rho^2 - c^2)}}.$$

Let us now conceive that β and γ are the two confocal surfaces which with a intersect in a point upon the right line R, which joins the duplicate centres, as has been stated in the passage to which we have referred; the equation will then become

$$(\rho^2 - a^2)\kappa^2\cos^2\iota = \xi_0^{\,2}.$$

If we now suppose that (cos i) remains constant while the point (x_0, y_0, z_0) moves upon the surface of the ellipsoid the confocal surfaces β and γ of course varying for e position of the point, we shall find that *the locus of the poi upon the surface of the ellipsoid will be a curve of the fourt degree.* From the equation (3) we can find the value of and attending to the known value of ξ_0, we may readily prov the equation of the locus to be

$$\frac{x^2}{a^2\rho^2}+\frac{y^2}{(a^2-b^2)(\rho^2-b^2)}+\frac{z^2}{(a^2-c^2)(\rho^2-c^2)}$$
$$=\cos i\left(\frac{x^2}{a^2}+\frac{y^2}{a^2-b^2}+\frac{z^2}{a^2-c^2}\right)\left\{\frac{x^2}{\rho^4}+\frac{y^2}{(\rho^2-b^2)^2}+\frac{z^2}{(\rho^2-c^2)^2}\right\}.$$

According to the different values we attribute to (cos i), is easy to trace the corresponding modifications of the curv We have now demonstrated the following theorem: *The loc of the points upon the surface of an ellipsoid, at which the successi normals make a constant angle with the right lines drawn from t successive points common tangents to any two confocal surfaces, whi with a third given confocal surface intersect in a point upon the su cessive radii vectores passing through the points of the locus, is a cur of the fourth degree.*

It is easy to demonstrate that, if from any point upon ellipsoid ρ, a cone be described enveloping a confocal surfa a, and if from the point a normal be drawn to the ellipsoid intersecting the plane of contact of the cone with the confoc surface a; then, if ρ_1 indicate the intercept upon the norm and ξ_0 the perpendicular from the centre upon the tange plane to the ellipsoid at the point vertex of the cone,

$$\rho^2-a^2=\xi_0\rho_1.$$

Let ϕ denote the angle between ρ_1 and ξ_1, where ξ_1 i dicates the perpendicular from the point vertex of the co upon the plane of contact, it is obvious that we shall have

$$(\rho^2-a^2)\cos\phi=\xi_0\xi_1.$$

We have however already seen (vol. v. p. 114) that

$$\xi_1=\frac{\dfrac{x^2}{a^2}+\dfrac{y^2}{a^2-b^2}+\dfrac{z^2}{a^2-c^2}-1}{\sqrt{\left\{\dfrac{x^2}{a^4}+\dfrac{y^2}{(a^2-b^2)^2}+\dfrac{z^2}{(a^2-c^2)^2}\right\}}},$$

where (x, y, z) denote the coordinates of the point fro which the perpendicular ξ_1 has been let fall upon the pla of contact determined by the point and the surface a. L

next suppose that $\cos\phi$ is constant, and we shall have for the locus of the point upon the surface of the ellipsoid ρ a *curve of the fourth degree;* its equation is

$$\frac{x^2}{a^2\rho^2} + \frac{y^2}{(a^2 - b^2)(\rho^2 - b^2)} + \frac{z^2}{(a^2 - c^2)(\rho^2 - c^2)}$$

$$= \cos\phi \left\{\frac{x^2}{a^4} + \frac{y^2}{(a^2 = b^2)^2} + \frac{z^2}{(a^2 - c^2)^2}\right\}^{\frac{1}{2}} \left\{\frac{x^2}{\rho^4} + \frac{y^2}{(\rho^2 - b^2)^2} + \frac{z^2}{(\rho^2 - c^2)^2}\right\}^{\frac{1}{2}}.$$

We have now demonstrated the following theorem: *The locus of the point upon the surface of an ellipsoid, at which the normal makes a constant angle with the normal to the plane of contact determined by the point and a given confocal surface of the second order, is a curve of the fourth degree.*

When the angle ϕ becomes ninety degrees, it is plain that the curve degenerates into the line of curvature determined by the given confocal surface a upon the surface of the ellipsoid ρ. The preceding locus upon *any* surface of the second order may in several respects be regarded as the generalization of the ordinary locus, known as the line of curvature. This part of our subject, however, we shall not delay to consider farther in detail.

6. We have already seen (vol. v. p. 129) that, if at the duplicate centre (x_0, y_0, z_0) we reciprocate the surface a with respect to a sphere of radius unity, its equation in coordinates (ξ, η, ζ) will be

$$(\rho^2 - a^2)\,\xi^2 + (\mu^2 - a^2)\,\eta^2 + (\nu^2 - a^2)\,\zeta^2 = 2\,(\xi\xi_0 + \eta\eta_0 + \zeta\zeta_0) - 1.$$

To determine the centre of this surface, we shall have

$$\xi' = \frac{\xi_0}{\rho^2 - a^2}, \quad \eta' = \frac{\eta_0}{\mu^2 - a^2}, \quad \zeta' = \frac{\zeta_0}{\nu^2 - a^2} \ldots\ldots (23),$$

where ξ', η', ζ' denote the coordinates of the centre. Hence we perceive that the coordinates of the centre are *the reciprocals of the intercepts upon the normals* to the three confocal surfaces (ρ, μ, ν), made by the plane of contact determined by the point (x_0, y_0, z_0), and the given confocal surface a. The equation may be now written in parallel coordinates referred to its centre

$$(\rho^2 - a^2)\,\xi^2 + (\mu^2 - a^2)\,\eta^2 + (\nu^2 - a^2)\,\zeta^2$$

$$= \frac{\xi_0^{\,2}}{\rho^2 - a^2} + \frac{\eta_0^{\,2}}{\mu^2 - a^2} + \frac{\zeta_0^{\,2}}{\nu^2 - a^2} - 1.$$

reduced to the corresponding problems relating to uncrystallized media. After the publication of the researches of M. de Senarmont, M. Duhamel was induced to resume the subject, and in a memoir printed in the 32nd *Cahier* of the above-mentioned *Journal*, he has deduced from theory a number of general consequences which are directly applicable to the experiments of M. de Senarmont.

In the following paper, I propose to present the theory of crystalline conduction in a form independent of the hypothesis of molecular radiation—a hypothesis which for my own part I regard as very questionable. The subject will thus be considerably simplified, for in fact the results flow readily from certain very general assumed laws, which no doubt follow as consequences of the hypothesis of molecular radiation, but which are of such simplicity that they would seem to follow from almost any reasonable hypothesis relating to the manner in which the passage of heat takes place in the interior of a solid body. As regards the mathematical deduction of consequences from the general formulæ, I have introduced the consideration of what may be called an *auxiliary solid*, by which means problems relating to crystallized bodies are reduced to corresponding problems relating to ordinary media. All the principal results of M. Duhamel, of which one at least was obtained by him in a very artificial manner, are thus rendered almost self-evident, or else directly reduced to known results relating to ordinary media; and some results of still greater generality follow with equal facility.

1. Let P be any point of a solid body, homogeneous or heterogeneous, crystallized or uncrystallized; suppose the temperature of the body to vary from point to point, and let dS be an elementary plane area drawn through P in a given direction. The quantity of heat which passes across the element dS in the elementary time dt will be ultimately proportional to $dSdt$, and may be expressed by $fdSdt$. This quantity f is the *flux* of heat referred to a unit of surface. Its value will depend upon the time, upon the position of the point P, and upon the direction of the elementary plane drawn through P. For the present, suppose the time and the position of the point P given, and consider only the variation of f in different directions about P.

If we suppose the values of f given in the direction of each of three planes, rectangular or not, passing through P, its value in the directi[illegible] [illegible]urth plane follows. For

ake P the vertex of a triangular pyramid, of which the des are in the direction of the first three planes, and the ase is parallel to the fourth, and then conceive the base, emaining parallel to itself, to approach indefinitely to P. he quantity of heat gained by the pyramid during the time t is equal to the quantity which enters by the faces, diminished by the quantity which escapes by the base. Now hen the pyramid is indefinitely diminished, the gain of eat in a given indefinitely short time will vary ultimately s the volume, or as the cubes of homologous lines, whereas e quantity which passes across any one of the four faces f the pyramid will vary ultimately as the area of the face, nd therefore as the squares of homologous lines. Hence in e limit the quantity of heat which escapes by the base will e equal to the sum of the quantities which enter by the des, and consequently if the flux across each be given, the ux across the base is determinate.

In particular, if we suppose the medium referred to the ectangular axes of x, y, z, and if f_x, f_y, f_z be the fluxes cross three planes drawn through P in directions perendicular to the axes of x, y, z; f the flux across a plane rawn in any other direction through P; l, m, n, the cosines f the angles which the normal to this plane makes with the xes, we have

$$f = lf_x + mf_y + nf_z \ldots\ldots\ldots\ldots (1).$$

This equation shews that if we represent the fluxes across planes perpendicular to the axes of x, y, z, by three forces or three velocities, the flux across any other plane will be represented by the resolved part of the forces or velocities along the normal to this plane. Hence the flux across one particular plane passing through P is a maximum, and the flux across any other plane is equal to this maximum flux multiplied by the cosine of the angle between the two planes.

2. Let u be the temperature at P at the end of the time t, and consider the portion of the solid which is contained in the elementary volume $dx\,dy\,dz$ adjacent to P. The quantity of heat which enters this element during the time dt by the first of the faces $dy\,dz$ is ultimately equal to $f_x dy\,dz\,dt$, and the quantity which escapes by the opposite face is ultimately equal to $\left(f_x + \dfrac{df_x}{dx}dx\right)dy\,dz\,dt$. Subtracting the former from the latter, and treating in the same way each of the other two pairs of opposite faces, we find that the loss of heat in

the element is ultimately equal to

$$\left(\frac{df_x}{dx}+\frac{df_y}{dy}+\frac{df_z}{dz}\right)dx\,dy\,dz\,dt.$$

But if ρ be the density, $\rho\,dx\,dy\,dz$ will be the mass; and c be the specific heat, the loss of heat will also be equal to

$$-\rho\,dx\,dy\,dz.c\,\frac{du}{dt}\,dt$$

ultimately. Equating the two results, and passing to th limit, we get

$$c\rho\,\frac{du}{dt}=-\left(\frac{df_x}{dx}+\frac{df_y}{dy}+\frac{df_z}{dz}\right)\ldots\ldots\ldots(2).$$

3. The formulæ (1) and (2) are general, but for the futu I shall suppose the medium to be homogeneous, and th temperature to differ by only a small quantity from a certa fixed standard which we may suppose to be the origin fro which u is measured. Since the medium is homogeneo ρ is constant,* and c moreover will be constant, except so f as relates to a change of specific heat produced by a chang of temperature. But since u is supposed to be small, th terms arising from the variation of c would be small qua tities of the second order, since c only appears multiplied b $\frac{du}{dt}$, and therefore c may be regarded as constant.

It remains to form the expressions for f_x, f_y, f_z. By th conduction of heat we mean that sort of communication whic takes place between the contiguous portions of bodies. I the case of bodies which are partially *diathermous*, that is t say, which behave with respect to heat, or at least heat o certain degrees of refrangibility, in the same way in whic semi-opake bodies behave with respect to light, or rather i which a green glass behaves with respect to red rays, he may be communicated from one portion of the body t another situated at a sensible distance. But this is, properl speaking, internal radiation, and not conduction. Again, i the solid be perfectly diathermous to heat of certain degree of refrangibility, a portion in the interior of the mass may b radiation send heat out of the solid altogether. For my ow part I believe conduction to be quite distinct from intern radiation, although the theory which makes conduction to b nothing more than molecular radiation and absorption seem

* The expansion of the solid produced by heat is not here taken int account.

be received by many philosophers with the most implicit reliance. No doubt, internal radiation may, and I believe generally if not always does, accompany conduction; and when the distance which a ray of heat can travel before it is absorbed is insensible, we may include internal radiation in the mathematical theory of conduction, and even, if we please, in our definition of the word *conduction*. Of course the distance which we may regard as insensible will depend partly on the dimensions of the body, partly upon certain lengths relating to the state of temperature in the interior, and depending upon the problem with which we have to deal. As an example of a length of this sort, we may take the distance between consecutive maxima, if we are considering the internal temperature of a solid of which the surface has a temperature that is subject to periodic variations.

4. Let us now confine ourselves to conduction, using that term with the extensions and restrictions above explained. The temperature u is supposed to be sufficiently small to allow us to superpose different systems of temperature without mutual disturbance. If the temperature were the same at all points, there would be equilibrium of temperature, and the flux at any point in any direction would be equal to zero. Let then a uniform temperature, equal and opposite to that of P, be superposed on the actual system. Then the temperature at P will be reduced to zero without any change being made in the fluxes f_x, f_y, f_z. Hence these quantities will depend, not upon the absolute temperature at P, but only on its variation in the neighbourhood of P. Since, by hypothesis, these fluxes have nothing to do with the temperatures at points situated at sensible distances from P, they may be assumed to depend only on the differential coefficients $\frac{du}{dx}$, $\frac{du}{dy}$, $\frac{du}{dz}$, which define the variation of temperature in the neighbourhood of P. Since moreover different systems of temperature may be superposed, it follows that f_x, f_y, f_z, are linear functions of the three differential coefficients above written. Hence equation (2) may be put under the form

$$c\rho\frac{du}{dt} = A'\frac{d^2u}{dx'^2} + B'\frac{d^2u}{dy'^2} + C'\frac{d^2u}{dz'^2}$$
$$+ 2D'\frac{d^2u}{dy'dz'} + 2E'\frac{d^2u}{dz'dx'} + 2F'\frac{d^2u}{dx'dy'}, \ldots (3),$$

where x', y', z', have been written for x, y, z.

Q 2

5. Let us now refer the solid to the rectangular ax Ox, Oy, Oz, instead of Ox', Oy', Oz'. Let l, m, n, be t cosines of the angles $x'Ox$, $x'Oy$, $x'Oz$; let l', m', n', be t same for y', and l'', m'', n'', the same for z'. Then

$$\frac{d}{dx'} = l\frac{d}{dx} + m\frac{d}{dy} + n\frac{d}{dz}.$$

But we have also

$$x' = lx + my + nz,$$

and similar formulæ hold good with respect to y' and Since symbols of differentiation combine with one anoth according to the same laws as factors, it follows that t right-hand member of equation (3) will be tranformed exact as if the symbols of differentiation were replaced by th corresponding coordinates. Hence there exists a system rectangular axes, namely, the principal axes of the surface,

$$A'x'^2 + B'y'^2 + C'z'^2 + 2D'y'z' + 2E'z'x' + 2F'x'y' = 1 \ldots (4),$$

for which the equation of motion of heat takes the form

$$c\rho\frac{du}{dt} = A\frac{d^2u}{dx^2} + B\frac{d^2u}{dy^2} + C\frac{d^2u}{dz^2} \ldots\ldots (5).$$

The system of axes of which the existence has just be established may conveniently be called the *thermic axes* the crystal. Since the left-hand member of equation (4) identical with $Ax^2 + By^2 + Cz^2$, it follows that not only do th principal axes of the surface (4) determine the directions the thermic axes, but the constants A, B, C, are the square reciprocals of the principal semi-axes of that surface.

6. Let us now take the thermic axes for axes of coo dinates, and investigate the general expression for the flu of heat. The general expressions for f_x, f_y, f_z, being line functions of the three differential coefficients of u wit respect to x, y, z, will contain altogether nine arbitrar constants. Substituting the general expressions in (2), an comparing with (5), we find three relations between the con stants, depending upon the choice of coordinate axes. The relations being introduced, the expressions may be put und the following form:

$$\left.\begin{aligned} -f_x &= A\frac{du}{dx} - F_1\frac{du}{dy} + E_1\frac{du}{dz} \\ -f_y &= B\frac{du}{dy} - D_1\frac{du}{dz} + F_1\frac{du}{dx} \\ -f_z &= C\frac{du}{dz} - E_1\frac{du}{dx} + D_1\frac{du}{dy} \end{aligned}\right\} \ldots\ldots\ldots (6).$$

[redacted]

I shall defer till towards the end of the paper a consideration of the reasons which make it probable that D, E, F are necessarily equal to zero. For the present it may be observed that if the medium be symmetrical with respect to two rectangular planes, these constants must vanish. For the planes of symmetry must evidently contain the thermic axes; and on account of the symmetry supposed, if the planes of symmetry be taken for those of xz and yz, f_x must change sign with x, while f_y and f_z remain unchanged; and similarly when the sign of y is changed, f_y must change sign, while f_x and f_z remain unchanged. Referring to (5), we see that this requires the constants D, E, F to vanish, so that

$$f_x = -A\frac{du}{dx},\quad f_y = -B\frac{du}{dy},\quad f_z = -C\frac{du}{dz}\ldots\ldots(7),$$

from whence the flux in any direction may be obtained by means of the formula (1). The formulæ (7) contain the expressions for the flux which result from the theory of M. Duhamel. The constants A, B, C, denote what may be called the *principal conductivities* of the crystal. The reader may suppose for the present that the following investigations are restricted to media which are symmetrical with respect to two rectangular planes.

7. It may be worth while to return to the coordinates x', y', z', which have a general direction, and examine the general expressions for the flux which correspond to the formulæ (7). Putting f_x', f_y', f_z', for the fluxes across planes perpendicular to the axes of x', y', z', we get from (1) and (7)

$$\left.\begin{aligned} -f_x' &= A'\frac{du}{dx'} + F'\frac{du}{dy'} + E'\frac{du}{dz'} \\ -f_y' &= B'\frac{du}{dy'} + D'\frac{du}{dz'} + F'\frac{du}{dx'} \\ -f_z' &= C'\frac{du}{dz'} + E'\frac{du}{dx'} + D'\frac{du}{dy'} \end{aligned}\right\}\ldots\ldots(8),$$

where $$\left.\begin{aligned} A' &= l^2A + m^2B + n^2C \\ D' &= l'l''A + m'm''B + n'n''C \end{aligned}\right\}\ldots\ldots\ldots(9);$$

from whence the expressions for B', E', and C', F', may be written down by symmetry.

8. So long as we are only concerned with the succession of temperatures in an infinite solid, we have no occasion to

consider the flux of heat, and the general equatio
enable us to perform all the requisite calculations
equation the *indestructibility** of heat is recognized,
its *identity*. If we discard the latter idea, it is n
talk of the heat gained, we will suppose, by a give
of the solid, as having come from this quarter ra
from that. If we denote by Δf_x, Δf_y, Δf_z the qu
which the values of f_x, f_y, f_z given by (6) exceed th
by (7), we have

$$\frac{d\Delta f_x}{dx} + \frac{d\Delta f_y}{dy} + \frac{d\Delta f_z}{dz} = 0,$$

which is analogous to the equation of continuity
compressible fluid.

If we suppose all possible systems of values a
succession to the constants D_1, E_1, F_1, the formul
express all possible modes of transfer, consistent
original assumption respecting the forms of f_x,
which the state of temperature of the solid at the c
time t can pass into its state at the end of the ti
Of course, if we suppose heat to be material, we ca
attaching to it the idea of *individuality*. But if w
heat to consist in motion of some sort, which fo
part I regard as by far the more probable hypo
require a definition of *sameness* of heat, supposin
it convenient to treat the subject in this way. I am
going to follow any further the subject which has
broached; but I thought it might be worth whil
out in what manner the additional arbitrary const
in the general expressions for the flux beyond what
in the equation of motion, or more properly *the eq*
successive distribution, corresponded to an attribut
which is necessarily involved in the idea of a flux,
is not necessarily involved in the idea of the succe
tributions of a given quantity of heat.

9. Besides the general equation (5), it is requisi
the equation of condition which has to be satisfi
surface when the solid radiates into a space at a

---

* According to the very important researches of Mr Jou
convertible into heat, from which there can be little doubt tha
heat is convertible into work. As regards the present investigatio
it is perfectly immaterial whether heat be indestructible, o
destroyed, or rather whether it be not convertible into any th
only not converted.

M. Duhamel did not attempt to investigate, nor am I going to attempt the investigation myself. If however the crystal be covered with a thin coating of some other substance, sufficient to stop all direct radiation from the crystal into the surrounding space, h will depend upon the nature of the coating. In either case h will be constant throughout any plane face by which the crystal may be bounded.

10. Let us return to the consideration of the propagation of heat in the interior of the mass. Imagine the coordinates x, y, z, of any point altered in the ratios of $\sqrt{A}$ to $\sqrt{K}$, $\sqrt{B}$ to $\sqrt{K}$, $\sqrt{C}$ to $\sqrt{K}$, where K is constant, and let ξ, η, ζ be the results. The equation (5) becomes

$$c\rho\frac{du}{dt} = K\left(\frac{d^2u}{d\xi^2} + \frac{d^2u}{d\eta^2} + \frac{d^2u}{d\zeta^2}\right) \ldots\ldots\ldots (11).$$

This will be true whatever be the value of K, but it will be convenient to suppose that

$$K^3 = ABC \ldots\ldots\ldots\ldots\ldots (12).$$

Now imagine a second solid formed from the first by altering all lines parallel to x in the ratio of $\sqrt{A}$ to $\sqrt{K}$, all lines parallel to y in the ratio of $\sqrt{B}$ to $\sqrt{K}$, and all lines parallel to z in the ratio of $\sqrt{C}$ to $\sqrt{K}$, nothing being as yet specified regarding the nature of the second solid, except that it is homogeneous. Imagine any number of points, lines, surfaces, or spaces, conceived as belonging to the first solid, and let the points, lines, &c. deduced from them by altering the coordinates in the ratios above-mentioned, and conceived as belonging to the second solid, be said to *correspond* to the others. On account of the particular magnitude of K chosen, it is evident that the volumes of corresponding spaces will be equal. Let the second solid be called the *auxiliary solid*, and the operation of deducing either solid from the other, *derivation;* and suppose the temperatures equal at corresponding points of the two solids.

The equation (11) shews that the successive distributions of temperature in the interior of the auxiliary solid will take place as if this solid were an ordinary medium in which the interior conductivity bears to K the same ratio that the product of the specific heat and density bears to $c\rho$.

The first of equations (7) gives

$$f_x = -\sqrt{(AK)}\,\frac{du}{d\xi}.$$

ow we refer f_x, not to a unit of surface, but to that of a plane perpendicular to the axis of x which is ged by derivation into a unit of surface, we must iply the above expression by $\surd(BC)$ and divide it by K. oting the result by f_ξ, using f_η, f_ζ to denote for y, z, f_ξ denotes for x, and taking account of (12), we get

$$f_\xi = -K\frac{du}{d\xi},\quad f_\eta = -K\frac{du}{d\eta},\quad f_\zeta = -K\frac{du}{d\zeta}\ldots(13).$$

t follows from these equations, that if we suppose not only temperatures at corresponding points of the two solids to lways the same, but also equal quantities of heat to flow equal times and in corresponding directions across corre-nding surfaces, the flow of heat in the auxiliary solid is t would naturally belong to an ordinary medium having interior conductivity K.

The density of the auxiliary solid being disposable, we y take it to be the same as that of the given solid, in ich case corresponding spaces will contain equal quantities matter. It is only necessary further to suppose the iliary solid to be an ordinary medium having a specific t c, and an interior conductivity K, in order that the tion of heat in the interior of the two solids should pre-ly correspond.

1. It remains only to investigate the condition which t be satisfied relatively to the surface of the auxiliary d, in order that the two solids should perfectly correspond every respect. Retaining the notation of Art. 9, let $d\sigma$ the element of surface which corresponds to dS; λ, μ, ν, direction-cosines corresponding to l, m, n. The quantity heat which escapes across dS during the time dt is mately equal to $h(u-v)dSdt$, and this must be equal the quantity which escapes across $d\sigma$. Hence it is suf-ent to attribute to the auxiliary solid an exterior con-tivity k, such that

$$k = \frac{dS}{d\sigma}h \ldots\ldots\ldots\ldots\ldots(14).$$

we have

$$\frac{\lambda}{\sqrt{A.l}} = \frac{\mu}{\sqrt{B.m}} = \frac{\nu}{\sqrt{C.n}} = \frac{1}{\sqrt{(Al^2+Bm^2+Cn^2)}}$$
$$= \sqrt{(A^{-1}\lambda^2 + B^{-1}\mu^2 + C^{-1}\nu^2)},$$

ldS, $\lambda d\sigma$, are the projections of dS, $d\sigma$, on

of yz, and these projections are proportional to $\sqrt{(BC)}$, K, or to $\sqrt{K}$, $\sqrt{A}$, whence we get from (15)

$$\frac{dS}{d\sigma} = \frac{\sqrt{K}}{\sqrt{(Al^2 + Bm^2 + Cn^2)}} = \sqrt{\{K(A^{-1}\lambda^2 + B^{-1}\mu^2 + C^{-1}\nu^2)\}} \ldots (16).$$

The first or second of these expressions will be employed according as we suppose l, m, n, or λ, μ, ν, given.

If the crystal be covered by a thin coating of a given substance, h will be constant, and k will be a function of l, m, n, or of λ, μ, ν, which is determined by (14) and (16). These formulæ shew that in the case supposed k will have the same value for the opposite faces of a plate bounded by parallel surfaces.

By means of the auxiliary solid, we may reduce problems relating to the conduction of heat in crystals to corresponding problems relating to ordinary media; or, conversely, from a set of self-evident or known results relating to ordinary media, we may deduce a set of corresponding results relating to crystals.

12. Let us first regard the crystal as infinite, in which case the auxiliary solid will be infinite likewise.

In an ordinary medium, if heat be introduced at one point according to any law, the isothermal surfaces will be a system of spheres, having the source of heat for their common centre, and the flow of heat at any point will take place in the direction of the radius rector drawn from the source. If the temperature be permanent, and the temperature at an infinite distance vanish, the temperature at any point will vary inversely as the distance from the source.

Hence, in a crystal, if heat be introduced at one point according to any law, the isothermal surfaces will be a system of similar and concentric ellipsoids, having their principal axes in the direction of the thermic axes drawn through the source, and proportional to the square roots of the principal conductivities. The flow of heat at any point will take place in the direction of the radius vector drawn from the source. If the temperature be permanent, and vanish at an infinite distance, the temperature along a given radius vector will vary inversely as the distance from the source.

It will frequently be convenient to refer to an ellipsoid constructed with its principal axes in the direction of the thermic axes, and equal to $2\sqrt{A}$, $2\sqrt{B}$, $2\sqrt{C}$, respectively. I shall call this ellipsoid the *thermic ellipsoid*.

13. In an ordinary medium, whether finite or infinite, in which the temperature varies from point to point, and may be either constant or variable as regards the time, the flow of heat at any point takes place in the direction of the normal to the isothermal surface passing through that point, that is, in a direction parallel to the radius vector drawn from the centre of the *thermic sphere* to the point of contact of a tangent plane drawn parallel to the isothermal surface at the point considered.

Now the tangency of two surfaces is evidently unchanged by derivation. Hence, in a crystal, if we have given the direction of the isothermal surface at any point, we may find that of the flow of heat by the following construction. In a direction parallel to the isothermal surface at the given point draw a tangent plane to the thermic ellipsoid, and join the centre with the point of contact: the flow of heat will take place in a direction parallel to this joining line. In other words, the flow of heat will take place in a direction parallel to the diameter which is conjugate to a plane parallel to the isothermal surface at the given point.

14. Conceive a plate bounded by parallel surfaces to be cut from a crystal, and heat to be applied towards its centre; and suppose the lateral boundaries sufficiently distant to produce no sensible influence on the result, so that we may regard the plate as infinite. In this case the auxiliary solid will likewise be an infinite plate bounded by parallel surfaces. Now if heat be supplied according to any law at one point of such a plate, or at any number of points situated in the same normal, the isothermal surfaces will be surfaces of revolution, having the normal drawn through the source or sources of heat for their axis, and the isothermal curves in which the parallel faces are cut by the isothermal surfaces will be circles, having their centres in the points in which the faces are cut by the normal above mentioned.

Hence, in a crystalline plate, if heat be supplied according to any law at one point, or at any number of points situated in a line parallel to the diameter of the thermic ellipsoid which is conjugate to the planes of the faces, (a line which for brevity I will call *the line of sources,*) any particular isothermal surface will be a surface generated by an ellipse which has its plane parallel to the faces, its centre in the line of sources, and its principal axes parallel and proportional to those of the ellipse in which the thermic ellipsoid is cut by a plane parallel to the faces. In particular, the isothermal

curves on the two faces are ellipses of the kind just men-tioned.* Hence

(1.) If the plate be cut in a direction perpendicular to one of the thermic axes, the line joining the centres of a pair of ellipses which correspond on the two faces to a given temperature (such as that of melting wax) will be normal to the plate. The principal axes of the ellipses will be in the direction of the two remaining thermic axes, and will be proportional to the square roots of the corresponding conductivities.

(2.) If the normal to the plate be not a thermic axis, the line joining the centres of the ellipses will be inclined to the normal, its direction being determined as above explained.

(3.) If the plate be cut in a direction parallel to either of the circular sections of the thermic ellipsoid (the three principal conductivities being supposed unequal,) the isothermal curves on both faces will be circles, but the line joining the centres of the two systems of circles will be inclined to the normal.

If the heat be communicated uniformly along the line of sources, or if there be only a single source situated midway between the faces, or more generally if the sources be alike two and two, those which belong to the same pair being situated at equal distances from the two faces respectively, the isothermal curves belonging to the same temperature in the two

* The problem solved in this article forms a good example of the advantage of considering the auxiliary solid. In M. Duhamel's memoir the plate is regarded as extremely thin, so that the variation of temperature in passing from one point to another of the same normal may be considered insensible—and it is remarked that the second case (in which a normal to the plate is not a thermic axis) is much more difficult than the first; whereas here the plate is not necessarily thin, and both cases follow immediately from what with regard to an uncrystalized body is self-evident. M. Duhamel has shewn that the isothermal curves on the two faces are ellipses, having their principal axes parallel and proportional to those of the ellipses in which the thermic ellipsoid is cut by planes parallel to the faces of the plate: but his demonstration that the line joining the centres of the two systems of ellipses has the direction assigned in the text does not seem altogether satisfactory, because the analysis only applies to the case in which the thickness of the plate is regarded as indefinitely small; whereas the space by which the ellipse corresponding on one face to a given temperature overlaps the ellipse corresponding on the other face to the same temperature is a small quantity of the order of the thickness of the plate, and ought for consistency's sake to be neglected.

The results contained in the remaining part of this paper are not found in the memoirs of M. Duhamel. It may here be remarked, that the results arrived at by the consideration of the auxiliary solid, such for example as that of Art. 17, might have been obtained by referring the crystal to oblique axes parallel to a system of conjugate diameters of the thermic ellipsoid.

tems will be of equal magnitude, provided that the ex-
ior conductivity *h* have the same value for the two faces.
e last condition is satisfied, according to what has been
eady remarked, when the two faces are covered by a thin
ting of the same substance, which regulates the exterior
nductivity; but it is probable that it may be satisfied
nerally even if the faces be left bare, provided that they
ve the same degree of polish.*

The experiments of M. de Senarmont bear directly on the first two cases mentioned above. In the case of crystals which exhibited three different conductivities, it was found that when three plates were cut in the directions of the three principal planes, the ratio of the principal axes of the ellipses formed on one plate, as determined by observation, agreed very closely with the result calculated from the ratios which had previously been determined by observation from the other two plates. An interesting experiment bearing on the second case is described by M. de Senarmont in his second paper (p. 187). A rather thick plate of quartz, inclined to the axis at an angle of 45°, was drilled in a direction perpendicular to its plane, and heated by means of a wire inserted into the hole, after its two faces had been covered with wax. The curves marked out on the two faces approximated to the two bases of an elliptic cylinder, symmetrical with respect to the principal plane, and having its axis inclined towards the axis of the crystal, (which in quartz is the direction of greatest conductivity,) so as to cross the wire, which was perpendicular to the plate. The curves however were not elliptical but egg-shaped, having their axes of symmetry situated in the principal plane, the end at which the curvature was least being that which was nearest to the wire, so that the blunt ends on the two faces were turned in opposite directions, the curves being in other respects alike. It will be seen at once that the symmetry of the curves with respect to the principal plane, the obliquity of the line joining their centres, and their equality combined with *dissymmetry*, follow immediately from theory. We learn too from theory, that in order to procure ellipses it would be necessary to drill the hole in the direction of that diameter of the thermic spheroid which is conjugate to the plane of the plate.

* This result follows readily from the theory of molecular radiation, according to the suppositions usually made.

15. Conceive a bar having a section with an arbitrary c
tour to be formed from an uncrystallized substance; let h
be applied in any manner at one or more places, and supp
the heat to escape again from the surface by radiation. C
sider only those portions of the bar which are situated a
sufficient distance from the source or sources of heat to re
der insensible any irregularity arising from the mode
which the heat is communicated. If the bar be sufficien
slender, we may regard the temperature throughout a secti
drawn in a direction perpendicular to its length as appro
mately constant, without assuming thereby that the isoth
mal surfaces are perpendicular to the length. Let x
measured in the direction of the length, and consider t
slice of the bar contained between the planes whose absci
are x and $x + dx$. Let u be the temperature of the bar
the distance of the first plane, p the perimeter, and Q t
area of the section, h the exterior conductivity, or rath
the mean of the exterior conductivities in case they shou
vary from one generating line to another; let c, ρ, K,
the same as before, and put $Q = ap$, so that $4a$ is the
of a square whose area divided by its perimeter is equ
to Qp^{-1}.

The excess of the quantity of heat which enters duri
the time dt by the first of the plane ends of the slice o
that which escapes by the second, is ultimately equal
$KQ\dfrac{d^2u}{dx^2}\,dx\,dt$. The quantity which the slice loses by radi
tion is ultimately equal to $hpu\,dx\,dt$, if we take the tempera
ture of the surrounding space, which is supposed to b
constant, for the origin of temperatures. But the gain of he
by the slice is also equal to $c\rho Q\dfrac{du}{dt}\,dx\,dt$. Hence we have

$$c\rho\frac{du}{dt} = K\frac{d^2u}{dx^2} - \frac{h}{a}u \;\ldots\ldots\ldots\ldots (17).$$

If we suppose the heat to be continually supplied, an
the temperature to have become stationary, we get from th
equation

$$u = Me^{\sqrt{\frac{h}{Ka}}\,x} + Ne^{-\sqrt{\frac{h}{Ka}}\,x} \;\ldots\ldots\ldots\ldots (18),$$

where M and N are arbitrary constants. If now a be sma
it follows from (18) that the lateral flux at the surface, whi
is equal to hu, as compared with the longitudinal flux, which
equal to $-K\dfrac{du}{dx}$, is a small quantity of the order $\sqrt{\dfrac{ha}{K}}$.

We may deduce the same consequence for the case of variable temperatures from the equation (17), without troubling ourselves with its solution. Conceive any number of bars of different sizes to be heated in a similar manner, and for greater generality, suppose the values of c, ρ, K, and h, as well as a, to be different for the different bars. Let x, x', x''... be corresponding lengths, and t, t', t'' corresponding times, relating to the several bars. The equation (17) shews that the temperatures at corresponding points and at the end of corresponding times may be the same in all the bars, provided

$$x \propto \sqrt{\frac{Ka}{h}}, \quad t \propto \frac{c\rho a}{h} \ldots\ldots\ldots\ldots (19).$$

These variations contain the definition of corresponding points and corresponding times. In order that the temperatures in the different bars should actually be the same at corresponding points and at the end of corresponding times, it is sufficient that the initial circumstances, or more generally the mode of communicating the heat, should be such as to give equal temperatures at the points and times defined by the variations (19), which in this point of view may be regarded as containing the definition of *similarity of heating*. Now, in comparing the longitudinal flux at corresponding points, if we take du the same, dx must vary as determined by (19), and therefore the flux will vary as $K\sqrt{(K^{-1}a^{-1}h)}$, or $\sqrt{(Ka^{-1}h)}$, and the ratio of the lateral flux at the surface to the longitudinal flux will vary as $\sqrt{(haK^{-1})}$; so that if we suppose a to decrease indefinitely, h and K being given, the ratio in question will be a small quantity of the order $\sqrt{\frac{ha}{K}}$ as before, and will ultimately vanish.

The second of the variations (19) shews that if we suppose the heat to be supplied to one bar in an irregular manner as regards the time, the fluctuations in the mode of communicating the heat must become more and more rapid as a decreases, in order that the similarity of temperatures may be kept up. If the fluctuations retain their original period, the motion of heat will tend indefinitely to become what we may regard at any instant as steady, and thus we fall back on the case first considered. We may conclude therefore generally, that if the bar be sufficiently slender, the direction of the maximum flux, even close to the surface, will sensibly coincide with that of the length of the bar; so that the isothermal surfaces, which are necessarily perpendicular to the direction of the flow of heat, will be planes perpendicular to the axis of the bar.

By supposing the bar to be the auxiliary solid belonging to a crystalline bar, we arrive at the following theorem. If a slender crystalline bar be heated at one end, and if we confine our attention to points of the bar situated at a sufficient distance from the source of heat to render insensible any irregularities attending the mode of communicating the heat, the isothermal surfaces will be sensibly planes parallel to the diametral plane of the thermic ellipsoid which is conjugate to the system of chords drawn parallel to the length of the bar. These planes will necessarily have an oblique position unless the direction of the length of the bar be a thermic axis of the crystal.

The same result might have been obtained without employing the auxiliary solid, by first shewing that when the bar is sufficiently slender the direction of the flow of heat sensibly coincides with that of the length of the bar. We should thus be led to a problem exactly the converse of that treated in Art. 13, namely, Given the direction of the flow of heat, to find that of the isothermal surface.

16. It may be shewn in a similar way, that if a thin plate be formed of an uncrystallized substance, and be heated at one or more places, or over a finite portion, if we consider only those parts of the plate which are situated at a sufficient distance from the sources of heat to render insensible any irregularities attending the mode in which the heat is communicated, the flow of heat will take place in a direction sensibly parallel to the plate, and therefore the isothermal surfaces will be cylindrical surfaces whose generating lines are perpendicular to the plate. It is here supposed that the lateral boundaries of the plate are situated at a sufficient distance to render their effect insensible.

Hence, in a thin crystalline plate heated in a similar manner, the isothermal surfaces, under similar restrictions, will be cylindrical surfaces whose generating lines are parallel to the diameter of the thermic ellipsoid which is conjugate to the plane of the plate.

17. The state of temperature, under given circumstances, of a rectangular parallelepiped formed of an uncrystallized substance, may be determined by certain known formulæ which it is not necessary here to describe.

Hence, the state of temperature of a parallelepiped cut from a crystal in such a manner that its edges are parallel to a system of conjugate [illegible] the thermic ellipsoid

be determined by the same formulæ. This parallele-d will of course be oblique-angled, except in the par-r case in which its edges are parallel to the thermic . It may be remarked that a parallelepiped for which tate of temperature shall be determinable by the formulæ estion may be cut from a crystal in a manner quite as ral as from an uncrystallized substance. In both cases direction of the first edge is arbitrary, and, when it is d on, the plane of the other two edges is determined in ction. The direction of the second edge having been en arbitrarily in the plane above mentioned, that of the rd edge is determined.

It does not seem worth while to notice the crystalline ures derived from spheres, &c., on account of the me-nical difficulty attending their execution. Besides, the rivation presents no theoretical difficulty.

Further consideration of the general expressions for the flux.

18. It has been already remarked, that if the crystal possess two planes of symmetry, the nine arbitrary constants which appear in the expressions for the flux in three rect-ngular directions, from which the flux in any other direc-ion may be derived, reduce themselves to six, and the xpressions for the flux take the form (8). I proceed now o consider what grounds we have for believing that these xpressions, with only six arbitrary constants, are the most eneral possible.

In the first place, it may be observed that this result ollows readily from the theory of molecular radiation. In his theory the extent of molecular radiation is supposed to e very great compared with the mean interval between the iolecules of a body, so that the body may be treated as ontinuous. If E, E' be any two elements of the body, tuated sufficiently near one another to render their mutual ifluence sensible, it is supposed that, during the time dt, a uantity of heat proportional to Edt radiates in all directions om E, whereof E' absorbs a portion proportional to E'. On e other hand, E' emits and E absorbs a quantity also pro-ortional, so far as regards only the magnitudes of E, E', and , to $EE'dt$. The exchange of heat between E and E' may erefore be expressed by $qEE'dt$. The quantity q is sup-osed to be proportional, so far as regards its dependence on e temperatures, to the small difference between the tempera-res of E and E'. It will also depend upon the nature of e body, upon the distance EE', and in the case of a crys-

talline body upon the direction of the line EE'; but we need not now consider its dependence upon these quantities. If the length $EE' = s$, and if we suppose the extent of internal radiation to be very small, we may express the difference between the temperatures of E and E' by $\frac{du}{ds}.s$. It follows then from the theory we are considering, that the total flux of heat arises from the exchange of heat between all possible pairs of elements, such as E, E'; the exchange between any pair E, E' being proportional to the rate of variation of temperature in the direction EE', and accordingly independent of the variation of temperature in other directions.

Now suppose the body referred to rectangular axes, and let P be the mathematical point whose coordinates are x, y, z. Conceive the body divided into an infinite number of infinitely small equal elements. Let E be the element which contains P, E any element in the neighbourhood of P, and consider the partial flux in the neighbourhood of P which arises from the exchange of heat between all pairs of elements which have the same relative position as E and E'. Through P draw an elementary plane S, which it will be convenient to consider as infinitely large compared with the dimensions of the elements such as E, and conceive S to assume in succession all possible directions by turning round P. The partial flux across S will vary as the number of points in which the lines, all equal and parallel to EE', which connect the pairs of elements, cut the plane S, or as the cosine of the angle between the normal to S and the direction EE'. Let $EE' = s$; let l', m', n', be the cosines of the angles which this line makes with the axes of x, y, z, and suppose S to be perpendicular to each of these axes in succession. We shall thus have for the partial fluxes f_x, f_y, f_z quantities proportional to $l'\frac{du}{ds}$, $m'\frac{du}{ds}$, $n'\frac{du}{ds}$, or to

$$l'^2\frac{du}{dx} + l'm'\frac{du}{dy} + l'n'\frac{du}{dz}, \quad l'm'\frac{du}{dx} + m'^2\frac{du}{dy} + m'n'\frac{du}{dz},$$

$$l'n'\frac{du}{dx} + m'n'\frac{du}{dy} + n'^2\frac{du}{dz}.$$

Hence, the coefficient of $\frac{du}{dy}$ in the expression for the partial flux f_x is equal to the coefficient of $\frac{du}{dx}$ in the expression for the partial flux f_y, and the same applies to y, z, and

to z, x. This being true for each partial flux, will be true likewise for the total flux, and therefore the general expressions for the flux in three rectangular directions, with nine arbitrary constants, will be reduced to the form (8), or the general expressions (6), referred to the thermic axes, to the form (7).

19. Let us further examine some of the consequences which would follow from the supposition that the expressions for the flux referred to the thermic axes have the general form (6). Conceive a crystalline mass, regarded as infinite, to be heated at one point according to any law, and let the source of heat be taken for origin. We have seen already that the succession of temperatures takes place in an infinite solid in exactly the same manner whether the expressions for the flux have the general form (6), or the more restricted form (7), and consequently, in the case supposed above, the temperature at a given time is some function of

$$A^{-1}x^2 + B^{-1}y^2 + C^{-1}z^2.$$

If x, y, z, be the coordinates of any point in a *line of motion*, or line traced at a given instant from point to point in the direction of the flow of heat, dx, dy, dz, will be proportional to f_x, f_y, f_z, which are given by (6), and in the present case $\frac{du}{dx}$, $\frac{du}{dy}$, $\frac{du}{dz}$, are proportional to $A^{-1}x$, $B^{-1}y$, $C^{-1}z$. Hence the differential equations of a line of motion are

$$\frac{dx}{x - F_1B^{-1}y + E_1C^{-1}z} = \frac{dy}{y - D_1C^{-1}z + F_1A^{-1}x}$$

$$= \frac{dz}{z - E_1A^{-1}x + D_1B^{-1}y} \ldots (20).$$

Taking ξ, η, ζ, to denote the same quantities as in Art. 10, and putting for shortness

$$D_1(BC)^{-\frac{1}{2}} = \omega', \quad E_1(CA)^{-\frac{1}{2}} = \omega'', \quad F_1(AB)^{-\frac{1}{2}} = \omega''' \ldots (21),$$

we get

$$\frac{d\xi}{\xi - \omega'''\eta + \omega''\zeta} = \frac{d\eta}{\eta - \omega'\zeta + \omega'''\xi} = \frac{d\zeta}{\zeta - \omega''\xi + \omega'\eta} \ldots (22).$$

Conceive an elastic solid to be fixed at the origin, and to expand alike in all directions and at all points with a velocity of expansion unity, so that a particle which at the end of the time t is situated at a distance r from the origin,

at the end of the time $t + dt$ is situated at a distance $r(1+$
Conceive this solid at the same time to turn, with an ang
velocity ω equal to $\sqrt{(\omega'^2 + \omega''^2 + \omega'''^2)}$, about an axis wh
direction-cosines are $\omega'\omega^{-1}$, $\omega''\omega^{-1}$, $\omega'''\omega^{-1}$. The direction
motion of any particle will represent the direction of t
flow of heat in what we may still call the *auxiliary* so
from whence the direction of the flow of heat in the giv
solid will be obtained by merely conceiving the whole fig
differently magnified or diminished in three rectangular d
rections.

This *rotatory* sort of motion of heat, produced by the mer
diffusion from the source outwards, certainly seems very
strange, and leads us to think, independently of the theory
of molecular radiation, that the expressions for the flux with
six arbitrary constants only, namely the expressions (8), or
the equivalent expressions (7), are the most general possible.

20. Let the auxiliary solid be referred to the rectangular axes of ξ', η', ζ', of which the last coincides with the axis to which ω refers. It may be seen immediately, without analytical transformation, that the differential equations to the lines of motion will be

$$\frac{d\xi'}{\xi' - \omega\eta'} = \frac{d\eta'}{\eta' + \omega\xi'} = \frac{d\zeta'}{\zeta'} \quad \ldots\ldots\ldots (23).$$

Taking polar coordinates r, θ in the plane of ξ', η', we have

$$(\sin\theta + \omega\cos\theta)(\cos\theta\, dr - \sin\theta\, r\, d\theta)$$
$$= (\cos\theta - \omega\sin\theta)(\sin\theta\, dr + \cos\theta\, r\, d\theta);$$

whence $$\omega dr = r d\theta \quad \ldots\ldots\ldots\ldots\ldots (24),$$

the differential equation of a system of equiangular spirals in which the angle between the tangent and radius vector is equal to $\tan^{-1}\omega$. We have also from (23)

$$\frac{d\zeta'}{\zeta'} = \frac{(\xi' - \omega\eta')\, d\xi' + (\eta' + \omega\xi')\, d\eta'}{(\xi' - \omega\eta')^2 + (\eta' + \omega\xi')^2}$$
$$= (1 + \omega^2)^{-1} \left\{ \frac{\xi' d\xi' + \eta' d\eta'}{\xi'^2 + \eta'^2} + \omega\, \frac{\xi' d\eta' - \eta' d\xi'}{\xi'^2 + \eta'^2} \right\}$$

whence $$(1 + \omega^2)\log\zeta' + \text{const} = \log r + \omega\theta$$

$= \log r + \omega^2 \log r$ from (24). We have therefore

$$\zeta' = mr \quad \ldots\ldots\ldots\ldots\ldots\ldots (25),$$

where m is an arbitrary

verse, unless the direction of transmission coincides with an axis, have still certain definite positions.

3. It is only in air and water, however, that such experiments are possible. For other substances, the best experiments which it is practicable for us to make are those upon the transmission of nearly-longitudinal vibrations along prismatic or cylindrical bodies. Were we able to ensure that the vibrations of those prisms and cylinders should be exactly longitudinal, we might compute from their velocity of transmission, as from that of such vibrations in an unlimited mass, the true longitudinal elasticity. This we can do for gaseous substances, as M. Wertheim has proved (*Ann. de Chimie et de Phys. Ser. III.* tom. XXIII.) by making the organ-pipes in which they vibrate of proper construction.

In liquid and solid columns, on the other hand, it is impossible to prevent a certain amount of lateral vibration of the particles, the effect of which is to diminish the velocity of transmission in a ratio depending on circumstances in the molecular condition of the superficial particles, which are yet almost entirely unknown.

4. It has indeed been sometimes supposed, that the coefficient of elasticity, as calculated from the vibrations of a solid rod, is that called the *weight of the modulus of elasticity;* that is to say, the reciprocal of the fraction by which the length of a rod is increased by a tension applied to its ends of unity of weight upon unity of area; that coefficient being less than the true coefficient of longitudinal elasticity, because the lateral collapsing of the particles enables them to yield more in a longitudinal direction to a given force than if their displacements were wholly longitudinal.

This conjecture, however, is inconsistent with the mechanics of vibratory movement; and accordingly, experiment has shewn that the elasticity corresponding to the velocity of sound in a rod agrees neither with the modulus of elasticity, nor with the true longitudinal elasticity; although it is in some cases nearly equal to the former of those quantities, and in others to the latter.

5. In liquids, it has been shewn by the experiments of M. Wertheim (*Ann. de Ch. et de Phys. Ser. III.* tom. XXIII.) that the velocity of sound in a mass contained in a trough, and set in motion through an organ-pipe, bears to that in an unlimited mass the ratio of $\sqrt{2}$ to $\sqrt{3}$. This has led him to form the conjecture, that liquids possess a momentary rigidity for very small molecular displacements, as great in comparison

with their other elastic forces as that of solids. This conjecture, paradoxical as it may seen, would indeed be necessary to account for the facts if the supposition I have already mentioned were true, that the velocity of sound in a rod depends upon the modulus of elasticity. I shall shew, however, in the sequel, that if we suppose that at the free surface of every mass of liquid, an atmosphere of its own vapour is retained by molecular attraction under certain conditions of equilibrium, the ratio $\sqrt{2} : \sqrt{3}$ between the velocities of sound in a prism and an unlimited mass is a consequence of the equations of motion in all cases in which the liquid has any rigidity whatsoever, even although so small as to be insensible by any means of observation; so that the supposition of a rigidity for small displacements equal to that of solids becomes unnecessary.

6. With respect to solids, all that theory is yet adequate to shew us is, that the velocity of sound along a rod must be less than in an unlimited mass—a conclusion in accordance with experiment. The precise ratio depends on properties of the superficial particles yet unknown.

General Equations of vibratory movement in homogeneous bodies.

7. Having now stated generally the objects of this paper, I shall proceed in the first place to the mathematical investigation of the integrals of the general differential equations of vibratory movement in homogeneous bodies; because, although those equations have already been integrated by many mathematicians, it will be necessary in this paper to introduce functions into the integrals which have hitherto been almost totally neglected in such researches; having been applied only to the theory of waves rolling by the influence of gravity, to that of total reflection, by Mr. Green (*Camb. Trans.* vol. VI.), and by Professor Stokes, to represent the gradual extinction of sound by its conversion into heat.

8. Let g represent the accelerating force of gravity:

D the weight of unity of volume of an homogeneous substance, having orthogonal axes of elasticity whose directions are the same throughout its extent:

A_1, A_2, A_3, the coefficients of longitudinal elasticity for the axes of x, y, z, respectively:

B_1, B_2, B_3, the coefficients of lateral elasticity, and

C_1, C_2, C_3, those of rigidity for the planes of yz, zx, xy respectively:

ξ, η, ζ, the displace[illegible]allel to x, y, z respectively.

Then it is well known that the differential equations of all vibratory movements are the following, when small quantities of the second order are neglected:

$$\left.\begin{aligned}
0 &= \left(-\frac{D}{g}\cdot\frac{d^2}{dt^2} + A_1\frac{d^2}{dx^2} + C_3\frac{d^2}{dy^2} + C_2\frac{d^2}{dz^2}\right)\xi \\
&\quad + (B_3 + C_3)\frac{d^2\eta}{dx\,dy} + (B_2 + C_2)\frac{d^2\zeta}{dz\,dx} \\
0 &= \left(-\frac{D}{g}\cdot\frac{d^2}{dt^2} + C_3\frac{d^2}{dx^2} + A_2\frac{d^2}{dy^2} + C_1\frac{d^2}{dz^2}\right)\eta \\
&\quad + (B_1 + C_1)\frac{d^2\zeta}{dy\,dz} + (B_3 + C_3)\frac{d^2\xi}{dx\,dy} \\
0 &= \left(-\frac{D}{g}\cdot\frac{d^2}{dt^2} + C_2\frac{d^2}{dx^2} + C_1\frac{d^2}{dy^2} + A_3\frac{d^2}{dz^2}\right)\zeta \\
&\quad + (B_2 + C_2)\frac{d^2\xi}{dz\,dx} + (B_1 + C_1)\frac{d^2\eta}{dy\,dz}
\end{aligned}\right\}\ldots(1),$$

of which the integrals are

$$\left.\begin{aligned}
\xi &= \Sigma\left\{L_1\phi(\sqrt{\epsilon}.t + \alpha x + \beta y + \gamma z + \kappa)\right\} \\
\eta &= \Sigma\left\{L_2\phi(\sqrt{\epsilon}.t + \alpha x + \beta y + \gamma z + \kappa)\right\} \\
\zeta &= \Sigma\left\{L_3\phi(\sqrt{\epsilon}.t + \alpha x + \beta y + \gamma z + \kappa)\right\}
\end{aligned}\right\}\ldots.(2).$$

The form of the function ϕ being arbitrary, subject to a restriction to be afterwards referred to, and Σ extending to any number of terms, the coefficients of which fulfil the following conditions. Let

$$\left.\begin{aligned}
\varpi_1 &= A_1\alpha^2 + C_3\beta^2 + C_2\gamma^2 \\
\varpi_2 &= C_3\alpha^2 + A_2\beta^2 + C_1\gamma^2 \\
\varpi_3 &= C_2\alpha^2 + C_1\beta^2 + A_3\gamma^2 \\
\rho_1 &= (B_1 + C_1)\beta\gamma \\
\rho_2 &= (B_2 + C_2)\gamma\alpha \\
\rho_3 &= (B_3 + C_3)\alpha\beta \\
\frac{\epsilon D}{g} &= E
\end{aligned}\right\}\ldots\ldots\ldots.(a).$$

Then the following equations must be satisfied by the coefficients of each set of terms in equation (2):

$$\left.\begin{aligned}
0 &= L_1(\varpi_1 - E) + L_2\rho_3 + L_3\rho_2 \\
0 &= L_1\rho_3 + L_2(\varpi_2 - E) + L_3\rho_1 \\
0 &= L_1\rho_2 + L_2\rho_1 + L_3(\varpi_3 - E)
\end{aligned}\right\}\ldots\ldots.(3).$$

[redacted]

By elimination we transform those equations as follow. Let

$$\left.\begin{aligned} G &= \varpi_1 + \varpi_2 + \varpi_3, \\ H &= \varpi_2\varpi_3 + \varpi_3\varpi_1 + \varpi_1\varpi_2 - \rho_1^2 - \rho_2^2 - \rho_3^2, \\ K &= \varpi_1\varpi_2\varpi_3 - 2\rho_1\rho_2\rho_3 - \varpi_1\rho_1^2 - \varpi_2\rho_2^2 - \varpi_3\rho_3^2, \end{aligned}\right\}\ldots(b).$$

Then for each set of values of α, β, γ, E has three values which are the roots of the cubic equation

$$0 = E^3 - GE^2 + HE - K \ldots\ldots\ldots\ldots(4);$$

so that s has six values, three positive and three negative, of equal arithmetical amount.

The absolute values of L_1, L_2, L_3, are arbitrary, but their mutual ratios are fixed by the following equations,

$$\left.\begin{aligned} L_1\{(\varpi_1 - E)\rho_1 - \rho_2\rho_3\} &= L_2\{(\varpi_2 - E)\rho_2 - \rho_3\rho_1\} \\ &= L_3\{(\varpi_3 - E)\rho_3 - \rho_1\rho_2\} \end{aligned}\right\}\ldots(5);$$

consequently they have in general three sets of ratios for each set of values of α, β, γ, corresponding to the three values of E.

6. The condition that the motions of the particles of the body must be *small oscillations* restricts the variations of the displacements ξ, η, ζ within certain limits. Now as the time t increases *ad infinitum*, this can be fulfilled only when each of those quantities is either a periodical circular function of t, or a function developable into a sum or definite integral of such functions. We may therefore make each of the functions ϕ a trigonometrical function of t. This being the case, those functions must be either trigonometrical or exponential with respect to x, y, and z, or compounded of both, being trigonometrical so far as α, β, γ, are real, and exponential so far as they are imaginary.

If we suppose each of these coefficients to consist of a real and an imaginary part, then each of their functions which enter into the equations of condition, will also consist of a real and an imaginary part. Each of the equations of condition thus becomes divided into two, which must be separately satisfied.

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$$\alpha = \frac{1}{\lambda}(\mp a - a'\sqrt{-1}),$$

$$\beta = \frac{1}{\lambda}(\mp b - b'\sqrt{-1}),$$

$$\gamma = \frac{1}{\lambda}(\mp c - c'\sqrt{-1}):$$

t $L_1 = l \mp l'\sqrt{-1}$, $L_2 = m \mp m'\sqrt{-1}$, $L_3 = n \mp n'\sqrt{-1}$;

t the displacements become

$$\left.\begin{array}{l} = \Sigma\left\{(l \mp l'\sqrt{-1})\, e^{\frac{2\pi}{\lambda}\{a'x+b'y+c'z\pm\sqrt{-1}(\sqrt{e}.t-ax-by-cz)\}}\right\} \\ = \Sigma\{\text{terms in } m, m'\}, \quad \zeta = \Sigma\{\text{terms in } n, n'\} \end{array}\right\}\dots(7).$$

:t the quantities in the equations of condition be thus sented:

$$\varpi_1 = p_1 \pm p'_1\sqrt{-1}, \text{ \&c.}; \quad \rho_1 = q_1 \pm q'_1\sqrt{-1}, \text{ \&c.};$$

$$G = \mathfrak{g} \pm \mathfrak{g}'\sqrt{-1}; \quad H = \mathfrak{h} \pm \mathfrak{h}'\sqrt{-1}; \quad K = \mathfrak{k} \pm \mathfrak{k}'\sqrt{-1}.$$

:quations of notation now become

$$\left.\begin{array}{l}
A_1(a^2 - a'^2) + C_3(b^2 - b'^2) + C_2(c^2 - c'^2) \\
C_3(a^2 - a'^2) + A_2(b^2 - b'^2) + C_1(c^2 - c'^2) \\
C_2(a^2 - a'^2) + C_1(b^2 - b'^2) + A_3(c^2 - c'^2) \\
(B_1 + C_1)(bc - b'c') \\
(B_2 + C_2)(ca - c'a') \\
(B_3 + C_3)(ab - a'b') \\
2(A_1aa' + C_3bb' + C_2cc') \\
2(C_3aa' + A_2bb' + C_1cc') \\
2(C_2aa' + C_1bb' + A_3cc') \\
(B_1 + C_1)(bc' + b'c) \\
(B_2 + C_2)(ca' + c'a) \\
(B_3 + C_3)(ab' + a'b) \\
E \text{ as before, or } \varepsilon = \dfrac{Eg}{D} \\
p_1 + p_2 + p_3, \quad \mathfrak{g}' = p_1' + p_2' + p_3' \\
p_2p_3 + p_3p_1 + p_1p_2 - q_1^2 - q_2^2 - q_3^2 \\
- p_2'p_3' - p_3'p_1' - p_1'p_2' + q_1'^2 + q_2'^2 + q_3'^2 \\
p_2p_3' + p_2'p_3 + p_3p_1' + p_3'p_1 + p_1p_2' + p_1'p_2 \\
- 2q_1q_1' - 2q_2q_2' - 2q_3q_3'
\end{array}\right\}\dots(\text{c}).$$

By elimination we transform those equations as follows. Let

$$\left.\begin{aligned} G &= \varpi_1 + \varpi_2 + \varpi_3, \\ H &= \varpi_2\varpi_3 + \varpi_3\varpi_1 + \varpi_1\varpi_2 - \rho_1^2 - \rho_2^2 - \rho_3^2, \\ K &= \varpi_1\varpi_2\varpi_3 + 2\rho_1\rho_2\rho_3 - \varpi_1\rho_1^2 - \varpi_2\rho_2^2 - \varpi_3\rho_3^2, \end{aligned}\right\}\ldots.(b).$$

Then for each set of values of α, β, γ, E has three values which are the roots of the cubic equation

$$0 = E^3 - GE^2 + HE - K\ldots\ldots\ldots.(4);$$

so that $\sqrt{\epsilon}$ has six values, three positive and three negative, of equal arithmetical amount.

The absolute values of L_1, L_2, L_3, are arbitrary, but their mutual ratios are fixed by the following equations,

$$\left.\begin{aligned} L_1\{(\varpi_1 - E)\rho_1 - \rho_2\rho_3\} &= L_2\{(\varpi_2 - E)\rho_2 - \rho_3\rho_1\} \\ &= L_3\{(\varpi_3 - E)\rho_3 - \rho_1\rho_2\} \end{aligned}\right\}\ldots(5);$$

consequently they have in general three sets of ratios for each set of values of α, β, γ, corresponding to the three values of E.

9. The condition that the motions of the particles of the body must be *small oscillations* restricts the variations of the displacements ξ, η, ζ within certain limits. Now as the time t increases *ad infinitum*, this can be fulfilled only when each of those quantities is either a periodical circular function of t, or a function developable into a sum or definite integral of such functions. We may therefore make each of the functions ϕ a trigonometrical function of t. This being the case, those functions must be either trigonometrical or exponential with respect to x, y, and z, or compounded of both, being trigonometrical so far as α, β, γ, are real, and exponential so far as they are imaginary.

If we suppose each of these coefficients to consist of a real and an imaginary part, then each of their functions which enters into the equations of condition, will also consist of a real and an imaginary part. Each of the equations of condition thus becomes divided into two, which must be separately satisfied.

Thus we arrive at the following results:

For the symbol $\phi\{\ \}$, put $e^{2\pi\sqrt{-1}\{\}}$; so as to make ξ, &c. trigonometrical with respect to t. Let λ be a line of such a length that

$$a^2 + b^2 + c^2 = 1,$$

and let
$$\alpha = \frac{1}{\lambda}(\mp a - a'\sqrt{-1}),$$
$$\beta = \frac{1}{\lambda}(\mp b - b'\sqrt{-1}),$$
$$\gamma = \frac{1}{\lambda}(\mp c - c'\sqrt{-1}):$$

also let $L_1 = l \mp l'\sqrt{-1}$, $L_2 = m \mp m'\sqrt{-1}$, $L_3 = n \mp n'\sqrt{-1}$;

so that the displacements become

$$\left.\begin{array}{l} \xi = \Sigma\{((l \mp l'\sqrt{-1})\, e^{\frac{2\pi}{\lambda}\{a'x+b'y+c'z+\sqrt{-1}(\sqrt{\epsilon}\,.\,t-ax-by-cz)\}} \\ \eta = \Sigma\{\text{terms in } m, m'\}, \quad \zeta = \Sigma\{\text{terms in } n, n'\} \end{array}\right\}\ldots(7).$$

Let the quantities in the equations of condition be thus represented:

$$\varpi_1 = p_1 \pm p_1'\sqrt{-1}, \text{ \&c.}; \quad \rho_1 = q_1 \pm q_1'\sqrt{-1}, \text{ \&c.};$$
$$G = \mathfrak{g} \pm \mathfrak{g}'\sqrt{-1}; \quad H = \mathfrak{h} \pm \mathfrak{h}'\sqrt{-1}; \quad K = \mathfrak{k} \pm \mathfrak{k}'\sqrt{-1}.$$

The equations of notation now become

$$\left.\begin{array}{l}
p_1 = A_1(a^2 - a'^2) + C_3(b^2 - b'^2) + C_2(c^2 - c'^2) \\
p_2 = C_3(a^2 - a'^2) + A_2(b^2 - b'^2) + C_1(c^2 - c'^2) \\
p_3 = C_2(a^2 - a'^2) + C_1(b^2 - b'^2) + A_3(c^2 - c'^2) \\
q_1 = (B_1 + C_1)(bc - b'c') \\
q_2 = (B_2 + C_2)(ca - c'a') \\
q_3 = (B_3 + C_3)(ab - a'b') \\
p_1' = 2(A_1aa' + C_3bb' + C_2cc') \\
p_2' = 2(C_3aa' + A_2bb' + C_1cc') \\
p_3' = 2(C_2aa' + C_1bb' + A_3cc') \\
q_1' = (B_1 + C_1)(bc' + b'c) \\
q_2' = (B_2 + C_2)(ca' + c'a) \\
q_3' = (B_3 + C_3)(ab' + a'b) \\
\dfrac{\epsilon D}{g} = E \text{ as before, or } \epsilon = \dfrac{Eg}{D} \\
\mathfrak{g} = p_1 + p_2 + p_3, \quad \mathfrak{g}' = p_1' + p_2' + p_3' \\
\mathfrak{h} = p_2p_3 + p_3p_1 + p_1p_2 - q_1^2 - q_2^2 - q_3^2 \\
\qquad - p_2'p_3' - p_3'p_1' - p_1'p_2' + q_1'^2 + q_2'^2 + q_3'^2 \\
\mathfrak{h}' = p_2p_3' + p_2'p_3 + p_3p_1' + p_3'p_1 + p_1p_2' + p_1'p_2 \\
\qquad - 2q_1q_1' - 2q_2q_2' - 2q_3q_3'
\end{array}\right\}\ldots(c).$$

$$\left.\begin{aligned}
\mathfrak{k} &= p_1p_2p_3 + 2q_1q_2q_3 - p_1q_1^2 - p_2q_2^2 - p_3q_3^2 \\
&\quad - p_1p_2'p_3' - p_1'p_2p_3' - p_1'p_2'p_3 \\
&\quad - 2(q_1q_2'q_3' + q_1'q_2q_3' + q_1'q_2'q_3) \\
&\quad + p_1q_1'^2 + p_2q_2'^2 + p_3q_3'^2 + 2(p_1'q_1q_1' + p_2'q_2q_2' + p_3'q_3q_3') \\
\mathfrak{k}' &= p_1'p_2'p_3' + 2q_1'q_2'q_3' - p_1'q_1'^2 - p_2'q_2'^2 - p_1'q_3'^2 \\
&\quad - p_1'p_2p_3 - p_1p_2'p_3 - p_1p_2p_3' \\
&\quad - 2(q_1'q_2q_3 + q_1q_2'q_3 + q_1q_2q_3') \\
&\quad + p_1'q_1^2 + p_2'q_2^2 + p_3'q_3^2 + 2(p_1q_1q_1' + p_2q_2q_2' + p_3q_3q_3').
\end{aligned}\right\} \ldots\text{(c)}.$$

Also let

$$\begin{aligned}
\mathfrak{r}_1 &= (p_1 - E)q_1 - p_1'q_1' - q_2q_3 + q_2'q_3' \\
\mathfrak{r}_2 &= (p_2 - E)q_2 - p_2'q_2' - q_3q_1 + q_3'q_1' \\
\mathfrak{r}_3 &= (p_3 - E)q_3 - p_3'q_3' - q_1q_2 + q_1'q_2' \\
\mathfrak{r}_1' &= (p_1 - E)q_1' + p_1'q_1 - q_2q_3' - q_2'q_3 \\
\mathfrak{r}_2' &= (p_2 - E)q_2' + p_2'q_2 - q_3q_1' - q_3'q_1 \\
\mathfrak{r}_3' &= (p_3 - E)q_3' + p_3'q_3 - q_1q_2' - q_1'q_2
\end{aligned}$$

Then the equations of condition relative to the coefficients become the following:

$$0 = E^3 - \mathfrak{g}E^2 + \mathfrak{h}E - \mathfrak{k} \ldots\ldots\ldots\ldots (8),$$

$$0 = \mathfrak{g}'E^2 - \mathfrak{h}'E + \mathfrak{k}' \ldots\ldots\ldots\ldots\ldots (9),$$

$$\left.\begin{aligned}
l\mathfrak{r}_1 + l'\mathfrak{r}_1' &= m\mathfrak{r}_2 + m'\mathfrak{r}_2' = n\mathfrak{r}_3 + n'\mathfrak{r}_3' \\
l\mathfrak{r}_1' - l'\mathfrak{r}_1 &= m\mathfrak{r}_2' - m'\mathfrak{r}_2 = n\mathfrak{r}_3' - n'\mathfrak{r}_3
\end{aligned}\right\} \ldots\ldots (10).$$

The three original equations of condition are transformed into the following six, to which (8), (9), (10) are equivalent:

$$\left.\begin{aligned}
0 &= l(p_1 - E) + l'p_1' + mq_3 + m'q_3' + nq_2 + n'q_2' \\
0 &= l'(p_1 - E) - lp_1' + m'q_3 - mq_3' + n'q_2 - nq_2' \\
0 &= lq_3 + l'q_3' + m(p_2 - E) + m'p_2' + nq_1 + n'q_1' \\
0 &= l'q_3 - lq_3' + m'(p_2 - E) - mp_2' + n'q_1 - nq_1' \\
0 &= lq_2 + l'q_2' + mq_1 + m'q_1' + n(p_3 - E) + n'p_3' \\
0 &= l'q_2 - lq_2' + m'q_1 - mq_1' + n'(p_3 - E) - np_3'
\end{aligned}\right\} \ldots\text{(10A)}.$$

To give an intelligible result, the terms of the series in equation (7) must be taken in pairs with the imaginary exponents in each pair of equal arithmetical value and opposite signs.

Hence equations (7) are equivalent to the following:

$$\left.\begin{aligned}\xi &= \Sigma\left[e^{\frac{2\pi}{\lambda}(a'x+b'y+c'z)}\left\{l.\cos\frac{2\pi}{\lambda}(\sqrt{\varepsilon}.t - ax - by - cz)\right.\right.\\&\qquad\left.\left.+\,l'.\sin\frac{2\pi}{\lambda}(\sqrt{\varepsilon}.t - ax - by - cz)\right\}\right]\\ \eta &= \Sigma\{\text{terms in } m, m'\},\quad \zeta = \Sigma\{\text{terms in } n, n'\}\end{aligned}\right\}\ldots(11).$$

The above equations (11), together with the equations of condition (8), (9), (10), (or their equivalent 10A), and the equations of notation (c), contain the complete representation of the laws of small molecular oscillations in a homogeneous body of any dimensions and figure; it being understood that in the symbol of summation Σ are included as many definite integrations as the problem may require with respect to independent variables of which the coefficients λ, $\sqrt{\varepsilon}$, a, b, c, a', b', c', l, m, n, l', m', n', are functions.

As there are fourteen coefficients, connected by seven equations, viz. $a^2 + b^2 + c^2 = 1$, and the six equations of condition, the greatest number of independent variables is limited to seven; therefore, in the most general case, the symbol $\Sigma\{\ldots\}$ in equations (11) may be replaced by

$$\Sigma\!\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int F(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6,\theta_7)\{..\}d\theta_1 d\theta_2 d\theta_3 d\theta_4 d\theta_5 d\theta_6 d\theta_7 \ldots(12),$$

θ_1, &c. being variables of which the coefficients are functions, and F an arbitrary function.*

10. Let us consider the physical meaning of a single set of terms of the sums in equations (11), containing but one set of values of the coefficients. It represents a system of plane waves, the wave surfaces, or planes of equal phase in which are normal to the line whose direction-cosines are a, b, c. λ is the *length of a wave* measured along that line.

$$\left.\begin{aligned}&\frac{2\pi}{\lambda}(\sqrt{\varepsilon}.t - ax - by - cz) + \tan^{-1}\frac{l}{l'}\\&\frac{2\pi}{\lambda}(\sqrt{\varepsilon}.t - ax - by - cz) + \tan^{-1}\frac{m}{m'}\\&\frac{2\pi}{\lambda}(\sqrt{\varepsilon}.t - ax - by - cz) + \tan^{-1}\frac{n}{n'}\end{aligned}\right\}\begin{gathered}\text{are the}\\ \textit{phases}\text{ of}\\ \text{vibration}\\ \text{for}\end{gathered}\left\{\begin{aligned}&\xi,\\&\eta,\\&\zeta.\end{aligned}\right.$$

* To make the functions in equations (11) satisfy the conditions of equilibrium, instead of those of oscillation, it is only necessary to make $\varepsilon = 0$, and to substitute $k = 0$, $k' = 0$, for equations (8) and (9). Some additio[nal] functions, however, are necessary in order to complete the values of ξ, η

$\sqrt{\epsilon} = \sqrt{\left(\frac{Eg}{D}\right)}$ is the normal velocity of propagation, $\frac{\lambda}{\sqrt{\epsilon}}$ is the periodic time of an oscillation of a particle

$$\sqrt{(l^2 + l'^2)} \cdot e^{\frac{2\pi}{\lambda}(a'x + b'y + c'z)},$$

and the corresponding expressions in m and n are the semi-amplitudes of vibration parallel to x, y, z, respectively; a', b', c', are proportional to the direction-cosines of a normal to a series of planes of equal amplitude of vibration.

The trajectory of each particle affected by a single series of plane waves is in general an ellipse, the position and magnitude of which are found as follows. Let ϕ_0 denote the value of $\frac{2\pi}{\lambda}(\sqrt{\epsilon}.t - ax - by - cz)$, which makes the total displacement $\sqrt{(\xi^2 + \eta^2 + \zeta^2)}$ a maximum or minimum. It is easily seen that

$$\tan \phi = -\frac{s}{2} \pm \sqrt{\left(1 + \frac{s^2}{4}\right)} \text{ .... } (13),$$

where

$$s = \frac{l^2 + m^2 + n^2 - l'^2 - m'^2 - n'^2}{ll' + mm' + nn'}.$$

The values of ξ, η, ζ, calculated from ϕ_0 by equations (11), are the coordinates of the extremities of the axes of the elliptic trajectory, referred to the natural position of the particle as origin.

The processes of summation and definite integration denote the representation of an arbitrary manner of oscillation by the combination of a definite or indefinite number of systems of plane waves.

Case of an indefinitely extended medium.

11. Let the medium, in the first place, be supposed to be indefinitely extended in all directions. This case having been thoroughly investigated by MM. Poisson, Cauchy, Green, MacCullagh, Haughton, Stokes, and others, I shall give merely an outline of the general results. The condition that the motion shall consist of small oscillations, here makes it necessary that the exponential factor in the displacements should in all cases be equal to unity, and therefore that

$$a' = 0; \quad b' = 0; \quad c' = 0;$$

and consequently each of the accented symbols in equations $(c) = 0$. Equation (9) vanishes, and the normal velocity of propagation for each set of direction-cosines a, b, c, has,

generally speaking, three values, corresponding to the three values of E, roots of equation (8). Equations (10) become

$$l : m : n :: l' : m' : n' :: \frac{1}{r_1} : \frac{1}{r_2} : \frac{1}{r_3} \ldots\ldots (14),$$

consequently the phases of ξ, η, ζ, are simultaneous; so that $\sqrt{(l^2 + l'^2)}$, $\sqrt{(m^2 + m'^2)}$, $\sqrt{(n^2 + n'^2)}$ are proportional to the direction-cosines of a rectilinear vibratory movement of the semi-amplitude $\sqrt{(l^2 + l'^2 + m^2 + m'^2 + n^2 + n'^2)}$, which cosines have in general three sets of values corresponding to the three values of E. It is easily shewn that those three directions are at right angles to each other. The number of coefficients being in this case reduced to eleven, connected by six equations, viz. $a^2 + b^2 + c^2 = 1$, equation (8), and the proportional equation (14) which is equivalent to four, the greatest number of definite integrations in the operation (12) is restricted to five.

Thus it appears that the velocity of transmission of vibratory movement through an indefinitely extended mass, has a set of definite values, not exceeding three, for each position of plane waves. When the direction of propagation coincides with an axis of elasticity, we find those values to be:

For vibrations parallel to	Velocity of propagation along			
	x	y	z	
x....	$\sqrt{\left(\frac{A_1 g}{D}\right)}$	$\sqrt{\left(\frac{C_3 g}{D}\right)}$	$\sqrt{\left(\frac{C_2 g}{D}\right)}$	...(15).
y....	$\sqrt{\left(\frac{C_3 g}{D}\right)}$	$\sqrt{\left(\frac{A_2 g}{D}\right)}$	$\sqrt{\left(\frac{C_1 g}{D}\right)}$	
z....	$\sqrt{\left(\frac{C_2 g}{D}\right)}$	$\sqrt{\left(\frac{C_1 g}{D}\right)}$	$\sqrt{\left(\frac{A_3 g}{D}\right)}$	

When the substance is equally elastic in all directions, we have simply,

Velocities of propagation in any direction for longitudinal vibrations .... $\sqrt{\left(\frac{Ag}{D}\right)}$.

The other normal pressures are found by substitut[ing] symbols according to the following table:

Plane.	Pressure.	Coefficients.		
yz	P_1	A_1,	B_3,	B_2,
zx	P_2	B_3,	A_2,	B_1,
xy	P_3	B_2,	B_1,	A_3.

Tangential Pressures.

Plane of Distortion.	Along	On the plane	
yz	y z	xy zx	$Q_1 = -C_1\left(\dfrac{d\eta}{dz} + \dfrac{d\zeta}{dy}\right)$
zx	z x	yz xy	$Q_2 = -C_2\left(\dfrac{d\zeta}{dx} + \dfrac{d\xi}{dz}\right)$ ...(18
xy	x y	zx yz	$Q_3 = -C_3\left(\dfrac{d\xi}{dy} + \dfrac{d\eta}{dx}\right)$

Case of an Uncrystallized Medium.

14. I shall now take the particular case of an un[crys]tallized medium, in which the coefficients of elasticit[y] the same for all axes, and may be represented thus:

$$\text{rigidity} = C; \quad \text{fluid elasticity} = J;$$

$$\text{longitudinal elasticity } A = 3C + J,$$

$$\text{lateral elasticity } B = C + J = A - 2C.$$

The position of the axes being in this case arbitra[ry] shall take the direction of propagation as the axis of as to make

$$a = 1, \quad b = 0, \quad c = 0.$$

To fulfil the condition that equations (8) and (9) have common roots, we must make

$$a' = 0,$$

being in this case a common factor of $\mathfrak{g}'$, $\mathfrak{h}'$, $\mathfrak{k}'$.

The equations of notation (c) now become

$$\left.\begin{aligned}
p_1 &= A - C(b'^2 + c'^2) \\
p_2 &= -Ab'^2 + C(1 - c'^2) \\
p_3 &= -Ac'^2 + C(1 - b'^2) \\
q_1 &= -(A - C)\,b'c'; \quad q_2 = 0; \quad q_3 = 0 \\
p_1' &= 0; \quad p_2' = 0; \quad p_3' = 0 \\
q_1' &= 0; \quad q_2' = (A - C)\,c'; \quad \ldots
\end{aligned}\right\} \ldots . ($$

[redacted]

$$\left.\begin{aligned}
&\mathfrak{f} = (A + 2C)(1 - b'^2 - c'^2)\\
&\mathfrak{g} = (2AC + C^2)(1 - b'^2 - c'^2)^2\\
&\mathfrak{h} = AC^2(1 - b'^2 - c'^2)^3\\
&\mathfrak{f}' = 0;\ \mathfrak{g}' = 0;\ \mathfrak{h}' = 0\\
&\mathfrak{r}_1 = (p_1 - E)q_1 + q_2'q_3';\ \mathfrak{r}_2 = 0;\ \mathfrak{r}_3 = 0;\\
&\mathfrak{r}_1' = 0;\ \mathfrak{r}_2' = (p_2 - E)q_2' - q_3'q_1;\ \mathfrak{r}_3' = (p_3 - E)q_3' - q_1q_2'.
\end{aligned}\right\}\ \ldots.\ (\text{d.})$$

Hence it appears, that for an uncrystallized medium, equation (8) has three roots, viz.

$$\left.\begin{aligned}
&\text{one root } E = A(1 - b'^2 - c'^2),\\
&\text{two equal roots, each, } E = C(1 - b'^2 - c'^2)
\end{aligned}\right\}\ldots(19).$$

So that the velocity of propagation is less than that in an unlimited mass in the ratio $\sqrt{(1 - b'^2 - c'^2)} : 1$. Equation (9) disappears.

Equations (10) become

$$\left.\begin{aligned}
&l\mathfrak{r}_1 = m'\mathfrak{r}_2' = n'\mathfrak{r}_3'\\
&-l'\mathfrak{r}_1 = m\mathfrak{r}_2' = n\mathfrak{r}_3',\\
\text{or}\quad &l : m' : n' :: -l' : m : n :: \frac{1}{\mathfrak{r}_1} : \frac{1}{\mathfrak{r}_2'} : \frac{1}{\mathfrak{r}_3'}
\end{aligned}\right\}\ldots.(20).$$

Equations (10A) become

$$\left.\begin{aligned}
0 &= l(p_1 - E) + m'q_3' + n'q_2'\\
0 &= l'(p_1 - E) - mq_3' - nq_2'\\
0 &= l'q_3' + m(p_2 - E) + nq_1\\
0 &= -lq_3' + m'(p_2 - E) + n'q_1\\
0 &= l'q_2' + mq_1 + n(p_3 - E)\\
0 &= -lq_2' + m'q_1 + n'(p_3 - E)
\end{aligned}\right\}\ \ldots.\ (20\text{A}).$$

15. It may be shewn that the vibrations corresponding to the roots $C(1 - b'^2 - c'^2)$ cannot take place in a body of which the surface is free, unless $b' = 0$, $c' = 0$, in which case they are reduced to ordinary transverse vibrations. (See *Appendix* No. II.)

Nearly-longitudinal Vibrations in an Uncrystallized Medium.

16. For the present, therefore, I shall confine the investigation to the root

$$E = A(1 - b'^2 - c'^2),$$

corresponding to the velocity of propagation

$$\sqrt{\varepsilon} = \sqrt{\left\{\frac{Ag}{D}(1 - b'^2 - c'^2)\right\}}\ \ldots\ldots..\ (21$$

The vibrations to which this root is applicable may be called *nearly-longitudinal;* because in them the longitudinal component predominates, and their velocity of transmission is a function of the longitudinal elasticity A.

This value being substituted for E in the expressions for r_1 &c., gives

$$\left.\begin{array}{l} m' = -b'l; \ \ n' = -c'l \\ m = b'l'; \ \ n' = c'l' \end{array}\right\} \ldots\ldots\ldots (22).$$

Which values being substituted in equations (11), (16), (17), (18), give the following results.

For brevity's sake, let

$$\frac{2\pi}{\lambda}(b'y + c'z) = \psi; \quad \frac{2\pi}{\lambda}(\sqrt{\epsilon}.t - x) = \phi;$$

also let $$\Phi = \Sigma\left\{\frac{\lambda}{2\pi}e^{\psi}(l'\cos\phi - l\sin\phi)\right\}.$$

Then the displacements are:

$$\left.\begin{array}{l} \xi = \Sigma\left\{e^{\psi}(l\cos\phi + l'\sin\phi)\right\} \ \ = \dfrac{d\Phi}{dx} \\ \eta = \Sigma\left\{b'e^{\psi}(l'\cos\phi - l\sin\phi)\right\} = \dfrac{d\Phi}{dy} \\ \zeta = \Sigma\left\{c'e^{\psi}(l'\cos\phi - l\sin\phi)\right\} = \dfrac{d\Phi}{dz} \end{array}\right\} \ldots (23).$$

The velocities of the particles are:

$$\left.\begin{array}{l} \dfrac{d\xi}{dt} = \Sigma\left\{\dfrac{2\pi}{\lambda}\sqrt{\epsilon}.e^{\psi}(l'\cos\phi - l\sin\phi)\right\} \\ \dfrac{d\eta}{dt} = -\Sigma\left\{\dfrac{2\pi}{\lambda}\sqrt{\epsilon}.b'e^{\psi}(l\cos\phi + l'\sin\phi)\right\} \\ \dfrac{d\zeta}{dt} = -\Sigma\left\{\dfrac{2\pi}{\lambda}\sqrt{\epsilon}.c'e^{\psi}(l\cos\phi + l'\sin\phi)\right\} \end{array}\right\} \ldots (24).$$

The longitudinal strains:

$$\left.\begin{array}{l} \dfrac{d\xi}{dx} = -\Sigma\left\{\dfrac{2\pi}{\lambda}e^{\psi}(l'\cos\phi - l\sin\phi)\right\} \\ \dfrac{d\eta}{dy} = \Sigma\left\{\dfrac{2\pi}{\lambda}b'^2e^{\psi}(l'\cos\phi - l\sin\phi)\right\} \\ \dfrac{d\zeta}{dz} = \Sigma\left\{\dfrac{2\pi}{\lambda}c'^2e^{\psi}(l'\cos\phi - l\sin\phi)\right\} \\ \text{The total change of volume:} \\ \dfrac{d\xi}{dx} + \dfrac{d\eta}{dy} + \dfrac{d\zeta}{dz} = -\Sigma\left\{\dfrac{2\pi}{\lambda}(1 - b'^2 - c'^2)e^{\psi}(l'\cos\phi - l\sin\phi)\right\} \end{array}\right\} \ldots (25).$$

[redacted]

The distortions:

$$\left.\begin{aligned}
\frac{d\eta}{dz} + \frac{d\zeta}{dy} &= 2\Sigma\left\{\frac{2\pi}{\lambda}\, b'c'e^{\psi}\,(l'\cos\phi - l\sin\phi)\right\}\\
\frac{d\zeta}{dx} + \frac{d\xi}{dz} &= 2\Sigma\left\{\frac{2\pi}{\lambda}\, c'e^{\psi}\,(l\cos\phi + l'\sin\phi)\right\}\\
\frac{d\xi}{dy} + \frac{d\eta}{dx} &= 2\Sigma\left\{\frac{2\pi}{\lambda}\, b'e^{\psi}\,(l\cos\phi + l'\sin\phi)\right\}
\end{aligned}\right\}\ldots(25).$$

The pressures due to the displacements are as follows:

Normal Pressures.

$$\left.\begin{aligned}
P_1 &= \Sigma\left[\frac{2\pi}{\lambda}e^{\psi}\left\{A(1-b'^2-c'^2)+2C(b'^2+c'^2)\right\}(l'\cos\phi - l\sin\phi)\right]\\
P_2 &= \Sigma\left[\frac{2\pi}{\lambda}e^{\psi}\left\{A(1-b'^2-c'^2)-2C(1-c'^2)\right\}(l'\cos\phi - l\sin\phi)\right]\\
P_3 &= \Sigma\left[\frac{2\pi}{\lambda}e^{\psi}\left\{A(1-b'^2-c'^2)-2C(1-b'^2)\right\}(l'\cos\phi - l\sin\phi)\right]
\end{aligned}\right\}\ldots(26).$$

The tangential pressures Q_1, Q_2, Q_3, are found by multiplying the distortions by $-C$.

Let R_1, R_2, R_3, be the three components of the pressure exerted by the particles of the body, in consequence of the molecular displacements, at any part of its external surface, the normal to which makes with the axes the angles α, β, γ. Then

$$\left.\begin{aligned}
R_1 &= P_1\cos\alpha + Q_3\cos\beta + Q_2\cos\gamma\\
R_2 &= Q_3\cos\alpha + P_2\cos\beta + Q_1\cos\gamma\\
R_3 &= Q_2\cos\alpha + Q_1\cos\beta + P_3\cos\gamma
\end{aligned}\right\}\ldots.(27).$$

Should there be any surface along which the particles are constrained to slide, it is obvious that at that surface the following condition must be fulfilled:

$$\left.\begin{aligned}
&0 = \xi\cos\alpha + \eta\cos\beta + \zeta\cos\gamma\,;\\
&\text{or if } z_1 = f(x, y) \text{ be the equation of the surface,}\\
&0 = \xi\frac{dz_1}{dx} + \eta\frac{dz_1}{dy} - \zeta
\end{aligned}\right\}\ldots(28).$$

Were we acquainted with the laws which determine the superficial pressures in vibrating bodies, equations (27) would enable us to determine the values which b' and c' must have, in virtue of those laws, during the transmission of sound in a limited mass of an uncrystallized material, and thence the ratio $\sqrt{(1 - b'^2 - c'^2)} : 1$ in which the velocity of sound in such

a body is less than in an unlimited mass of the same material. Those laws, however, are as yet a matter of conjecture only.

Transmission of a definite musical tone.

17. When the body transmits one or more definite musical tones (which is the case in all experiments capable of yielding useful results), the velocity of propagation must be the same for all the elementary vibrations into which the motion may be resolved: that is to say, $1 - b'^2 - c'^2$ must have the same value in all the terms of the sums Σ. This affords the means of simplifying the equations. Let

$$b'^2 + c'^2 = h^2; \quad b' = h\cos\theta; \quad c' = h\sin\theta;$$

h being the same for all the terms in the sums Σ. Then the velocity of propagation is

$$\sqrt{\epsilon} = \sqrt{\left\{\frac{Ag}{D}(1 - h^2)\right\}} \quad \dots\dots\dots\dots (29),$$

and this factor may be removed outside the sign of summation.

When but one musical tone is transmitted, the factor $\frac{2\pi}{\lambda}$ also may be removed outside that sign, and for $\Sigma\{\ \}$ may be substituted a definite integration

$$\left.\begin{array}{l} \Sigma \int F\theta \{\dots\}\, d\theta \\ F\theta \text{ being arbitrary.} \\ \text{We have also} \\ e^{\psi} = e^{\frac{2\pi}{\lambda} h (y\cos\theta + z\sin\theta)} \end{array}\right\} \dots\dots\dots\dots (30);$$

in which $\frac{2\pi}{\lambda}$ h, y, and z are independent of θ, and may be treated as constants in the definite integration.

Introducing these modifications into equations (23) &c., we find

$$\left.\begin{array}{l} \Phi = \dfrac{\lambda}{2\pi}(l'\cos\phi - l\sin\phi)\, \Sigma \int e^{\psi} F\theta\, d\theta \\ \qquad\qquad \textit{Displacements.} \\ \xi = (l\cos\phi + l'\sin\phi)\, \Sigma \int e^{\psi} F\theta\, d\theta \\ \eta = (l'\cos\phi - l\sin\phi)\, h.\Sigma \int \cos\theta\, e^{\psi} F\theta\, d\theta \\ \zeta = (l'\cos\phi - l\sin\phi)\, h.\Sigma \int \sin\theta\, e^{\psi} F\theta\, d\theta \end{array}\right\} \dots (31).$$

[redacted]

Velocities of the Particles.

$$\frac{\pi}{}\sqrt{\varepsilon}.(l'\cos\phi - l\sin\phi)\,\Sigma\int e^{\psi}F\theta\, d\theta$$

$$\frac{2\pi}{\lambda}\sqrt{\varepsilon}.(l\cos\phi + l'\sin\phi)\,h.\Sigma\int \cos\theta e^{\psi}F\theta\, d\theta$$

$$\frac{2\pi}{\lambda}\sqrt{\varepsilon}.(l\cos\phi + l'\sin\phi)\,h.\Sigma\int \sin\theta e^{\psi}F\theta\, d\theta.$$

Longitudinal Strains.

$$\frac{2\pi}{\lambda}(l'\cos\phi - l\sin\phi)\,\Sigma\int e^{\psi}F\theta\, d\theta$$

$$(l'\cos\phi - l\sin\phi)\,h^2.\Sigma\int \cos^2\theta e^{\psi}F\theta\, d\theta$$

$$(l'\cos\phi - l\sin\phi)\,h^2.\Sigma\int \sin^2\theta e^{\psi}F\theta\, d\theta.$$

Cubic Dilatation.

$$\frac{d\zeta}{dz} = -\frac{2\pi}{\lambda}(l'\cos\phi - l\sin\phi)(1-h^2)\,\Sigma\int e^{\psi}F\theta\, d\theta.$$

Distortions. ...(31).

$$= \frac{4\pi}{\lambda}(l'\cos\phi - l\sin\phi)\,h^2\Sigma\int \cos\theta\sin\theta e^{\psi}F\theta\, d\theta$$

$$= \frac{4\pi}{\lambda}(l\cos\phi + l'\sin\phi)\,h\Sigma\int \sin\theta e^{\psi}F\theta\, d\theta$$

$$= \frac{4\pi}{\lambda}(l\cos\phi + l'\sin\phi)\,h\Sigma\int \cos\theta e^{\psi}F\theta\, d\theta$$

h being multiplied by $-C$, give the tangen- sures Q_1, Q_2, Q_3, on the coordinate planes.

Pressures on the coordinate planes, due to the displacements.

$$l'\cos\phi - l\sin\phi)\,\{A(1-h^2) + 2Ch^2\}\,\Sigma\int e^{\psi}F\theta\, d\theta$$

$$l'\cos\phi - l\sin\phi)\,[\{A(1-h^2) - 2C\}\,\Sigma\int e^{\psi}F\theta\, d\theta + 2Ch^2\Sigma\int \sin^2\theta e^{\psi}F\theta\, d\theta]$$

$$l'\cos\phi - l\sin\phi)\,[\{A(1-h^2) - 2C\}\,\Sigma\int e^{\psi}F\theta\, d\theta + 2Ch^2\Sigma\int \cos^2\theta e^{\psi}F\theta\, d\theta]$$

Let R_1, R_2, R_3, be the components of the pressure exerted by the body, in consequence of the molecular displacements at a point of its surface normal to the direction (α, β, γ). Also let

$$\cos\beta = \sin\alpha\cos\chi,$$
$$\cos\gamma = \sin\alpha\sin\chi,$$

so as to make x the axis of polar coordinates, and xy the plane from which longitudes χ are measured. Then

$$\left.\begin{aligned}
R_1 &= \frac{2\pi}{\lambda}[\cos\alpha(l'\cos\phi - l\sin\phi)\{A(1-h^2)+2Ch^2\}\Sigma\int e^{\psi}F\theta d\theta \\
&\quad - 2\sin\alpha(l\cos\phi + l'\sin\phi)Ch\Sigma\int\cos(\theta-\chi)e^{\psi}F\theta d\theta] \\
R_2 &= \frac{2\pi}{\lambda}[-2\cos\alpha(l\cos\phi + l'\sin\phi)Ch\Sigma\int\cos\theta e^{\psi}F\theta d\theta \\
&\quad + \sin\alpha(l'\cos\phi - l\sin\phi)\{\cos\chi(A(1-h^2)-2C)\Sigma\int e^{\psi}F\theta d\theta \\
&\qquad + 2Ch^2\Sigma\int\sin\theta\sin(\theta-\chi)e^{\psi}F\theta d\theta\}] \\
R_3 &= \frac{2\pi}{\lambda}[-2\cos\alpha(l\cos\phi + l'\sin\phi)Ch\Sigma\int\sin\theta e^{\psi}F\theta d\theta \\
&\quad + \sin\alpha(l'\cos\phi - l\sin\phi)\{\sin\chi\big(A(1-h^2)-2C\big)\Sigma\int e^{\psi}F\theta d\theta \\
&\qquad - 2Ch^2\Sigma\int\cos\theta\sin(\theta-\chi)e^{\psi}F\theta\, d\theta\}]
\end{aligned}\right\}\ldots(32).$$

Let P' represent the normal pressure at the given point of the surface due to molecular displacements: then

$$\left.\begin{aligned}
P' &= R_1\cos\alpha + \sin\alpha(R_2\cos\chi + R_3\sin\chi) \\
&= P_1\cos^2\alpha + P_2\sin^2\alpha\cos^2\chi + P_3\sin^2\alpha\sin^2\chi \\
&\quad + 2Q_1\sin^2\alpha\cos\chi\sin\chi + 2Q_2\cos\alpha\sin\alpha\sin\chi \\
&\qquad + 2Q_3\cos\alpha\sin\alpha\cos\chi \\
&= \frac{2\pi}{\lambda}[(l'\cos\phi - l\sin\phi)\{\big(A(1-h^2) \\
&\quad + 2C(h^2\cos^2\alpha - \sin^2\alpha)\big)\Sigma\int e^{\psi}F\theta\, d\theta \\
&\quad + 2Ch^2\sin^2\alpha\,\Sigma\int\sin^2(\theta-\chi)e^{\psi}F\theta\, d\theta\} \\
&\quad - 4(l\cos\phi + l'\sin\phi)Ch\cos\alpha\sin\alpha\,\Sigma\int\cos(\theta-\chi)e^{\psi}F\theta d\theta]
\end{aligned}\right\}\ldots(32\text{A}).$$

Propagation of sound by nearly-longitudinal vibrations along a horizontal prism of liquid contained in a rectangular trough, investigated according to a peculiar hypothesis.

18. I shall now suppose the vibrating body to be a rectangular horizontal prism of liquid contained in a trough of some substance so dense hard and smooth, that the particles at the sides and bott[illegible] are constrained to slide along those surfac[illegible] cal ends of the trough

be capable of perfectly reflecting a wave of sound travelling horizontally; so that the propagation of that wave may take place as if in a trough of indefinite length: and I shall investigate the velocity of such a wave according to a peculiar hypothetical view of the molecular condition of the upper surface of the liquid.

The axis of x being the horizontal axis of the trough, and parallel to the direction of propagation, let that of y be transverse and that of z vertical. Let the middle of the bottom of the trough be the origin of coordinates, $2y_1$ being its breadth, and z_1 the depth of liquid in it.

The conditions to be fulfilled at the bottom are when

$$z = 0, \quad \alpha = \tfrac{1}{2}\pi, \quad \text{and} \quad \chi = -\tfrac{1}{2}\pi.$$

Let $\quad \Sigma \int \sin\theta e^{\psi} F\theta d\theta = \Sigma \int \sin\theta e^{\frac{2\pi}{\lambda} hy\cos\theta} F\theta d\theta = 0$

at the sides when

$$y = \pm y_1 \quad \alpha = \tfrac{1}{2}\pi, \quad \text{and} \quad \chi = 0 \text{ or } \pi.$$

Let $\quad \Sigma \int \cos\theta e^{\psi} F\theta d\theta = \Sigma \int \cos\theta e^{\frac{2\pi}{\lambda} h(\pm y_1\cos\theta + z\sin\theta)} F\theta d\theta = 0;$

which conditions are fulfilled by making

$$\cos\theta = 0, \quad \sin\theta = \pm 1,$$

and putting for $\Sigma\int F\theta d\theta$ a summation of two terms in which the signs of the exponent are respectively positive and negative.

Thus we obtain

$$\left.\begin{aligned} \xi &= (l\cos\phi + l'\sin\phi)\left(e^{\frac{2\pi}{\lambda}hz} + e^{-\frac{2\pi}{\lambda}hz}\right) \\ \eta &= 0 \\ \zeta &= (l'\cos\phi - l\sin\phi)h\left(e^{\frac{2\pi}{\lambda}hz} - e^{-\frac{2\pi}{\lambda}hz}\right) \end{aligned}\right\} \ldots (33).$$

The trajectory of each particle is an ellipse in a vertical longitudinal plane; the motion being *direct* in the upper part of the ellipse, because the sign of $\frac{d\xi}{dt}$ is the same with that of ζ. The axes are vertical and horizontal respectively, and have the following values:

$$\text{horizontal axis} = 2\sqrt{(l^2 + l'^2)}.\left(e^{\frac{2\pi}{\lambda}hz} + e^{-\frac{2\pi}{\lambda}hz}\right),$$

$$\text{vertical axis} = 2\sqrt{(l^2 + l'^2)}.h.\left(e^{\frac{2\pi}{\lambda}hz} - e^{-\frac{2\pi}{\lambda}hz}\right);$$

so that the motion is analogous to that of waves propagated by gravitation, being entirely horizontal at the bottom of

trough, and elliptical elsewhere, the ellipse being larger and less eccentric as the height above the bottom increases. The ratio of the axes, however, instead of approaching equality as the depth of the trough increases (which is the case with waves of gravitation), approaches $1 : h$.

19. To determine this ratio, upon which the velocity of sound along such a mass of liquid must depend, I shall assume the following hypothetical principles respecting the state of the particles at the upper surface.

First, that (as laid down in a previous paper, *Cambridge and Dublin Mathematical Journal, February*, 1851,) the elasticity of bodies is due partly to the mutual actions of atomic centres producing elasticity both of volume and figure, and partly to a mere fluid elasticity resisting change of volume only, and exerted by atmospheres surrounding those centres; and that the effect of the mutual actions of the atomic centres in producing pressure is very small in liquids, and absolutely inappreciable in gases and vapours.

Secondly, that every liquid maintains at its surface, by molecular attraction, an atmosphere of its own vapour, under these conditions—that the total pressures of the liquid and vapour, and also their fluid pressures, shall be equal at the bounding surface. (From this hypothesis I have already deduced the form of an approximate equation between the pressure and temperature of vapour at saturation.) The total pressure of the vapour on the liquid is sensibly equal to its fluid pressure: the total pressure of the liquid on the vapour consists of its fluid pressure, and a pressure due to atomic centres; the latter quantity must therefore be null.

Thirdly, that the pressure of the vapour follows that of the liquid throughout its variations during the propagation of sound; so that the portion of the pressure of the liquid on the vapour, due to atomic centres, must continue null throughout these variations.

Let ϖ be the mutual pressure of the liquid and its vapour in a state of rest; then $\varpi + P'$ is their momentary mutual pressure during the passage of a wave of sound horizontally along the trough. The portion of P' depending on the coefficient of rigidity C being made $= 0$, we shall obtain an equation from which the value of h may be deduced.

Making the proper substitutions in equation (32A), viz.

$$\cos\alpha = 0, \quad \sin\alpha = 1, \quad \cos\chi = 0, \quad \sin\chi = 1, \quad \psi = \pm\frac{2\pi}{\lambda}hz,$$

$$\cos\theta = 0, \quad \sin\theta = \pm 1, \quad F0 = 1, \quad z = z_1, \text{ \&c.},$$

we find

$$p + P' = \varpi + \frac{2\pi}{\lambda}(l'\cos\phi - l\sin\phi)\{A(1-h^2)-2C\}\left(e^{\frac{2\pi}{\lambda}hz_1}+e^{-\frac{2\pi}{\lambda}hz_1}\right).$$

The part of this depending on mere fluid elasticity, in which the liquid is followed by the vapour, is

$$\varpi - J\left(\frac{d\xi}{dx}+\frac{d\zeta}{dz}\right)$$

$$= \varpi + \frac{2\pi}{\lambda}(l'\cos\phi - l\sin\phi).J(1-h^2)\left(e^{\frac{2\pi}{\lambda}hz_1}+e^{-\frac{2\pi}{\lambda}hz_1}\right),$$

which being subtracted, there remains for the part depending on atomic centres,

$$0 = \frac{2\pi}{\lambda}(l'\cos\phi - l\sin\phi)C(1-3h^2)\left(e^{\frac{2\pi}{\lambda}hz_1}+e^{-\frac{2\pi}{\lambda}hz_1}\right).$$

Consequently $\quad 1 - 3h^2 = 0, \quad$ or $\quad h = \sqrt{\tfrac{1}{3}}$ ....(33),

is the equation of condition sought, arising from the state of the free surface; and this equation is independent of the amount of rigidity of the liquid, requiring only that it shall be *something*, however small, while that of the vapour is null.

It follows from this equation, that the velocity of propagation of sound along a trough of liquid of the density D, and longitudinal elasticity A, is

$$\sqrt{\varepsilon} = \sqrt{\left\{\frac{Ag}{D}(1-h^2)\right\}} = \sqrt{\left(\frac{2}{3}\cdot\frac{Ag}{D}\right)} \dots (34),$$

or less than the velocity in an unlimited mass in the ratio of $\sqrt{2}$ to $\sqrt{3}$.

20. This is precisely the result arrived at by M. Wertheim from a comparison of his numerous experiments on the propagation of sound in water at various temperatures, from 15° to 60° centigrade, in solutions of various salts, in alcohol, turpentine, and ether (*Ann. de. Chim. Ser. III.* tom. XXIII.), with those of M. Grassi on the compressibility of the same substances (*Comptes Rendus* XIX. p. 153), and with the experiments of MM. Colladon and Sturm on the velocity of sound in an expanse of water.

M. Wertheim having given this comparison in detail, I shall quote one example only.

The velocity of sound in an unlimited mass of water at the temperature of 16° centigrade, as ascertained by MM. Colladon and Sturm, was 1435 mètres per second.

That of sound in water contained in a trough, the vibrations of which were regulated by an organ-pipe, was found by M. Wertheim, at 15° centigrade, to be 1173 mètres per second.

The ratio of the squares of those quantities is 0.6686 : 1, differing from $\frac{2}{3}$ by 0.0009 only.

Remarks on the propagation of sound along solid rods.

21. I refrain from giving in the body of this paper detailed investigations of particular problems respecting the propagation of sound along a solid prism or cylinder; for in the present state of our knowledge of the condition of the superficial particles of such bodies, the conclusions would be almost entirely speculative and conjectural.

I may mention briefly, however, the following general results. If we adopt for solids the same hypothesis as for liquids, then the ratio of the velocity of sound in a rod of an uncrystallized material to that in an unlimited mass has the following values:

For a rectangular prismatic rod, the lateral vibrations of the particles of which are confined to planes parallel to one pair of faces of the prism, but are perfectly free in other respects, the ratio is $\sqrt{2} : \sqrt{3}$, being the same as for a liquid.

For a cylindrical rod, the surface being perfectly free, the ratio has various values from $\sqrt{\frac{1}{2}}$ to $\sqrt{\frac{2}{3}}$, approaching the less value as the diameter of the rod diminishes, and the greater as it increases *ad infinitum*. (See *Appendix*, No. I.)

22. These conditions, however, cannot be realized in practice; and the lateral vibrations being more or less confined by the means used in fixing the rods, we find that the ratio generally exceeds $\sqrt{2} : \sqrt{3}$, and sometimes approaches equality.

The following table illustrates this fact. I have selected in the first place, the experiments of M. Wertheim on tubes of crystal (*Ann. de Chim. Ser. III.* tom. XXIII), because in them the coefficients of elasticity and the velocities of sound were ascertained by experiments on the same pieces of the material. To these I have added a calculation founded on a comparison of the experiments of M. Wertheim on the elasticity of brass, with those of M. Savart on the velocity of sound in it, as being the only other data of the kind now existing from which a satisfactory conclusion can be drawn.

The coefficients of longitudinal elasticity, calculated by myself from M. Wertheim's experiments, are extracted from my paper on Elasticity in ... *and Dublin Mathe-*

...atical *Journal* for February, 1851. The quantities $\frac{\varepsilon D}{g}$ for ...rystal are given as calculated by M. Wertheim. For brass, I have used the following data:

$\sqrt{\varepsilon}$ = velocity of sound in brass rods; mean of many experiments by M. Savart = 3560 mètres per second.

D = density, in kilogrammes per cubic mètre, 8395.

TABLE.

	Longitudinal Elasticity A Kilogrammes per square millimètre.	$\frac{\varepsilon D}{g}$ Kilogrammes per square millimètre.	Ratio $1 - h^2$
Crystal.			
Tube No. I.	5514.2	5354.0	0.970,
" " III.	5611.0	5476.7	0.976,
" " IV.	6183.1	5597.3	0.905,
" " V.	6659.9	5489.8	0.824,
Brass.	15625	10847	0.694.

Concluding Remarks.

23. The chief positive results arrived at in this paper may be summed up as follows—

(I.) In liquid and solid bodies of limited dimensions, the freedom of lateral motion possessed by the particles causes vibrations to be propagated less rapidly than in an unlimited mass.

(II.) The symbolical expressions for vibrations in limited bodies are distinguished by containing exponential functions of the coordinates as factors; and the retardation referred to depends on the coefficients of the coordinates in the exponents of those functions, which coefficients depend on the molecular condition of the body's surface; a condition yet imperfectly understood.

(III.) If we adopt the hypothetical principle, *that at the free surface of a vibrating mass of liquid the normal pressure depending on the actions of atomic centres is always null*, then we deduce from theory that the ratio of the velocity ... along a mass of liquid contained in a rectangular ... that in an unlimited mass is $\sqrt{2} : \sqrt{3}$, that ratio be... pendent of the specific rigidity of the liquid; a c... agreeing with our present experimental knowledge.

24. I do not put forward the hypothetical part of the researches as more than a probable conjecture; nor should be justified in so doing in the present state of our knowledg of molecular forces. I think, however, that these investi gations are sufficient to prove that we are not warranted i concluding from M. Wertheim's experiments (as he is dis posed to do) that liquids possess a momentary rigidity great as that of solids, seeing that any amount of rigidity howsoever small, will account for the phenomena if we adop certain suppositions as to molecular forces; and to shew, th our knowledge of those forces is not yet sufficiently ad vanced to enable us to use experiments on sound as means of determining the coefficients of elasticity of solids.

London, February, 1851.

Appendix.—No. I.

Propagation of sound by nearly-longitudinal vibrations along a cylindric uncrystallized rod.

Let the vibrating body be cylindrical round the axis of and let the vibrations of all particles in a given circle roun that axis be assumed to be equal and simultaneous. Let represent the distance of any particle from the axis of and χ, the angle $\hat{yr}$.

Then $$e^{\psi} = e^{\frac{2\pi}{\lambda} rh\cos(\theta-\chi)} \quad\text{..............} \quad (35).$$

To make the results of the definite integrations $\Sigma\int F\theta$ independent of the angle χ, we must have $F\theta$ = constant, an the limits of integration 0 and 2π.

The following are the definite integrals which enter int the solution of this problem.

Let $$\frac{2\pi}{\lambda} hr = k,$$

$$\left.\begin{aligned}\Theta &= \int_0^{2\pi} e^{k\cos\theta} d\theta = 4\int_0^{\frac{1}{2}\pi} \frac{e^{k\cos\theta} + e^{-k\cos\theta}}{2}\, d\theta \\ &= 4\int_0^{\frac{1}{2}\pi} d\theta \left\{1 + \Sigma\left(\frac{k^{2n}\cos^{2n}\theta}{\Gamma(2n+1)}\right)\right\} \\ &= 2\pi\left[1 + \Sigma\left\{\frac{k^{2n}}{2^{2n}(\Gamma(n+1))^2}\right\}\right]\end{aligned}\right\} \quad ...(36)$$

$$\left.\begin{aligned}\Theta' &= \frac{d\Theta}{dk} = \int_0^{2\pi} \cos\theta\, e^{k\cos\theta} d\theta \\ &= 2\pi.\Sigma\left\{\frac{k^{2n-1}}{2^{2n-1}.\Gamma(n).\Gamma(n+1)}\right\} \\ \Theta'' &= \frac{d^2\Theta}{dk^2} = \int_0^{2\pi} \cos^2\theta\, e^{k\cos\theta} d\theta \\ &= 2\pi\left[\tfrac{1}{2} + \Sigma\left\{\frac{(2n+1)k^{2n}}{2^{2n+1}\Gamma(n+1)\,\Gamma(n+2)}\right\}\right]\end{aligned}\right\}\ldots(36);$$

the values of n comprehending all integers from 1 inclusive.

Those series have the following properties:

(I.) The term (n) of Θ = term $(n-1) \times \frac{k^2}{4n^2}$; therefore this series always becomes convergent at the term for which $n > \frac{1}{2}k$.

(II.) Term (n) of Θ' = term $(n-1) \times \frac{k^2}{4(n-1)n}$; therefore this series becomes convergent when $n^2 - n > \frac{1}{4}k^2$.

(III.) Term (n) of Θ'' = term $(n-1) \times \frac{(2n+1)k^2}{4(2n-1).n.(n+1)}$; therefore it begins to converge when $n^2 - \frac{n}{2n-1} > \frac{k^2}{4}$.

(IV.) $\Theta' = k(\Theta - \Theta'')$.

(V.) Term (n) of Θ'' = term (n) of $\Theta \times \frac{2n+1}{2n+2}$; a ratio which is $\frac{1}{2}$ for the first term $(n = 0)$, and approaches equality as n increases; therefore when $\frac{1}{4}k^2$ is an inappreciably small fraction, $\frac{\Theta''}{\Theta} = \frac{1}{2}$ sensibly.

And the larger k is, the more nearly is $\frac{\Theta''}{\Theta} = 1$.

The following table of a few numerical results illustrates this:

$\frac{k^2}{4}$	$\frac{\Theta}{2\pi}$	$\frac{\Theta''}{2\pi}$	$\frac{\Theta''}{\Theta}$
0	1.0000	0.5000	0.5000,
$\frac{1}{4}$	1.2661	0.7010	0.5537,
$\frac{1}{3}$	1.3622	0.7741	0.5683,
$\frac{1}{2}$	1.5661	0.9302	0.5490,
1	2.2796	1.4843	0.6511,
2	4.2523	3.0550	0.7160,
3	7.1590	5.4238	0.7576,
4	11.3019	9.3620	0.8284.

The displacements in this case are as follows—

$$\left.\begin{aligned}\xi &= (l\cos\phi + l'\sin\phi)\Theta\\ \eta &= (l'\cos\phi - l\sin\phi)\,h\Theta'\cos\chi\\ \zeta &= (l'\cos\phi - l\sin\phi)\,h\Theta'\sin\chi\end{aligned}\right\}\ldots\ldots(37);$$

whence it appears that the two transverse displacements η and ζ compose a radial displacement

$$\rho = (l'\cos\phi - l\sin\phi)\,h\Theta'\ldots\ldots\ldots(37\text{A}).$$

Therefore the trajectory of each particle is an ellipse, in a plane passing through the axis of the cylinder; and the axes of the ellipse are longitudinal and radial, and have the following values:

$$\left.\begin{aligned}\text{longitudinal axis} &= 2\sqrt{(l^2 + l'^2)}.\Theta\\ \text{radial axis} &= 2\sqrt{(l^2 + l'^2)}.h\Theta'\end{aligned}\right\}\ldots\ldots(38).$$

If we now adopt the same hypotheses with respect to the outer surface of the cylinder that have been used in the problem respecting liquids, we shall have for the mutual pressure of the solid and its atmosphere of vapour,

$$\varpi + P' = \varpi + \frac{2\pi}{\lambda}(l'\cos\phi - l\sin\phi)\{(A - 2C)(1 - h^2)\Theta_1 - 2Ch^2\Theta_1''\}$$

Θ_1, Θ_1'' being the values of those integrals corresponding t the radius of the cylinder.

The portion of this pressure depending on mere flui elasticity is

$$\varpi - J\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right) = \varpi + \frac{2\pi}{\lambda}(l'\cos\phi - l\sin\phi)\,J(1 - h^2).\Theta_1$$

which being subtracted, leaves

$$0 = \frac{2\pi}{\lambda}(l'\cos\phi - l\sin\phi)\,C\{\Theta_1 - h^2(\Theta_1 + 2\Theta_1'')\};$$

therefore, according to the hypothesis adopted,

$$h^2 = \frac{1}{1 + 2\dfrac{\Theta_1''}{\Theta_1}}\ldots\ldots\ldots\ldots(39):$$

and the velocity of sound along the cylinder is

$$\sqrt{s} = \sqrt{\left\{\frac{Ag}{D}(1 - h^2)\right\}} = \sqrt{\left(\frac{Ag}{D}\cdot\frac{2}{2 + \dfrac{\Theta_1}{\Theta_1''}}\right)}\ldots(40).$$

Now the limits of the ratios in the above formulæ are the following:

$$\begin{array}{llccc}
\text{limits of} & \dfrac{2\pi h r_1}{\lambda} = k_1 \ldots\ldots & 0 \ldots\ldots & \infty, \\
\text{“} \quad \text{“} & \dfrac{\Theta_1}{\Theta_1''} \ldots\ldots\ldots\ldots & 2 \ldots\ldots & 1, \\
\text{“} \quad \text{“} & h \ldots\ldots\ldots\ldots & \sqrt{\tfrac{1}{2}} \ldots\ldots & \sqrt{\tfrac{1}{3}}, \\
\text{“} \quad \text{“} & \sqrt{(1-h^2)} \ldots\ldots & \sqrt{\tfrac{1}{2}} \ldots\ldots & \sqrt{\tfrac{2}{3}}.
\end{array}$$

That is to say, if the hypothesis already explained with reference to liquids is applicable to a solid cylinder of an uncrystallized material, the velocity of sound along such a cylinder, when its surface is perfectly free, will be less than that in an unlimited mass in some ratio between $\sqrt{\frac{1}{2}}$ and $\sqrt{\frac{2}{3}}$.

Appendix.—No. II.

General Equations of nearly-transverse Vibrations.

The two equal roots of equation (8) in uncrystallized bodies, viz.

$$E = C(1 - b'^2 - c'^2)$$

correspond to what may be called *nearly-transverse* vibrations, propagated with the velocity

$$\sqrt{\varepsilon} = \sqrt{\left\{\frac{Cg}{D}(1 - b'^2 - c'^2)\right\}} \ldots\ldots\ldots (41).$$

Equations (20) in this case give no result; but equations (20A) are reducible to the following two:

$$\left.\begin{aligned} l &= -m'b' - n'c' \ldots \\ l' &= mb' + nc' \ldots\ldots \end{aligned}\right\} \ldots\ldots\ldots\ldots (42);$$

the ratios $m : n$ and $m' : n'$ are arbitrary.

Equations (11), (16), (17), (18), become the following:

Displacements.

$$\left.\begin{aligned} \xi &= \Sigma\left[e^{\psi}\left\{-(m'b' + n'c')\cos\phi + (mb' + nc')\sin\phi\right\}\right] \\ \eta &= \Sigma\left\{e^{\psi}(m\cos\phi + m'\sin\phi)\right\} \\ \zeta &= \Sigma\left\{e^{\psi}(n\cos\phi + n'\sin\phi)\right\} \end{aligned}\right\} \ldots (43).$$

Velocities of the Particles.

$$\frac{d\xi}{dt} = \Sigma\left[\frac{2\pi}{\lambda}\sqrt{\epsilon}.e^{\psi}\left\{(m'b'+n'c')\sin\phi+(mb'+nc')\cos\phi\right\}\right]$$

$$\frac{d\eta}{dt} = \Sigma\left\{\frac{2\pi}{\lambda}\sqrt{\epsilon}.e^{\psi}(-m\sin\phi+m'\cos\phi)\right\}$$

$$\frac{d\zeta}{dt} = \Sigma\left\{\frac{2\pi}{\lambda}\sqrt{\epsilon}.e^{\psi}(-n\sin\phi+n'\cos\phi)\right\}$$

Longitudinal Strains.

$$\frac{d\xi}{dx} = -\Sigma\left[\frac{2\pi}{\lambda}e^{\psi}\left\{(mb'+nc')\cos\phi+(m'b'+n'c')\sin\phi\right\}\right]$$

$$\frac{d\eta}{dy} = \Sigma\left\{\frac{2\pi}{\lambda}e^{\psi}(mb'\cos\phi+m'b'\sin\phi)\right\}$$

$$\frac{d\zeta}{dz} = \Sigma\left\{\frac{2\pi}{\lambda}e^{\psi}(nc'\cos\phi+n'c'\sin\phi)\right\}.$$

Cubic Dilatation.

$$\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz} = 0.$$

Distortions.

$$\frac{d\eta}{dz}+\frac{d\zeta}{dy} = \Sigma\left[\frac{2\pi}{\lambda}e^{\psi}\left\{(mc'+nb')\cos\phi+(m'c'+n'b')\sin\phi\right\}\right]$$

$$\frac{d\zeta}{dx}+\frac{d\xi}{dz} = \Sigma\left[\frac{2\pi}{\lambda}e^{\psi}\left\{-\left(n'(1+c'^2)+m'b'c'\right)\cos\phi + \left(n(1+c'^2)+mb'c'\right)\sin\phi\right\}\right]$$

$$\frac{d\xi}{dy}+\frac{d\eta}{dx} = \Sigma\left[\frac{2\pi}{\lambda}e^{\psi}\left\{-\left(m'(1+b'^2)+n'b'c'\right)\cos\phi + \left(m(1+b'^2)+nb'c'\right)\sin\phi\right\}\right]$$

Which being multiplied by $-C$ give the tangential pressures Q_1, Q_2, Q_3.

Normal Pressures on the coordinate planes due to the displacements.

$$P_1 = 2C.\Sigma\left[\frac{2\pi}{\lambda}e^{\psi}\left\{(mb'+nc')\cos\phi+(m'b'+n'c')\sin\phi\right\}\right]$$

$$P_2 = -2C.\Sigma\left[\frac{2\pi}{\lambda}e^{\psi}\left\{mb'\cos\phi+m'b'\sin\phi\right\}\right]$$

$$P_3 = -2C.\Sigma\left[\frac{2\pi}{\lambda}e^{\psi}\left\{nc'\cos\phi+n'c'\sin\phi\right\}\right]$$

$$P_1+P_2+P_3 = 0.$$

The normal pressure due to the displacements at any point of the surface of a prism or cylinder described round x is

$$P' = -2C.\Sigma\left[\frac{2\pi}{\lambda}e^{\psi}\{(mb'\cos^2\chi + nc'\sin^2\chi + (mc' + nb')\cos\chi\sin\chi)\cos\phi\right.$$
$$\left.+(m'b'\cos^2\chi + n'c'\sin^2\chi + (m'c' + n'b')\cos\chi\sin\chi)\sin\phi\}\right] \ldots (44).$$

If this pressure is to be null at all points of the surface, we must have $b' = 0$, $c' = 0$, and consequently $l = 0$, $l' = 0$; and the motion is restricted to common exactly-transverse vibrations, for which

$$E = C \quad \text{and} \quad \sqrt{\epsilon} = \sqrt{\left(\frac{Cg}{D}\right)}.$$

Nearly-transverse vibrations therefore cannot be transmitted along a cylindrical or prismatic uncrystallized body whose surface is absolutely free.

ON THE CONNEXION OF INVOLUTE AND EVOLUTE IN SPACE.

By Professor De Morgan.

All that has been done on this subject, so far as I know, is due to Monge, and is found in the tenth volume of the Memoirs of the Academy, pp. 511-550, as presented in 1771. This communication is in great part repeated, word for word, in the *Application de l'Analyse &c.* It is shewn that every curve has an infinite number of evolutes, all lying on its polar surface, or surface of ultimate intersection of normal planes: and that each such evolute is a line of shortest distance, or *stretched thread*, on the polar surface. These theorems are made clear by infinitesimal geometry: and an algebraic mode of finding differential equations to the evolutes is indicated, involving that perfect bar to further general proceedings, indefinite elimination before integration. I cannot find that anything has been done since. The object of this paper is to deduce all the known properties of the evolute in one connected algebraical system, to present an explicit process for its determination, and to shew the connexion of plane and spherical curves.

The time may come when every surface will have its *geometry*, founded on its *right line*, or shortest line between any two points, and its *circle*, or locus of points at equal

shortest distances from a given point. Gauss will be regarded as the founder of this geometry, by his celebrated memoir which shews that the *circle* of any surface has tangents perpendicular to its radii, and that the sum of the angles in a *rectilinear triangle* differs from two right angles by a function of all its curvatures. If *right lines* be drawn in the surface perpendicular to a curve, through every point of it, the ultimate intersections of those right lines (when they do* intersect) give an evolute from which the original curve may be unrolled. The great desideratum, at present, is the further continuation of the coordinate system. Professors Graves and Gudermann have commenced it by their spherical coordinates.

Every curve has, or may have, an evolute on every surface on which it can be placed. But no more is in our power at present, than the consideration of cases in which the *right line* of the surface, connecting the involute and evolute, is *straight*. Hence it may easily be seen that our evolute will be the cuspidal edge of a developable surface, and our involute a *curved* line of curvature of that surface. This point, though clear enough, is not in our definition, but must be deduced from it. That definition is: When two curves have every tangent of one normal to the other, the tangent curve is called the *evolute*, the normal curve the *involute*.

Let v be a subsidiary variable, by which the position of the connecting straight line is determined. Let (x, y, z) and (ξ, η, ζ) be corresponding points on the evolute and involute. Let s and σ be arcs, measured from points taken at pleasure. Let t be the line joining (x, y, z) and (ξ, η, ζ). Let accentuation denote differentiation with respect to v, of which every letter used is a function. The equations of definition are

$$\xi'(\xi - x) + \eta'(\eta - y) + \zeta'(\zeta - z) = 0 \quad \text{.........(1)},$$

$$\xi = x + x'w, \quad \eta = y + y'w, \quad \zeta = z + z'w \quad \text{......(2)},$$

from which

$$x'\xi' + y'\eta' + z'\zeta' = 0 \quad \text{.................(3)}.$$

From (1) and (3)

$$\xi''(\xi - x) + \eta''(\eta - y) + \zeta''(\zeta - z) + \sigma'^2 = 0 \quad \text{...... (4)}.$$

---

* It must remain to be settled whether there exists a surface having curves without evolutes upon it. If we take a ruled surface which is not developable, and draw a curve perpendicular to the generating straight lines (which are certainly *right* lines), there is certainly a curve to which *certain* normal right lines do not contribute an evolute. But through every point of the curve an infinite number of right lines may be drawn, of which others, besides the generating straight lines, may be normal to the curve.

From (1) and (4), the evolute is on the polar surface of the involute. And (1), (4), and any two of (2), by elimination of v and w, give one primitive and one differential equation of the first order, by which to determine the evolute: ξ, η, ζ, being given *a priori* in terms of v. At this point terminates Monge's algebraical treatment of the subject.

Let $dz = pdx + qdy$ on the polar surface of the involute. By (4) we know that p and q are of the same form as if v were constant in (1) so far as it enters in ξ, η, ζ. We have then

$$\xi' + \zeta' p = 0, \quad \eta' + \zeta' q = 0 \text{ ....(5).}$$

Let $P = 0$ be the polar surface, and $dP = P_x dx + \&c.$ (5) gives

$$P_x : P_y : P_z :: \xi' : \eta' : \zeta' \text{ .... (6).}$$

From (2), by substitution of ξ', η', ζ', in (3)

$$s'^2 + s'^2 w' + s's'' w = 0, \quad \text{or } w = \frac{c-s}{s'} \text{ ....(7),}$$

whence $t^2 = (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2 = s'^2 w^2 = (c-s)^2$,

or $$t = \pm (c - s) \text{ ....(8),}$$

from which the mode of unrolling the involute from the evolute is proved in the usual way.

By (7), the set (2) gives

$$\xi = x + (c-s)\frac{dx}{ds}, \quad \eta = y + (c-s)\frac{dy}{ds}, \quad \zeta = z + (c-s)\frac{dz}{ds} \text{ ... (9),}$$

whence

$$\frac{d\xi}{ds} = (c-s)\frac{d^2x}{ds^2}, \quad \frac{d\eta}{ds} = (c-s)\frac{d^2y}{dx^2}, \quad \frac{d\zeta}{ds} = (c-s)\frac{d^2z}{ds^2} \text{ ...(10),}$$

or (6) $$P_x : P_y : P_z :: \frac{d^2x}{ds^2} : \frac{d^2y}{ds^2} : \frac{d^2z}{ds^2};$$

from which, by the known properties of the shortest line on a surface, it follows that any arc of the evolute is the shortest distance on the polar surface between its two ends. Hence, to draw a surface through a given curve on which that curve may be a shortest line, we must take either the common polar surface of its involutes, or a surface which touches that polar surface in the given curve.

Given the evolute, to find the involutes. It is granted throughout that a curve is found when its coordinates are severally expressed in terms of one variable. Fi terms of v by common integration of $ds = \sqrt{(x'^2 + y'^2 +}$ and substitute in (9).

Given the involute, to find the evolutes. Let $t = +(c - s)$. The set (9) gives

$$\xi = x - t\frac{dx}{dt}, \quad d\xi = -t\,d\left(\frac{dx}{dt}\right) \text{ \&c.,}$$

$$\text{or } \frac{dx}{dt} = -\int\frac{d\xi}{t} \text{ \&c.: but } \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 1,$$

$$\text{whence } x = \xi - t\int\frac{d\xi}{t}, \; y = \eta - t\int\frac{d\eta}{t}, \; z = \zeta - t\int\frac{d\zeta}{t} \ldots (11),$$

$$\left(\int\frac{d\xi}{t}\right)^2 + \left(\int\frac{d\eta}{t}\right)^2 + \left(\int\frac{d\zeta}{t}\right)^2 = 1 \;\ldots\ldots\ldots\; (12);$$

from which last t is to be determined in terms of v, of which ξ, η, ζ, are given functions. Let these integrals be called A, B, C; we have then

$$\left.\begin{array}{l} A^2 + B^2 + C^2 = 1, \;\; tA' = \xi', \;\; tB' = \eta', \;\; tC' = \zeta' \\ A\xi' + B\eta' + C\zeta' = 0, \quad \text{since } A' : B' : C' :: \xi' : \eta' : \zeta' \end{array}\right\} \ldots (13),$$

$$A'\xi' + B'\eta' + C'\zeta' + A\xi'' + B\eta'' + C\zeta'' = 0.$$

Hence we get

$$A'\sigma'^2 + \xi'(A\xi'' + B\eta'' + C\zeta'') = 0, \quad B'\sigma'^2 + \eta'(A\xi'' + \text{\&c.}) = 0,$$

$$C'\sigma'^2 + \zeta'(A\xi'' + \text{\&c.}) = 0 \ldots\ldots\ldots\ldots\ldots (15).$$

From (13) obtain B and C in terms of A, ξ', η', ζ', and substitute in the first of (15). We have then a differential equation of the first order to determine A, and thence B and C. The equations of the evolute are then

$$x = \xi - \frac{\xi' A}{A'}, \quad y = \eta - \frac{\eta' B}{B'}, \quad z = \zeta - \frac{\zeta' C}{C'} \ldots\ldots (16).$$

But the following process gives a better chance of obtaining a result. Assume

$$\lambda A + \mu B + \nu C = k \;\ldots\ldots\ldots\ldots\ldots\ldots\; (17),$$

λ, μ, ν being new functions of v to be determined, and k an arbitrary constant. Differentiate and substitute from (15), and it will be found that the result is identically satisfied if we can obtain λ, μ, ν, so as to satisfy

$$\left.\begin{array}{l} \lambda'\sigma'^2 - \xi''(\lambda\xi' + \mu\eta' + \nu\zeta') = 0 \\ \mu'\sigma'^2 - \eta''(\lambda\xi' + \mu\eta' + \nu\zeta') = 0 \\ \nu'\sigma'^2 - \zeta''(\lambda\xi' + \mu\eta' + \nu\zeta') = 0 \end{array}\right\} \ldots\ldots\ldots\; (18).$$

We have already two relations between A, B, C; if then we can obtain a third, even by a *partial solution* of (18), other than $\lambda = \xi', \mu = \eta', \nu = \zeta'$, which is already in use, we have completely solved the question.

Since $\lambda : \mu' : \nu' :: \xi'' : \eta'' : \zeta''$, the result must be of the form $\lambda = \int Q d\xi'$, $\mu = \int Q d\eta'$, $\nu = \int Q d\zeta'$, and substitution of these reduces each of the set (18) to

$$\xi'\int\xi' dQ + \eta'\int\eta' dQ + \zeta'\int\zeta' dQ = 0,$$

which we clearly see cannot be integrated independently of all specific relation between ξ, η, ζ, except by $Q =$ const. Hence one partial solution of (18) dependent upon the connexion of ξ, η, ζ, is the necessary and sufficient condition for the determination of the evolute of the locus of (ξ, η, ζ).

I shall proceed to apply this process to the case of a spherical curve, drawn upon the surface of the sphere $\xi^2 + \eta^2 + \zeta^2 = a^2$. It will appear that, in separating the plane curve from all curves of double curvature, much analogy has been lost. The sphere of finite radius is a surface on which every shortest line is a plane section: on the sphere of infinite radius the converse is also true. But this convertibility is, for our present purpose, rather the accident of an individual case than the difference of a species. From the theorem that *every* shortest line is a plane *and normal* section (a property which belongs to the sphere only, that of infinite radius being included) it follows that the polar surface of any spherical curve cuts the sphere in its spherical evolute. The simplest considerations will shew this. If P and P' be points infinitely near on a spherical curve, and if the great circles perpendicular to the curve through P and P' meet in Q, the planes of those great circles are normals to the curve, and Q is a point in the polar line of P. The polar surface of the curve is then a cone, of which the vertex is the centre of the sphere, and the directrix the spherical evolute of the curve. When the radius is infinite, the spherical evolute becomes the ordinary plane evolute, and the polar surface becomes the cylinder described on that evolute perpendicular to the plane of the involute. The evolutes, as is well known, are the screws of that cylinder, or curves which always make the same angle with the generating lines of the cylinder. These generating lines are radii of the sphere of infinite radius, and it is very easily shewn that the propert[illegible] of spheres of finite radius; namely, that the ang[illegible] the radius drawn to a point in the involute with [illegible] line drawn from thence to touch the evolute [illegible]

Let C be the centre, P and P' two points infinitely near on the involute, CM and CM' their polar lines. Take any point T in CM, join PT, $P'T$, and produce the latter to meet CM' in T': then TT' is the element of an evolute by which PP' is described. Now $CP = CP'$, and TP is known to differ from TP' only by a quantity of the second order; while CT is common. Hence the angles CPT, $CP'T$ only differ by a quantity of the second order, or $d(CPT) = 0$, or $CPT =$ const, as asserted. Hence it follows that PT is determined in direction by being perpendicular to the tangent, and at a given angle with CP.

A little consideration will now enable us to predict the results which the previous method will establish. Let the tangent be denoted by G, the line PT by T, the radius CP by A, and let the cosines of the angles they make with the axes be denoted by x, y, z, suffixed. Let κ be the constant angle made by T with A. We have then, over and above the necessary relations $T_x^2 + T_y^2 + T_z^2 = 1$, &c. the following equations,

$$T_xA_x + T_yA_y + T_zA_z = \cos\kappa,$$

$$G_xA_x + G_yA_y + G_zA_z = 0,$$

$$G_xT_x + G_yT_y + G_zT_z = 0;$$

whence $\qquad G_x \sin\kappa = \pm(A_yT_z - A_zT_y)$ &c.

$$A_x \cos\kappa = T_x + A_y(A_xT_y - A_yT_x) + A_z(A_xT_z - A_zT_x)$$

$$= T_x \pm \sin\kappa(A_yG_z - A_zG_y),$$

or $\qquad T_x = \cos\kappa . A_x \pm \sin\kappa(A_yG_z - A_zG_y),$

or $\qquad \dfrac{dx}{ds} = \cos\kappa . \dfrac{\xi}{a} \pm \sin\kappa\, \dfrac{1}{a}\left(\eta\dfrac{d\zeta}{d\sigma} - \zeta\dfrac{d\eta}{d\sigma}\right),$

and similarly for y and z. But $dx : ds$ is the A in equation (12) of the preceding process.

We now return to equations (18). If we assume

$$\lambda = \int v d\xi' = \xi'v - \xi, \quad \mu = \eta'v - \eta, \quad \nu = \zeta'v - \zeta,$$

we find each of (18) reduced to $\xi\xi' + \eta\eta' + \zeta\zeta' = 0$, which is true on every spherical curve. Accordingly (17), allowing for the second in (13), becomes the last of the following set:

$$A^2 + B^2 + C^2 = 1, \quad A\xi' + B\eta' + C\zeta' = 0, \quad A\xi + B\eta + C\zeta = k \ldots (19),$$

e two first of which are already established. Combining ese with the equation of the sphere, we find

$$Aa^2 = k\xi + \sqrt{(a^2 - k^2)}\,\frac{\eta\zeta' - \zeta\eta'}{\sigma'} \text{ \&c. .... (20),}$$

om which we deduce

$$\frac{A'}{\xi'} = \frac{B'}{\eta'} = \frac{C'}{\zeta'} = \frac{1}{a^2}\left\{k - \sqrt{(a^2 - k^2)}\,\frac{\xi_{,,}\xi + \eta_{,,}\eta + \zeta_{,,}\zeta}{\sigma'^2}\right\} \text{...(21),}$$

where $\xi_{,,} = \eta'\zeta'' - \zeta'\eta''$, &c. Now $\xi''\xi$ + &c. is $M\sqrt{(\xi_{,,}^2 + \&c.)}$, where M is the perpendicular let fall from the centre upon he osculating plane at (ξ, η, ζ). But $\sigma'^3 : \sqrt{(\xi''^2 + \&c.)}$ is R, he radius of curvature of the involute. From this, t being $\xi' : A'$, we have

$$\frac{1}{t} = \frac{1}{a^2}\left\{k - \sqrt{(a^2 - k^2)}\,.\,\frac{M}{R}\right\}.$$

Returning to the diagram just now described, from P draw PR perpendicular upon CT: then $CR = M$, $PR = R$. Let $\angle CPR = \alpha$, and let $k : a = \cos\kappa$. Then

$$t = \frac{a\cos\alpha}{\cos(\alpha \pm \kappa)},$$

whence it is seen that the constant angle κ is no other than CPT.

Returning to the set (16) we have three equations for the volute of a spherical curve, in which it will be found that $\sqrt{(a^2 - k^2)}$ is a factor of each of x, y, z. When $k = a$, the volute is then the vertex: and it is clear that the centre is n evolute of every spherical curve. Moreover, when $M = 0$, that is, when the osculating plane happens to pass through the centre, $kt = a^2$. But $k < a$, whence $t > a$. This, however, is not a condition the failure of which gives imaginary evolutes, but is fulfilled by the figure.

There is one evolute which we may call the *principal* evolute, being that which, when the radius becomes infinite, is the plane evolute of the limiting plane curve. It is that in which κ is a right angle, or in which we proceed from the involute to the evolute along a tangent of the sphere.

Though it may be desirable to deduce the particular case om the general one, yet it must be remembered that the ndamental equations become

$$\xi' x + \eta' y + \zeta' z = 0 \text{ .... (1),}$$

$$\xi'' x + \eta'' y + \zeta'' z = 0 \text{ .... (4).}$$

whence
$$\frac{x}{\xi_{\prime\prime}} = \frac{y}{\eta_{\prime\prime}} = \frac{z}{\zeta_{\prime\prime}}$$
is the equation of the polar line of (ξ, η, ζ). The elimination of v from the last pair gives the polar surface, and this combination with $x^2 + y^2 + z^2 = a^2$ gives the spherical evolute.

April 23, 1851.

ON A MECHANICAL EXPERIMENT CONNECTED WITH THE ROTATION OF THE EARTH.

By Henry Wilbraham.

Since much attention has been excited by the late successful experiment, shewing that if a weight suspended by a string be set oscillating as a pendulum the plane of oscillation will not revolve with the revolution of the earth, and consequently, to a person unconscious of the earth's rotation, will appear to revolve, the following investigation relating to it may perhaps have some interest.

It occurred to me on reading the account of the experiment, that supposing the pendulum to be put in motion in the natural way, that is to say, by drawing it aside and setting it loose to swing, some actual rotation might be communicated to the plane of oscillation, and therefore some variation made in the velocity of its apparent rotation, by the horizontal motion communicated to it on setting it loose in consequence of the hand or instrument at the instant holding it, though apparently at rest, being affected with the rotation of the earth. The following is a calculation of the nature and amount of the effect thus produced.

As the velocity communicated is in a direction perpendicular to the plane joining the point from which the weight is loosed with the position of the string when at rest, the motion will not be accurately an oscillation in that plane, but will be a curve of double curvature described on a sphere whose centre is the point of suspension, and its projection on a horizontal plane will be a very eccentric ellipse, to investigate which the component, parallel to the horizontal plane, of the force acting on the weight need alone be considered. If a be the length of the string, r the perpendicular distance of the weight at any point from the position of the string when at rest, c the va[illegible] the weight is loosed,

e angular velocity, due to the earth's rotation, of the al plane (which angular velocity is equal to that of the 's rotation multiplied by the sine of the latitude of the), and therefore ωc the horizontal velocity communicated ting loose the weight; it is easily seen from the equa- of motion, that the whole force acting on the body may olved into—1st, a vertical force, which is immaterial to esent purpose; 2ndly, a force

$$\frac{3g\sqrt{(a^2-r^2)}+\omega^2c^2-2g\sqrt{(a^2-c^2)}}{a^2}\,r$$

ds the vertical line drawn through the point of suspen- which, supposing c to be much less than a, ω being very , is nearly

$$\left(1+\frac{c^2}{a^2}\right)\frac{gr}{a}-\frac{3gr^3}{2a^3}\text{, or } \mu r-\lambda r^3,$$

$$\mu=\left(1+\frac{c^2}{a^2}\right)\frac{g}{a}\text{, and } \lambda=\frac{3g}{2a^3}.$$

ffect of the last term is to make the apses of the ellipse, if μr were the only force would be described, to pro- If we take the common equation of motion in a plane

$$\frac{h^2}{r^2}\left\{\frac{d\left(\frac{1}{r}\right)}{d\theta^2}+\frac{1}{r}\right\}+\mu r-\lambda r^3=0,$$

it may be written, putting z for $\frac{1}{r^2}$,

$$\frac{d^2z}{d\theta^2}-\frac{1}{2z}\left(\frac{dz}{d\theta}\right)^2+2z+\frac{2\mu}{h^2}z^{-1}-\frac{2\lambda}{h^2}z^{-2}=0,$$

e the method of variation of parameters, taking as the of the solution $z=A'+B'\cos 2(\theta+\epsilon')$, where

$$A'^2-B'^2=\frac{4\mu}{h^2},$$

erefore $$A'\frac{dA'}{d\theta}=B'\frac{dB'}{d\theta},$$

, on differentiating twice in the usual way and elimi- $\frac{dA'}{d\theta}$ and $\frac{dB'}{d\theta}$,

$$\frac{d\epsilon'}{d\theta} = -\frac{\lambda}{2B'h^2z^2}\,\frac{\dfrac{B'}{A'} + \cos 2(\theta + \epsilon')}{1 + \dfrac{B'}{A'}\cos 2(\theta + \epsilon')},$$

$$= -\frac{\lambda}{A'^2B'h^3}\,\frac{\dfrac{B'}{A'} + \cos 2(\theta + \epsilon')}{\left\{1 + \dfrac{B'}{A'}\cos 2(\theta + \epsilon')\right\}^3};$$

therefore, putting A, B, ϵ for A', B', ϵ' in the small term, and integrating from 0 to π,

$$\epsilon' = \epsilon + \frac{\lambda\pi}{4(A^2 - B^2)^{\frac{3}{2}}h^3}.$$

The latter term is the progress of the apse in one semi-revolution of the body, or, as it may otherwise be called, the actual angular motion of the plane of oscillation during one oscillation. The whole actual angular motion of the plane of oscillation during the period in which from the earth's rotation a vertical plane fixed with respect to the earth revolves through 360°, will be to that during one oscillation as the time of such revolution through 360° to the time of an oscillation, that is, as $\frac{2\pi}{\omega}$ to $\pi\sqrt{\frac{a}{g}}$. This whole actual angular motion then is

$$\frac{\lambda\pi}{2(A^2 - B^2)^{\frac{3}{2}}\omega h^3}\sqrt{\frac{g}{a}},$$

which, on substituting for $A^2 - B^2$ its value $\frac{4\pi}{h^2}$ for h its value ωc^2, and for λ and μ the values given above, becomes

$$\frac{3\pi ac^2}{32(a^2 + c^2)^{\frac{3}{2}}}, \text{ or } \frac{3\pi c^2}{32a^2} \text{ nearly,}$$

which, reduced to degrees and minutes, is $16° \, 52' \, 30'' \times \frac{c^2}{a^2}$.

Hence the plane of oscillation will in T hours revolve through

$$\left(360° - \frac{c^2}{a^2}\,16° \, 52' \, 30''\right)\frac{T \sin . \text{latitude}}{24},$$

if there be no resistance of the air. The resistance being a tangential force, has no direct permanent effect on the line of apses of the ellipse, but as it continually diminishes the major axis and the velocity in direction of the major axis, and has an exceedingly sm[illegible]t on the minor axis and

the velocity in that direction, the latter velocity may be supposed constant while c is diminished to c'; and the result will be the same as if $\omega c'$ had been substituted for ωc as the original transverse velocity. Consequently, when c becomes c', the angular velocity of the line of apses is to its primitive velocity as c' to c, and the line of apses will in T hours revolve through

$$\left(360^\circ - \frac{cc''}{a^2}\, 16^\circ\, 52'\, 30''\right) \frac{T \sin . \text{latitude}}{24},$$

where c'' is between the maximum values of r at the beginning of the motion and at the end of the observation respectively.

April 15, 1851.

ON THE INDEX SYMBOL OF HOMOGENEOUS FUNCTIONS.

By Robert Carmichael, A.B., Trinity College, Dublin.

It is proposed in the following paper to extend a method of analysis adopted by Professor Boole, in the *Philosophical Transactions* of the year 1844, for the solution of linear differential equations of a certain class, to the solution of partial differential equations of a corresponding class. The solutions of such partial differential equations will be found to be unaffected by the number of independent variables which the equation may contain, but more especial reference is made to ordinary partial differential equations, containing but two independent variables, x and y. In the next place, the same symbol by which this extension is effected, is employed for the discovery of general theorems. Finally, by its application to the subject of Definite Integrals, it will be seen that valuable results can be obtained, and some examples are furnished.

The relation which the result of the operation of the symbol employed upon any homogeneous function bears to the degree of the function, seems to give ground for the appellation Index Symbol.

In conclusion, the writer begs to express in the most ample manner his acknowledgments to the author above named.

1. In general, if $u_m = f(x_1, x_2 \ldots\ldots x_n)$

be a homogeneous function of m^{th} degree between $x_1, x_2 \ldots x_n$,

$$x_1 \frac{du_m}{dx_1} + x_2 \frac{du_m}{dx_2} + \ldots + x_n \frac{du_m}{dx_n} = mu_m,$$

or, putting the operating symbol

$$x_1 \frac{d}{dx_1} + x_2 \frac{d}{dx_2} + \ldots + x_n \frac{d}{dx_n} = \nabla,$$

we have $$\nabla . u_m = mu_m,$$

and generally $$\nabla^p . u_m = m^p . u_m.$$

Hence the theorem

$$f(\nabla).u_m = f(m).u_m \ldots\ldots\ldots\ldots (1),$$

which is an extension of the theorem

$$f(xD).x^m = f(m).x^m.$$

In fact, x^m is a particular homogeneous function of m^{th} degree, and xD is the first term of ∇.

2. Now if U be any mixed rational function of $x_1\ x_2 \ldots x_n$, it can, in general, be put under the form

$$U = u_0 + u_1 + u_2 + \ldots + u_m,$$

and we have a theorem for mixed rational functions corresponding to (1), namely

$$f(\nabla).U = f(0)\,u_0 + f(1)\,u_1 + f(2)\,u_2 + \ldots + f(m)\,u_m \ldots (2).$$

As an example, let the result of the operation of e^∇ on U be investigated. Then

$$e^\nabla . U = u_0 + eu_1 + e^2u_2 + \ldots + e^m u_m.$$

3. Since x_2, x_3, &c. are constant relative to x_1, and therefore $\frac{d}{dx_1}$, $\frac{d}{dx_2}$, &c. commutative, writing

$$\nabla_2 = x_1^2 \frac{d^2}{dx_1^2} + x_2^2 \frac{d^2}{dx_2^2} + \ldots\ldots + 2x_1x_2 \frac{d^2}{dx_1 dx_2} + \ldots\ldots$$

$$\nabla_3 = x_1^3 \frac{d^3}{dx_1^3} + x_2^3 \frac{d^3}{dx_2^3} + \ldots\ldots + 3x_1^2x_2 \frac{d^3}{dx_1^2 dx_2} + \ldots\ldots$$
$$+ 3x_1x_2^2 \frac{d^3}{dx_1 dx_2^2} + \ldots\ldots$$

&c.,

we have $$\nabla(\nabla - 1) = \nabla_2,$$

∇'

and generally

$$\nabla(\nabla-1)\ldots\ldots(\nabla-n+1)=\nabla_n \ldots\ldots (3),$$

a theorem analogous to Professor Boole's

$$xD(xD-1)\ldots(xD-n+1)=x^nD^n.$$

As an example of the operation of this latter symbol, let the subject be x^n, and

$$xD(xD-1)\ldots(xD-n+1).x^n=1.2.3\ldots n.x^n.$$

To which we have the corresponding theorem for homogeneous functions

$$\nabla(\nabla-1)\ldots(\nabla-m+1).u_m=1.2.3\ldots m.u_m.$$

4. Again, as the symbol xD furnishes solutions of the class of ordinary differential equations represented by

$$Ax^\alpha\frac{d^\alpha y}{dx^\alpha}+Bx^\beta\frac{d^\beta y}{dx^\beta}+\ldots.=X,$$

in the form $\quad y=F(xD).X+F(xD).0,$

where $F(xD)=\left\{\begin{array}{l}A.xD(xD-1)\ldots(xD-\alpha+1)\\+B.xD(xD-1)\ldots(xD-\beta+1)\\+\&c.\end{array}\right\}^{-1}$:

in like manner we obtain the solutions of the particular class of partial differential equations represented by

$$\left.\begin{array}{l}A\left(x^\alpha\dfrac{d^\alpha z}{dx^\alpha}+\alpha x^{\alpha-1}y\dfrac{d^\alpha z}{dx^{\alpha-1}dy}+\dfrac{\alpha(\alpha-1)}{1.2}x^{\alpha-2}y^2\dfrac{d^\alpha z}{dx^{\alpha-2}dy^2}+\ldots\right)\\+B\left(x^\beta\dfrac{d^\beta z}{dx^\beta}+\beta x^{\beta-1}y\dfrac{d^\beta z}{dx^{\beta-1}dy}+\dfrac{\beta(\beta-1)}{1.2}x^{\beta-2}y^2\dfrac{d^\beta z}{dx^{\beta-2}dy^2}+\ldots\right)\\\qquad+\&c.\end{array}\right\}=\Theta,$$

where Θ is a given function of x and y, in the form

$$z=F(\nabla).\Theta+F(\nabla).0 \ldots\ldots\ldots\ldots (4),$$

in which

$$F(\nabla)=\{A\nabla(\nabla-1)..(\nabla-\alpha+1)+B\nabla(\nabla-1)..(\nabla-\beta+1)+\&c.\}^{-1},$$

and in which the value of *the first* term can be had at once by formula (2). It appears, then, that as far as equations of this class are concerned, the *number* and *character* of *the arbitrary functions* in a solution are unaffected by the number of independent variables which the equation may contain, and are solely dependent on its order.

When the roots of the equation

$$A.\nabla(\nabla-1)\ldots(\nabla-\alpha+1)+B.\nabla(\nabla-1)\ldots(\nabla-\beta+1)+\&c.=0$$

are all real and unequal, the arbitrary portion of the soluti[on] is of the form

$$u_m + u_n + u_p + \&c.$$

When it contains a equal roots, whose common value is m, its form is

$$u_m \{\log x + \log y\}^{a-1} + v_m \{\log x + \log y\}^{a-2} + \&c.$$
$$+ u_n + u_p + \&c.,$$

where u_m, v_m, &c. are different arbitrary homogeneous func-tions of the same degree. Finally, when this equation con-tains pairs of imaginary roots, the form of the arbitrary portion of the solution is

$$u_{m+n\sqrt{-1}} + u_{m-n\sqrt{-1}} + \&c. + u_p + \&c.$$

5. As an example of this method of solution of partial differential equations, let it be required to find the integral of

$$x^2r + 2xys + y^2t - n(xp + yq - z) = 0.$$

When thrown into the symbolic shape, this equation becomes

$$\nabla(\nabla - 1)z - n(\nabla - 1)z = 0,$$

and the solution is given by

$$z = \frac{1}{(\nabla - n)(\nabla - 1)} . 0 = \frac{N}{\nabla - n} . 0 + \frac{N'}{\nabla - 1} . 0,$$

or is at once, including N and N' in the homogeneous func-tions, *which are given in degree but arbitrary in form,*

$$z = u_n + u_1.^*$$

As *a second* example, required the integral of

$$x^2r + 2xys + y^2t = \Theta_m + \Theta_n,$$

where Θ_m, Θ_n are given homogeneous functions in x and y

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* If $n = -\dfrac{3}{m-1}$, this value of z renders the integral

$$\iint (px + qy - z)^m \, dx\, dy,$$

a maximum or a minimum within certain assigned limits (Jellett's *Calculus of Variations*, p. 253).

In general, by the method stated above, it can be readily seen that the form of the function w, which, for certain assigned limits, renders the sym-metrical multiple integral containing p independent variables

$$\int dx \int dy \int dz \,.\left(x\frac{dw}{dx} + y\frac{dw}{dy} + z\frac{dw}{dz} + \&c. - w\right)^m,$$

a maximum or a minimum is, as before,

$$w = u_n + u_1,$$

where $$n = -\frac{p+1}{m-1}.$$

[redacted]

f the m^{th} and n^{th} degrees respectively. Then

$$z = \frac{1}{\nabla(\nabla-1)}\{\Theta_m + \Theta_n\} + \frac{1}{\nabla(\nabla-1)}\cdot 0,$$

r, by (1),

$$z = \frac{\Theta_m}{m(m-1)} + \frac{\Theta_n}{n(n-1)} + u_0 + u_1,$$

which is the required solution.

As *a third* example, let the integral of the partial differential equation of the third order in three independent variables $x, y, z,$

$$\left.\begin{aligned} & x^3\frac{d^3u}{dx^3} + y^3\frac{d^3u}{dy^3} + z^3\frac{d^3u}{dz^3} \\ & + 3\left(x^2y\frac{d^3u}{dx^2dy} + x^2z\frac{d^3u}{dx^2dz} + xy^2\frac{d^3u}{dxdy^2} + \&c.\right)\end{aligned}\right\} = \Phi_m + \Phi_n$$

be investigated, Φ_m, Φ_n being given homogeneous functions in $x, y, z,$ of the m^{th} and n^{th} degrees, respectively.

The required solution is

$$u = \frac{\Phi_m}{m(m-1)(m-2)} + \frac{\Phi_n}{n(n-1)(n-2)} + u_0 + u_1 + u_2.$$

6. Supposing two of the independent variables to vanish in the last example and one in each of the preceding, we are at once furnished with the solutions of the following ordinary linear differential equations:

$$\left.\begin{aligned} x^3\frac{d^3y}{dx^3} &= ax^m + by^n, \\ x^2\frac{d^2y}{dx^2} &= ax^m + by^n, \\ x^2\frac{d^2y}{dx^2} - nx\frac{dy}{dx} + ny &= 0,\end{aligned}\right\}$$

which are, respectively,

$$\left.\begin{aligned} & y = \frac{ax^m}{m(m-1)(m-2)} + \frac{bx^n}{n(n-1)(n-2)} + C_0 + C_1x + C_2x^2, \\ & y = \frac{ax^m}{m(m-1)} + \frac{bx^n}{n(n-1)} + C_0 + C_1x, \\ & y = C_nx^n + C_1x.\end{aligned}\right\}$$

Now it must be remembered that the solutions given by the symbol ∇ are the same in form, no matter how large the

number of independent variables may be. For instance, the solution of

$$\left.\begin{array}{l} x_1^2 \dfrac{d^2w}{dx_1^2} + x_2^2 \dfrac{d^2w}{dx_2^2} + x_3^2 \dfrac{d^2w}{dx_3^2} \text{ \&c.} \\ + 2\left(x_1x_2 \dfrac{d^2w}{dx_1dx_2} + x_2x_3 \dfrac{d^2w}{dx_2dx_3} + \text{\&c.}\right) \end{array}\right\} = \Psi_m + \Psi_n \dots (e),$$

is exactly the same in form as that of

$$x^2 \frac{d^2z}{dx^2} + 2xy \frac{d^2z}{dxdy} + y^2 \frac{d^2z}{dy^2} = \Theta_m + \Theta_n \dots \dots (b);$$

namely
$$w = \frac{\Psi_m}{m(m-1)} + \frac{\Psi_n}{n(n-1)} + u_0 + u_1,$$

the only difference being in the number of independent variables contained in u_0, u_1.

Hence, in order to find *the form* of the integral of an equation of the class (a) containing any number n of independent variables, it is sufficient to have found the form of the integral of a corresponding equation (b) containing any lower number of independent variables. Hence is derived the following conclusion, which seems to be of some importance.

The solution of an ordinary linear differential equation of the class represented by (No. 4)

$$Ax^\alpha \frac{d^\alpha y}{dx^\alpha} + Bx^\beta \frac{d^\beta y}{dx^\beta} + \text{\&c.} = X \dots\dots\dots (c),$$

being given, we can at once write down the solution of a partial differential equation of the class represented by

$$\left.\begin{array}{l} A\left(x^\alpha \dfrac{d^\alpha z}{dx^\alpha} + \alpha x^{\alpha-1}y \dfrac{d^\alpha z}{dx^{\alpha-1}dy} + \text{\&c.}\right) \\ + B\left(x^\beta \dfrac{d^\beta z}{dx^\beta} + \beta x^{\beta-1}y \dfrac{d^\beta z}{dx^{\beta-1}dy} + \text{\&c.}\right) \\ + \text{\&c.} \end{array}\right\} = \Theta \dots (d),$$

by substituting for each term in the solution of the ordinary linear differential equation in which *an arbitrary constant* is introduced, such as $C_m x^m$, *a homogeneous function of the same degree, but of arbitrary form in x and y.*

And thus the solution of partial differential equations of the class (d) is reduced to the solution of the corresponding class (c) of ordinary li ... tions, which is in all cases furnished ... Boole.

7. The subject of the application of the symbol ∇ to the solution of partial differential equations having been now examined at some length, it may be well to investigate whether it may not be possible to discover, by the aid of the same symbol, extensions of known theorems. Two examples will suffice.

(I.) It is proposed to commence with those of which such important use is made in the calculus of operations, namely

$$\phi(D)\, PQ = P\phi(D)\, Q + P'.\phi'(D)\, Q + \frac{P''}{1.2}.\phi''(D)\, Q + \&c.$$

and

$$P\phi(D)\, Q = \phi(D)\, PQ - \phi'(D).\, P'Q + \frac{\phi''(D)}{1.2}.\, P''Q - \&c.$$

where P and Q only contain x and D is $\dfrac{d}{dx}$; and to shew that they admit of the following extensions,

$$\Phi(\nabla)\, PQ = P\Phi(\nabla)\, Q + \nabla P.\Phi'(\nabla)\, Q + \frac{\nabla^2 P}{1.2}.\Phi''(\nabla)\, Q + \&c.,$$

and

$$P\Phi(\nabla)\, Q = \Phi(\nabla)\, PQ - \Phi'(\nabla).\nabla P.Q + \frac{\Phi''(\nabla)}{1.2}.\nabla^2 P.Q - \&c.,$$

where P and Q are functions of the n independent variables

$$x_1, x_2, \ldots x_n,$$

and ∇ is the symbol before employed.

Both these theorems are obvious upon the substitutions

$$x_1 = e^{\theta_1},\ \ x_2 = e^{\theta_2},\ \ \&c.;$$

since, then,

$$\nabla.PQ = \left(\frac{d}{d\theta_1} + \frac{d}{d\theta_2} + \ldots.\right) PQ = P\nabla Q + Q\nabla P.$$

(II.) By the same substitutions is obtained the theorem

$$F(\nabla + \Theta' + \Phi')\, e^{-(\Theta+\Phi)}\, u = e^{-(\Theta+\Phi)}\, F(\nabla)\, e^{\Theta+\Phi}\, u,$$

which obviously admits of still further generalization, but which in its present form may be made use of for the solution of partial differential equations, in the same manner as the theorem

$$F(D + \Theta')\, e^{-\Theta} u = e^{-\Theta} F(D)\, e^{\Theta} u$$

is made use of for the solution of ordinary linear differential tions.

The subject of Definite Integrals furnishes many in- results from the employment of the symbol ∇; but be impossible here, adequately, to follow up such

an investigation in its details. A general example, with its application to a particular case, will serve to illustrate its importance.

If

$$\int dx \int dy \int dz \ldots a^{\phi(x,\, y,\, z,\, \&c.)} . b^{\chi(x,\, y,\, z,\, \&c.)} . c^{\psi(x,\, y,\, z,\, \&c.)} \ldots = K,$$

the quantities a, b, c, &c. being unconnected with the limits,

$$\int dx \int dy \int dz \ldots F(\phi + \chi + \psi + \&c.)\, a^{\phi} . b^{\chi} . c^{\psi} \ldots = F(\nabla) K,$$

where now $\quad \nabla = a \dfrac{d}{da} + b \dfrac{d}{db} + c \dfrac{d}{dc} + \&c.$

This is obvious since, from the suppositions made relative to a, b, c, &c., we can operate with the symbol ∇ under the integral signs. It will be seen that this result bears a strong resemblance to Liouville's well-known extension of Dirichlet's Integral. As a particular case of the above, it can be easily proved that

$$\int_0^\infty \int_0^\infty \int_0^\infty a^{-x^2} . b^{-y^2} . c^{-z^2} . dx\, dy\, dz = \tfrac{1}{8} \pi^{\frac{3}{2}} \frac{1}{\{\log a . \log b . \log c\}^{\frac{1}{2}}},$$

whence

$$\int_0^\infty \int_0^\infty \int_0^\infty \phi(x^2 + y^2 + z^2)\, a^{-x^2} . b^{-y^2} . c^{-z^2} . dx\, dy\, dz$$

$$= \tfrac{1}{8} \pi^{\frac{3}{2}} \phi(-\nabla) \frac{1}{\{\log a . \log b . \log c\}^{\frac{1}{2}}}.$$

And, in general, it may be observed that, where the element of the integral is of such a form as to exhibit the variables $x, y, z,$ &c. only in the indices of known quantities $a, b, c,$ &c., which are at the same time unconnected with the limits employed, an extension of the integral in question may be had by the aid of the symbol ∇.

June, 1851.

MATHEMATICAL NOTES.

I.—To the Editor of the Cambridge and Dublin Mathematical Journal.

Sir,—I hope that you will permit me, for the sake not of controversy but of peace, to say a few explanatory words upon the Note to which Mr. Sylvester has replied in the last Number of the *Journal.*

Most willingly do I acknowledge that in the sense stated by Mr. Sylvester in the Reply, his theorem is perfectly original. But it appeared to me that it was not thus announced; and further, that special comparison was invited between it and my owr ter said, " As

I have alluded to Prof. Boole's theorem relative to Linear Transformations, it may be proper to mention my theorem on the subject which is of a much more general character, and includes Mr. Boole's (so far as it refers to Quadratic Functions) as a corollary to a particular case."

The remainder of Mr. Sylvester's Postscript was occupied with the comparison, and no reference was therein made to the Theory of Linear Transformations, of which the selected theorem of mine formed a part. Now I had published that Theory as a general one. It appeared to me then that it was incumbent upon me to shew that it included the case contemplated in Mr. Sylvester's theorem—included that case, I mean, as concerned the discovery of the algebraic relations among the constants of the transformed functions, the avowed object of the theory. To shew this was the design of my Note. If my language in relation to Mr. Sylvester's theorem bears any construction unwarranted by this view, I declare that such construction was not designed.

I shall not enter into any defence of the particular points of my Note referred to by Mr. Sylvester, because their importance does not appear to me sufficient to warrant further controversy, but it is due to myself to say that the opinions which I expressed, whether right or wrong, were founded upon careful examination. In asserting that my method was practically more convenient than Mr. Sylvester's, I rested upon the evidence of examples. In affirming the equivalence of Mr. Sylvester's theorem to the result afforded by my method, I had the general proof before me. Both these I will forward to the *Journal* if required. Of any wish either to impose upon others by authority, or to detract from the just claims of a fellowlaborer I am incapable, and it would cause me deep regret if I thought that traces of such a feeling were really manifest in the language of my Note. But I cannot claim this consideration for my own motives without fully according the same to those of Mr. Sylvester, convinced that the present misunderstanding is simply the result of hasty judgment.

It is gratifying to me to see from the announcements made in Mr. Sylvester's last paper and from its references to foreign memoirs which I have not the opportunity of consulting, how imperfect is the sketch which I lately endeavoured to give of the progress of one branch of analysis.

I am, Sir, your obedient Servant,

GEORGE BOOLE.

May 20, 1851.

II.—*Proposed Question in the Theory of Probabilities.* By George Boole.

Of those rigorous consequences of the first principles of the theory of probabilities the general utility of which has caused them to be ranked by Laplace among the great secondary principles of the science, none is more important than the following:—If an event E can only happen as the result of some one of certain conflicting causes $A_1, A_2, \ldots A_n$, then if c_i represent the probability of A_i, and p_i the probability that if A_i happen E will happen, the total probability of the event E will be represented by the sum $\Sigma c_i p_i$.

I am desirous of calling the attention of mathematicians to a question closely analogous to that of which the answer is conveyed in the above theorem; like it also, admitting of rigorous solution and susceptible of wide application. The question is the following:—If an event E can only happen as a consequence of some one or more of certain causes $A_1, A_2 \ldots A_n$, and if generally c_i represent the probability of the cause A_i, and p_i the probability that if the cause A_i exist the event E will exist, then the series of values $c_1, c_2 \ldots c_n, p_1, p_2 \ldots p_n$, being given, required the probability of the event E.

It is to be noted that in this question the quantity c_i represents the total probability of the existence of the cause A_i, not the probability of its exclusive existence; and p_i the total probability of the existence of the event E when A_i is known to exist, not the probability of E's existing as a *consequence* of A_i. By the cause A_i is indeed meant the event A_i with which in a proportion p_i of the cases of its occurrence the event E has been associated.

The motives which have led me, after much consideration, to adopt with reference to this question a course unusual in the present day, and not upon slight grounds to be revived, are the following. First, I propose the question as a test of the sufficiency of received methods. Secondly, I anticipate that its discussion will in some measure add to our knowledge of an important branch of pure analysis. However, it is upon the former of these grounds alone that I desire to rest my apology.

While hoping that some may be found who, without departing from the line of their previous studies, may deem this question worthy of their attention, I wholly disclaim the notion of its being offered as a trial of personal skill or knowledge, but desire that it may be viewed solely with reference to those public and scientific ends for the sake of which alone it is proposed.

III.—*Solutions of some Elementary Problems in Geometry of Three Dimensions.*

By W. Walton.

The following are simple solutions of three elementary problems, solutions of which are given in most works on geometry of three dimensions.

1. To find the length of the perpendicular from a given point on a given straight line.

Let (x', y', z') be the coordinates of the given point P; (a, β, γ) those of a given point C in the given line. PQ, meeting the given line in Q at right angles, is the required length. Join CP.

If l, m, n, be the direction-cosines of CQ; then the projections of CP upon the coordinate axes are

$$x' - a, \quad y' - \beta, \quad z' - \gamma;$$

the projections of these upon CQ are

$$l(x' - a), \quad m(y' - \beta), \quad n(z' - \gamma),$$

and their sum $l(x' - a) + m(y' - \beta) + n(z' - \gamma)$

will be equal to the projection of CP upon CQ, that is, to CQ itself. Hence

$$PQ^2 = CP^2 - CQ^2$$
$$= (x'-a)^2 + (y'-\beta)^2 + (z'-\gamma)^2 - \{l(x'-a) + m(y'-\beta) + n(z'-\gamma)\}^2.$$

2. To find the perpendicular distance between two straight lines not in the same plane.

Let one of the lines pass through a given point C, of which the coordinates are a, β, γ, the direction-cosines of the line being l, m, n. Let the other line pass through C', the coordinates of which are a', β', γ', and let its direction-cosines be l', m', n'. Let the required perpendicular distance cut the lines in P, P', and let λ, μ, ν, be its direction-cosines.

Then, PP' being perpendicular to both the lines, the projection of CC' upon PP' is equal to PP'. Hence

$$PP' = \lambda(a - a') + \mu(\beta - \beta') + \nu(\gamma - \gamma') \ldots . \ (1).$$

But, by the conditions of perpendicularity,

$$l\lambda + m\mu + n\nu = 0, \quad l'\lambda + m'\mu + n'\nu = 0,$$

and therefore $\dfrac{\lambda}{mn' - m'n} = \dfrac{\mu}{nl' - n'l} = \dfrac{\nu}{lm' - l'm} = \rho \ldots . . \ (2),$

ρ representing the value of each ratio.

Hence, by the equation

$$\lambda^2+\mu^2+\nu^2=1,$$

we see that

$$\rho^2\{(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2\}=1.$$

From (1), (2), (3), we see that

$$PP'=\frac{(mn'-m'n)(\alpha-\alpha')+(nl'-n'l)(\beta-\beta')+(lm'-l'm)(\gamma-\gamma')}{\{(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2\}^{\frac{1}{2}}}$$

3. To find the equations to the straight line which cuts at right angles two given straight lines.

Let the equations to the given lines be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r\ldots\ldots\ldots(1),$$

$$\frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}=r'\ldots\ldots\ldots(2).$$

Let X, Y, Z, be the coordinates of any point whatever in the line which is normal to both; (x, y, z) and (x', y', z') being the points in which this normal cuts (1) and (2) respectively.

Now the projection of the distance between (X, Y, Z) and (x, y, z) upon each of the lines (1) and (2) is zero. Hence

$$l(X-x)+m(Y-y)+n(Z-z)=0,$$

$$l'(X-x)+m'(Y-y)+n'(Z-z)=0,$$

and therefore, by (1),

$$l(X-\alpha)+m(Y-\beta)+n(Z-\gamma)=r,$$

and, θ being the inclination of (1) to (2),

$$l'(X-\alpha)+m'(Y-\beta)+n'(Z-\gamma)=r(ll'+mm'+nn')$$
$$=r\cos\theta.$$

Eliminating θ between the last two equations, we have

$$(l'-l\cos\theta)(X-\alpha)+(m'-m\cos\theta)(Y-\beta)+(n'-n\cos\theta)(Z-\gamma)=0$$
$$\ldots\ldots(3).$$

Similarly, by projecting the distance between (X, Y, Z) and (x', y', z') upon each of the given lines, we shall get

$$(l-l'\cos\theta)(X-\alpha')+(m-m'\cos\theta)(Y-\beta')+(n-n'\cos\theta)(Z-\gamma')=0$$
$$\ldots\ldots(4).$$

The equations (3) and (4) are those to the common normal to the two given lines.

April 26, 1851.

On the General Theory of Associated Algebraical Forms.

By J. J. Sylvester, M.A., F.R.S.

ollowing brief exposition of the general theory of d Forms, as far as it has been as yet developed by ars or genius of mathematicians, is intended as y and, to a certain extent, emendative of some of ments in my paper on Linear Transformations, in ding number of the *Journal*.

first place, let a linear equivalent of any given eous function be understood to mean what that becomes when linear functions of the variables are ed in place of the variables themselves, subject to ition of the modulus of transformation (*i.e.* the value eterminant formed by the coefficients of transforma- ing unity.

lly, let two square arrays of terms (the determinants nding to each of which are unity) be said to be entary when each term in the one square is equal alue of what the determinant represented by the uare becomes when the corresponding term itself is ity, but all the other terms in the same line and with it are taken zero. This relation between the ares is well known to be reciprocal. Thus, for

$$\begin{matrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{matrix} \quad \text{and} \quad \begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{matrix}$$

aid to be reciprocally complementary to one another e two determinants which they represent are each d when we have

$$a = \begin{matrix} 1 & 0 & 0 \\ 0 & \beta' & \gamma' \\ 0 & \beta'' & \gamma'' \end{matrix} \qquad \alpha = \begin{matrix} 1 & 0 & 0 \\ 0 & b' & c' \\ 0 & b'' & c'' \end{matrix}$$

$$b = \begin{matrix} 0 & 1 & 0 \\ \alpha' & 0 & \gamma' \\ \alpha'' & 0 & \gamma'' \end{matrix} \qquad \beta = \begin{matrix} 0 & 1 & 0 \\ a' & 0 & c' \\ a'' & 0 & c'' \end{matrix}$$

$$b' = \begin{matrix} \alpha & 0 & \gamma \\ 0 & 1 & 0 \\ \alpha'' & 0 & \gamma'' \end{matrix} \qquad \beta' = \begin{matrix} a & 0 & c \\ 0 & 1 & 0 \\ a'' & 0 & c'' \end{matrix}$$

&c. &c.

Accordingly, two transformations, say of $F(x, y, z)$ and $G(u, v, w)$ respectively, may be said to be concurrent when in F for x, y, z, we write

$$\begin{aligned} &ax + by + cz, \\ &a'x + b'y + c'z, \\ &a''x + b''y + c''z; \end{aligned}$$

and in G for u, v, w, we write

$$\begin{aligned} &au + bv + cw, \\ &a'u + b'v + c'w, \\ &a''u + b''v + c''w; \end{aligned}$$

but complementary when for u, v, w, we write

$$\begin{aligned} &\alpha u + \beta v + \gamma w, \\ &\alpha' u + \beta' v + \gamma' w, \\ &\alpha'' u + \beta'' v + \gamma'' w; \end{aligned}$$

a, b, c, &c., α, β, γ, &c. being related in the manner antecedently explained.

Two forms, each of the same number of variables, may be said to be associate forms when the coefficients of the one are functions of those of the other; and when it happens that the coefficients of the first are all explicit functions of those of the second, the latter may be termed the originant and the former the derivant.

If now all the linear equivalents of one of two associated forms are similarly related to corresponding linear equivalents of the other, so that each may be derived from each by the same law, the forms so associated will be said to be concomitant each to the other. This concomitance may be of two kinds, and very probably, in the nature of things, only of the two kinds about to be described.

The first species of concomitance is defined by the corresponding equivalents of the two associated forms being deduced by precisely similar, or, as we have expressed it, concurrent transformations or substitutions, each from its given primitive. The second species of concomitance is defined by the corresponding equivalents being deduced not by similar but by contrary, *i.e.* reciprocal or complementary substitutions. Concomitants of the first kind may be called co-variants; concomitants of the second kind may be called contra-variants. When of the two associated forms one is a constant, the distinction between co- and contra-variants disappears, and the constant may be termed an invariant of the

with which it is associated.* It follows readily from these itions that a covariant of a co-variant and a contravariant contra-variant are each of them covariants; but a covariant contra-variant and a contra-variant of a co-variant are of them contra-variants; and also that an invariant, ther of a co-variant or of a contra-variant, is an invariant he original function †

t will also readily be seen that as regards functions of two ers a contra-variant becomes a co-variant by the simple rchange of x, y with $-y$, x, respectively. Co-variants are r. Cayley's hyperdeterminants; contra-variants include, but not coincident with, M. Hermite's formes-adjointes, if we derstand by the last-named term such forms as may be ived by the process described by M Hermite in the third his letters to M. Jacobi, "Sur différents objets de la éorie des Nombres," (which process is an extension of t employed for determining the polar reciprocal of an ebraical locus).‡ M. Hermite appears, however, elsewhere have used the term forme-adjointe in a sense as wide as

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* Accordingly an invariant to a given form may be defined to be such nction of the coefficients of the form, as remains absolutely unaltered n instead of the given form any linear equivalent thereto is substituted. ourse if the determinant of the coefficients of the transformations corondent to the respective equivalents be not taken unity as supposed his definition, the effect will be merely to introduce as a multiplier e power of the determinant formed by the coefficients of transformation.

† It may likewise be shewn that linear equivalents of co-variants and ravariants are themselves related to one another as covariants and conariants respectively, the transformations by which the equivalents are ined being taken concurrent in the one case and contrary or reciprocal he other; and of course any algebraic function of any number of riants is a covariant and of contravariants a contravariant.

‡ This has been further generalized by me in the theorem given in the number of this *Journal*, where I have shewn in effect that any invariant spect to ξ, η, θ of

$$f(\xi, \eta, \ldots \theta) + (x\xi + y\eta + \ldots + \theta t + \rho)\,\rho^{n-1},$$

eing supposed to be of the degree n) is a contra-variant of $f(x, y \ldots t)$. en this invariant is the determinant of f, it may be shewn that we obtain Hermite's theorem. It is somewhat remarkable that contra-variants ld have been in use among mathematicians as well in geometry as the ry of numbers (although their character as such was not recognized) re co-variants had ever made their appearance. Invariants of course came up with the theory of the equation to the squares of the differences e roots of equations, the last term in such equation being an invariant. ieve that I am correct in saying that covariants first made their appear- in one of Mr. Boole's papers, in this *Journal*, but Hesse's brilliant ication of one from among the infinite variety of these forms to the overy of the points of inflexion in a curve of the third order, in other s, to the Canonical Reduction of the Cubic Function of Three Letters, ars to have been the first occasion of their being turned to practical nt.

that in which I employ contra-variants. For instance, he has given a most remarkable theorem, which admits of being stated as follows:

If we have a function of any number of letters, say of x, y, z, as

$$ax^m + m.bx^{m-1}.y + m.cx^{m-1}.z + \frac{m.(m-1)}{2} d.x^{m-1}.y^2 + \&c.,$$

and if I be any invariant of this function, then will

$$\left(x^m.\frac{d}{da} + x^{m-1}y\frac{d}{db} + x^{m-2}z\frac{d}{dc} + d^{m-2}y^2\frac{d}{dd}\ \&c.\right)^r I$$

be a "*forme-adjointe*" of the given function. It is perfectly true and admits of being very easily proved, as I shall shew in your next number, that this is a contra-variant of the given function;* but it is not (as far as I can see) a forme-adjointe in the sense in which the use of that word is restricted in the letter alluded to. If, however, we adopt as the *definition* of *formes-adjointes* generally, that property in regard to their transformées which M. Hermite has demonstrated of the particular class treated of by him in the letter alluded to, then his formes-adjointes become coincident with my contra-variants. It will thus be seen that co-variants and contra-variants form two distinct and co-extensive species of associated forms, which divide between them the wide and fertile empire of linear transformations so far as its provinces have been as yet laid open by the researches of analysts. In your next number I purpose to enter much more largely into the subject generally. More particularly I shall describe the new method of Permutants, including the theory of Intermutants and Commutants (which latter are a species of the former, but embrace Determinants as a particular case), and their application to the theory of Invariants. I shall also exhibit the connexion between the theory of Invariants and that of Symmetrical Functions, and some remarkable theorems on Relative Invariants.†

Some of your readers may like to be informed that a Supplement to my last paper, under the title of "An Essay on Canonical Forms," has been since published;‡ and that

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* This is also true if I be taken any *covariant* instead of an invariant of the function.

† It will be readily apprehended that the definitions and conceptions above stated, respecting covariants and contravariants of two single functions, may be extended so as to comp... of functions covariantive or contravariantive to one a...

‡ By Mr. George Bell, U... Street.

I have there given a much simpler method of solution of the problem of the reduction of quintic functions to their canonical form than in the original memoir, and extended the method successfully to the reduction of all odd-degreed functions to their canonical form. I may take this occasion to state that the Lemma given in Note B of the Supplement, upon which this method of reduction is based, is an immediate deduction from the well-known theorem for the multiplication of Determinants.

There is a numerical error in "The Cubical Hyperdeterminant of the Twelfth Degree," worked out after the method of commutants by Mr. Spottiswoode, given at the end of my paper in the May No. The correct result will be stated in the next number of the *Journal*, where I hope also to be able to fix the number of distinct solutions of the problem of reducing a Sextic Function to its canonical form

$$u^6 + v^6 + w^6 + mu^2v^2w^2.$$

For odd-degreed functions there is never more than one solution possible, as shewn in the Supplement referred to.

POSTSCRIPT.

Since the above was sent to press, I have discovered an uniform mode of solution for the canonical reduction of functions, whether of odd or even degrees. The canonical form however, except for the fourth and eighth degrees, requires to be varied from that assumed in my previous paper. Thus, for the sixth degree the canonical form will be

$$au^6 + bv^6 + cw^6 + muvw(u - v)(v - w)(w - u),$$

where u, v, w are supposed to be connected by the identical equation $u + v + w = 0$. And there will be only *two* solutions—a remarkable and most unexpected discovery. For functions of the eighth degree there are five distinct solutions, and in general there is the strongest reason for believing (indeed it may be positively affirmed) that *when the canonical form has been rightly assumed* for a function of the even degree n, the number of solutions will be $\frac{1}{2}(n+2)$ when $\frac{1}{2}n$ is even, but $\frac{1}{4}(n+2)$ when $\frac{1}{2}n$ is odd. It turns out therefore that the theory for functions of the sixth degree is in some respects simpler than for those of the fourth. The investigation into canonical forms here referred to has led me to the discovery of a most unexpected theorem for finding all the invariants of a certain class, belonging to functions of two letters of an even degree. See *London and Edinburgh Philosophical Magazine* for the present month.

END OF VOL. VI.

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Rev. William A. Willock.

Trinity College, Dublin.

Mathematical Notes.*

* These articles are also referred to under their Authors' names, except such of them as are anonymous.

[redacted]

THE CAMBRIDGE AND DUBLIN MATHEMATICAL JOURNAL.

EDITED BY W. THOMSON, M.A.

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VOL. VII.

(BEING VOL. XI. OF THE CAMBRIDGE MATHEMATICAL JOURNAL.)

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Notes and other passages enclosed in brackets [ ], are insertions ma the Editor.

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CAMBRIDGE AND DUBLIN
MATHEMATICAL JOURNAL.

ON THE INDETERMINATE CURVATURE OF SURFACES.

By William Walton.

If, through an indefinitely small arc ds at any point x, y, z, of a surface $F = 0$, a normal plane be drawn cutting the surface, the ordinary formula for the determination of the radius of curvature ρ of the section at the point is

$$\rho^2 = \frac{U^2 + V^2 + W^2}{(l^2u + m^2v + n^2w + 2mnu' + 2nlv' + 2lmw')^2},$$

where l, m, n, are the direction-cosines of ds, and where

$$U = \frac{dF}{dx}, \qquad V = \frac{dF}{dy}, \qquad W = \frac{dF}{dz},$$

$$u = \frac{d^2F}{dx^2}, \qquad v = \frac{d^2F}{dy^2}, \qquad w = \frac{d^2F}{dz^2},$$

$$u' = \frac{d^2F}{dy\,dz}, \qquad v' = \frac{d^2F}{dz\,dx}, \qquad w' = \frac{d^2F}{dx\,dy};$$

l, m, n, being connected by the condition

$$lU + mV + nW = 0.$$

Suppose, however, that U, V, W, are simultaneously zero at any point; then the second differential of the equation to the surface gives us

$$Ud^2x + Vd^2y + Wd^2z + udx^2 + vdy^2 + wdz^2 + 2u'dydz + 2v'dzdx + 2w'dxdy = 0,$$

and therefore

$$l^2u + m^2v + n^2w + 2mnu' + 2nlv' + 2lmw' = 0.$$

Thus the expression for ρ^2 assumes the indeterminate form $\frac{0}{0}$ omes inapplicable. This indeterminateness belon

to a sing point or to any point on a singular line surface. object in this paper is to supply this defe the theory curvature by the investigation of a for applicable to such anomalous cases.

The tangent plane at any point $x+\delta x$, $y+\delta y$, $z+\delta z$, the surface, is represented by the equation

$$(x_1 - x - \delta x)(U+\delta U) + (y_1 - y - \delta y)(V+\delta V) + (z_1 - z - \delta z)(W+\delta W) = 0.$$

If x, y, z, be a p... 0, $V=0$, $W=0$; then

$$(x_1 - x - \delta x)\,\delta \ldots V + (z_1 - z - \delta z)\,\delta W = 0,$$

or, in the limit,

$$(x_1 - x)\,\frac{d \ldots}{} \ldots + (z_1 - z)\,\frac{dW}{ds} = 0.$$

The equation to ... through ds, is

$$L(x_1 - x) + \ldots + N(z_1 - z) = 0 \;\ldots\ldots (1),$$

where L, M, N, are subject to the conditions

$$L\frac{dU}{ds} + M\frac{dV}{ds} + N\frac{dW}{ds} = 0 \;\ldots\ldots\ldots (2),$$

$$Ll \quad + Mm \quad + Nn \quad = 0 \;\ldots\ldots\ldots (3),$$

l, m, n, being connected together by the equation

$$l^2 + m^2 + n^2 = 1 \;\ldots\ldots\ldots\ldots\ldots (4).$$

Eliminating L, M, N, between (1), (2), (3), we get, for the equation to the normal plane,

$$(x_1 - x)\left(n\frac{dV}{ds} - m\frac{dW}{ds}\right) + (y_1 - y)\left(l\frac{dW}{ds} - n\frac{dU}{ds}\right) + (z_1 - z)\left(m\frac{dU}{ds} - l\frac{dV}{ds}\right) = 0 \;\ldots\ldots\ldots (5).$$

At the intersection of the surface

$$F = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

with the plane (5), at the point x, y, z, if x', y', z', be the centre of the osculating circle, and λ, μ, ν, be taken to represent $\frac{dl}{ds}$, $\frac{dm}{ds}$, $\frac{dn}{ds}$, respectively, we know that

$$\left.\begin{aligned} x' - x = \lambda\rho^2,\quad y' - y = \mu\rho^2,\quad z' - z = \nu\rho^2, \\ \rho^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2 \end{aligned}\right\} \ldots (7).$$

[redacted]

rentiating the equation (4), we have

$$l\lambda + m\mu + n\nu = 0 \text{ .... } (8).$$

rentiating the equation (6) twice, we get

$$l\frac{dU}{ds} + m\frac{dV}{ds} + n\frac{dW}{ds} + \lambda U + \mu V + \nu W = 0,$$

h becomes, at the point x, y, z,

$$l\frac{dU}{ds} + m\frac{dV}{ds} + n\frac{dW}{ds} = 0 \text{ .... } (9),$$

by a third differentiation,

$$\frac{dU}{ds} + 2\mu\frac{dV}{ds} + 2\nu\frac{dW}{ds} + l\frac{d^2U}{ds^2} + m\frac{d^2V}{ds^2} + n\frac{d^2W}{ds^2} = 0$$

$$\text{......} (10).$$

differentiating the equation (5) twice with regard to z_1, and then making x_1, y_1, z_1 coincide with x, y, z, tain

$$\lambda\left(n\frac{dV}{ds} - m\frac{dW}{ds}\right) - \mu\left(l\frac{dW}{ds} - n\frac{dU}{ds}\right)$$
$$+ \nu\left(m\frac{dU}{ds} - l\frac{dV}{ds}\right) = 0 \text{ .... } (11).$$

(8) and (11) we have

$$\mu\left\{(l^2 + m^2)\frac{dW}{ds} - n\left(l\frac{dU}{ds} - m\frac{dV}{ds}\right)\right\}$$
$$= \nu\left\{(l^2 + n^2)\frac{dV}{ds} - m\left(l\frac{dU}{ds} - n\frac{dW}{ds}\right)\right\},$$

herefore, by (9),

$$\mu\frac{dW}{ds} = \nu\frac{dV}{ds},$$

by symmetry,

$$\frac{1}{\lambda}\frac{dU}{ds} = \frac{1}{\mu}\frac{dV}{ds} = \frac{1}{\nu}\frac{dW}{ds} \text{ .... } (12),$$

$$\frac{dU}{ds} = lu + mv + nw,$$

$$\frac{d^2U}{ds^2} = l\frac{du}{ds} + m\frac{dv}{ds} + n\frac{dw}{ds} + \lambda u + \mu v + \nu w,$$

and, similarly,

$$\frac{d^2V}{ds^2} = m\frac{dv}{ds} + n\frac{du'}{ds} + l\frac{dw'}{ds} + \mu v + \nu u' + \lambda w',$$

and $$\frac{d^2W}{ds^2} = n\frac{dw}{ds} + l\frac{dv'}{ds} + m\frac{du'}{ds} + \nu w + \lambda v' + \mu u'.$$

Hence
$$l\frac{d^2U}{ds^2} + m\frac{d^2V}{ds^2} + n\frac{d^2W}{ds^2}$$

$$= l^2\frac{du}{ds} + m^2\frac{dv}{ds} + n^2\frac{dw}{ds}$$

$$+ 2mn\frac{du'}{ds} + 2nl\frac{dv'}{ds} + 2lm\frac{dw'}{ds}$$

$$+ \lambda(lu + mw' + nv')$$

$$+ \mu(mv + nu' + lw')$$

$$+ \nu(nw + lv' + mu')$$

$$= l^2\frac{du}{ds} + m^2\frac{dv}{ds} + n^2\frac{dw}{ds}$$

$$+ 2mn\frac{du'}{ds} + 2nl\frac{dv'}{ds} + 2lm\frac{dw'}{ds}$$

$$+ \lambda\frac{dU}{ds} + \mu\frac{dV}{ds} + \nu\frac{dW}{ds}.$$

Hence the equation (10) becomes

$$3\lambda\frac{dU}{ds} + 3\mu\frac{dV}{ds} + 3\nu\frac{dW}{ds}$$

$$+ l^2\frac{du}{ds} + m^2\frac{dv}{ds} + n^2\frac{dw}{ds}$$

$$+ 2mn\frac{du'}{ds} + 2nl\frac{dv'}{ds} + 2lm\frac{dw'}{ds} = 0 \quad \ldots\ldots (13)$$

From (12) and (13) there is

$$\frac{3\left(\frac{dU^2}{ds^2} + \frac{dV^2}{ds^2} + \frac{dW^2}{ds^2}\right)}{l^2\frac{du}{ds} + m^2\frac{dv}{ds} + n^2\frac{dw}{ds} + 2mn\frac{du'}{ds} + 2nl\frac{dv'}{ds} + 2lm\frac{dw'}{ds}}$$

$$= -\frac{1}{\lambda}\frac{dU}{ds} = -\frac{1}{\mu}\frac{dV}{ds} = -\frac{1}{\nu}\frac{dW}{ds},$$

nd therefore, by (7),

$$i = \frac{\left(l^2 \frac{du}{ds} + m^2 \frac{dv}{ds} + n^2 \frac{dw}{ds} + 2mn \frac{du'}{ds} + 2nl \frac{dv'}{ds} + 2lm \frac{dw'}{ds}\right)^2}{\frac{dU^2}{ds^2} + \frac{dV^2}{ds^2} + \frac{dW^2}{ds^2}}.$$

If we substitute for $\frac{du}{ds}$ its value

$$l \frac{du}{dx} + m \frac{du}{dy} + n \frac{du}{dz},$$

id make like substitutions for $\frac{dv}{ds}, \frac{dw}{ds}, \ldots$, the square root f the numerator of this fraction will assume the form

$$\begin{aligned}
&l^2 \left\{l \frac{du}{dx} + m \frac{du}{dy} + n \frac{du}{dz} + 2m \frac{dw'}{dx} + 2n \frac{dv'}{dx}\right\} \\
&+ m^2 \left\{m \frac{dv}{dy} + n \frac{dv}{dz} + l \frac{dv}{dx} + 2n \frac{du'}{dy} + 2l \frac{dw'}{dy}\right\} \\
&+ n^2 \left\{n \frac{dw}{dz} + l \frac{dw}{dx} + m \frac{dw}{dy} + 2l \frac{dv'}{dz} + 2m \frac{du'}{dz}\right\} \\
&+ 2lmn \left\{\frac{du'}{dx} + \frac{dv'}{dy} + \frac{dw'}{dz}\right\},
\end{aligned}$$

the denominator, when expressed in terms of partial differential coefficients, being equal to

$$(lu + mw' + nv')^2 + (mv + nu' + lw')^2 + (nw + lv' + mu')^2.$$

Also the equation (9) may be written

$$l^2u + m^2v + n^2w + 2mnu' + 2nlv' + 2lmw' = 0 \ldots (14).$$

Ex. To find the radius of the osculating circle of any normal section of the Cono-cuneus of Wallis at any point of its singular line.

The equation to the surface is

$$F = a^2y^2 + x^2(z^2 - c^2) = 0.$$

We have therefore

$$\begin{aligned}
U &= 2x(z^2 - c^2), & V &= 2a^2y, & W &= 2x^2z, \\
u &= 2(z^2 - c^2), & v &= 2a^2, & w &= 2x^2, \\
u' &= 0, & v' &= 4xz, & w' &= 0,
\end{aligned}$$

$$\frac{du}{dx} = 0, \qquad \frac{du}{dy} = 0, \qquad \frac{du}{dz} = 4z,$$

$$\frac{dv}{dx} = 0, \qquad \frac{dv}{dy} = 0, \qquad \frac{dv}{dz} = 0,$$

$$\frac{dw}{dx} = 4x, \qquad \frac{dw}{dy} = 0, \qquad \frac{dw}{dz} = 0,$$

$$\frac{du'}{dx} = 0, \qquad \frac{du'}{dy} = 0, \qquad \frac{du'}{dz} = 0,$$

$$\frac{dv'}{dx} = 4z, \qquad \frac{dv'}{dy} = 0, \qquad \frac{dv'}{dz} = 4x,$$

$$\frac{dw'}{dx} = 0, \qquad \frac{dw'}{dy} = 0, \qquad \frac{dw'}{dz} = 0.$$

The quantities U, V, W, all vanish if $x = 0$, $y = 0$, whatever be the value of z; the axis of z being therefore a singular line.

Putting $x = 0$, $y = 0$, in these expressions for the partial differential coefficients of F, we have

$$\frac{9}{\rho^2} = \frac{(12l^2nz)^2}{4l^2(z^2 - c^2)^2 + 4m^2a^4},$$

$$\frac{1}{\rho^2} = 4z^2 \frac{l^4n^2}{l^2(z^2 - c^2)^2 + m^2a^4} \; \ldots\ldots\ldots\ldots (1)'.$$

Also the equation (14) becomes

$$l^2(z^2 - c^2) + m^2a^2 = 0 \ldots\ldots\ldots\ldots (2)'.$$

If from the equation

$$l^2 + m^2 + n^2 = 1,$$

combined with (2)', we obtain l^2, m^2, in terms of n^2, and substitute in (1)', we shall eventually get

$$\frac{1}{\rho^2} = n^2(1 - n^2)\frac{4a^2z^2}{(c^2 - z^2)(a^2 + c^2 - z^2)^2}.$$

It is easily seen, from this result, that, for any proposed value of z, the curvature will be a maximum when $n = \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$, and a minimum when $n = 0$.

Thus the maximum value of the radius of curvature is ∞, and each of its two minimum values is equal to the square root of

$$\frac{(c^2-z^2)(a^2+c^2-z^2)^2}{a^2z^2},$$

Cambridge, July 21, 1851.

ON THE CIRCULAR SECTIONS OF SURFACES OF THE SECOND ORDER.

By William Walton.

In a paper, in the fourth volume of the *Cambridge Mathematical Journal*, I have applied the method of Indeterminate Maxima and Minima, among other questions, to the determination of the positions of the circular sections through the centre of an ellipsoid referred to its axes. The same method may be conveniently adopted in investigating the positions of such sections and of their centric loci in surfaces of the second order referred to any system of rectangular axes. The results at which I have arrived in this paper coincide with those given by Mr. Frost and Professor Thomson in the first volume of the second series of the *Mathematical Journal*.

The equation to any surface of the second order is

$$\phi(x, y, z) = ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy$$
$$+ 2a''x + 2b''y + 2c''z + f = 0 \dots\dots\dots (1).$$

Let x_1, y_1, z_1, be the coordinates of the centre of any plane section, of which l, m, n, are the direction-cosines and r a radius vector originating in the centre of the section. Then, l', m', n', being the direction-cosines of r in any position, and x, y, z, the coordinates of its extremity,

$$x = x_1 + l'r, \quad y = y_1 + m'r, \quad z = z_1 + n'r.$$

If we substitute these values of x, y, z, in (1), we shall have a quadratic in r. The coefficient of r in this quadratic must be zero, the origin of the radii vectores being the centre of the section. Hence we see that, ϕ being written for $\phi(x_1, y_1, z_1)$,

$$r^2(al'^2 + bm'^2 + cn'^2 + 2a'm'n' + 2b'n'l' + 2c'l'm') + \phi = 0 \dots (2),$$

and $$l'\frac{d\phi}{dx_1} + m'\frac{d\phi}{dy_1} + n'\frac{d\phi}{dz_1} = 0 \dots\dots\dots\dots (3).$$

The quantities l', m', n', are subject to the two relations

$$ll' + mm' + nn' = 0 \ldots\ldots\ldots\ldots (4),$$

$$l'^2 + m'^2 + n'^2 = 1 \ldots\ldots\ldots\ldots (5).$$

Now, generally, r^2 will have a maximum and a minimum value by virtue of the equations (2), (4), (5), for any system of values of l, m, n. Our object is to proceed with the investigation for finding the corresponding values of l', m', n', and then to assign such values to l, m, n, as shall render l', m', n', indeterminate.

Before proceeding with this investigation, we may observe that, when l', m', n', are indeterminate, the two equations (3) and (4) must be identical, and that accordingly

$$\frac{1}{l}\frac{d\phi}{dx_1} = \frac{1}{m}\frac{d\phi}{dy_1} = \frac{1}{n}\frac{d\phi}{dz_1} \ldots\ldots\ldots\ldots (6),$$

which are the equations to the locus of the centres corresponding to a system of values of l, m, n, determined in the manner above described.

If we put

$$P = al'^2 + bm'^2 + cn'^2 + 2a'm'n' + 2b'n'l' + 2c'l'm' \ldots (7),$$

it is plain that r^2 will have a maximum or minimum value whenever P has a minimum or a maximum.

Now, from (4), we see that

$$2m'n' = \frac{l^2l'^2 - m^2m'^2 - n^2n'^2}{mn}, \quad 2n'l' = \frac{m^2m'^2 - n^2n'^2 - l^2l'^2}{nl},$$

$$2l'm' = \frac{n^2n'^2 - l^2l'^2 - m^2m'^2}{lm}.$$

Substituting these values of $2m'n'$, $2n'l'$, $2l'm'$, in (7), we hav

$$P = Hl'^2 + Km'^2 + Ln'^2 \ldots\ldots\ldots\ldots (8),$$

where

$$\left.\begin{aligned} H &= a + \frac{l}{mn}(la' - mb' - nc') \\ K &= b + \frac{m}{nl}(mb' - nc' - la') \\ L &= c + \frac{n}{lm}(nc' - la' - mb') \end{aligned}\right\} \ldots\ldots\ldots (9).$$

Differentiating (4), (5), (8), considering l', m', n', variable and putting $dP = 0$, we have, by the aid of indeterminat multipliers λ, μ,

$$\lambda l + (\mu + H)l' = 0, \quad \lambda m + (\mu + K)m' = 0, \quad \lambda n + (\mu + L)n' = 0$$

Multiplying these equations in order by l', m', n', adding, and attending to (4), (5), (8), we have

$$\mu + P = 0,$$

and therefore

$$l' = \frac{\lambda l}{P - H}, \quad m' = \frac{\lambda m}{P - K}, \quad n' = \frac{\lambda n}{P - L}.$$

Assuming that l', m', n', are indeterminate, we have

$$\lambda = 0, \quad P = H = K = L.$$

Thus the equations (9) become, α, β, γ, representing $a - P$, $b - P$, $c - P$, respectively,

$$\alpha mn + l^2 a' - lmb' - lnc' = 0,$$

$$\beta nl + m^2 b' - mnc' - mla' = 0,$$

$$\gamma lm + n^2 c' - nla' - nmb' = 0.$$

Multiplying the first of these three equations by n, the third by l, adding, and dividing by m, we get

$$\left.\begin{array}{ll} & \alpha n^2 + \gamma l^2 - 2lnb' = 0. \\ \text{Similarly} & \beta l^2 + \alpha m^2 - 2mlc' = 0, \\ \text{and} & \gamma m^2 + \beta n^2 - 2nma' = 0. \end{array}\right\} \quad \ldots\ldots\ldots (11).$$

Eliminating mn, m^2, n^2, from the first and last three of the last six equations, we obtain

$$\left.\begin{array}{l} (a'^2 - \beta\gamma)\, l + (c'\gamma - a'b')\, m + (b'\beta - c'a')\, n = 0. \\ \text{Similarly we may shew that} \\ (b'^2 - \gamma\alpha)\, m + (a'\alpha - b'c')\, n + (c'\gamma - a'b')\, l = 0, \\ (c'^2 - \alpha\beta)\, n + (b'\beta - c'a')\, l + (a'\alpha - b'c')\, m = 0. \end{array}\right\} \ldots (12).$$

Eliminating n between the first two of the equations (12), we get

$$(a'l - b'm)\, S = 0,$$

where
$$S = a'^2\alpha + b'^2\beta + c'^2\gamma - \alpha\beta\gamma - 2a'b'c'.$$

Similarly there is also

$$(b'm - c'n)\, S = 0,$$

$$(c'n - a'l)\, S = 0.$$

Thus we see that either $\quad S = 0 \ldots\ldots\ldots\ldots\ldots\ldots (13),$

or $\quad a'l = b'm = c'n \ldots\ldots\ldots\ldots\ldots (14).$

If $S = 0$, the equations (12) must be identical, and therefore

$$(b'\beta - c'a')(c'\gamma - a'b') = (a'\alpha - b'c')(a'^2 - \beta\gamma).$$

This relation reduces the first of the equations (12) to the fo

$$\frac{l}{a'a - b'c'} + \frac{m}{b'\beta - c'a'} + \frac{n}{c'\gamma - a'b'} = 0 \ldots\ldots (15).$$

This equation, combined with any one of the equations (11), and the equation (13), will determine the values of l, m, n.

If the equations (14) be true, then, by (9),

$$a = \frac{b'c'}{a'}, \quad \beta = \frac{c'a'}{b'}, \quad \gamma = \frac{a'b'}{c'} \ldots\ldots (16),$$

and the equation (13) is still satisfied, the equation (15) assuming a nugatory form. In this case the equations to the centric loci become

$$a'\frac{d\phi}{dx_1} = b'\frac{d\phi}{dy_1} = c'\frac{d\phi}{dz_1},$$

or, writing these equations in full, and attending to the relations (16),

$$a'(Px_1 + a'') = b'(Py_1 + b'') = c'(Pz_1 + c'').$$

These equations shew that the circular sections, of which the direction-cosines are given by (14), are all perpendicular to their centric locus. Thus the surface is one of revolution.

Cor. Suppose that $a = b$ and $c' = 0$. Then also $a = \beta$. The equations (11) then become

$$an^2 + \gamma l^2 - 2lnb' = 0, \quad a(l^2 + m^2) = 0, \quad \gamma m^2 + an^2 - 2nma' = 0.$$

These equations are satisfied by

$$l = 0, \quad m = 0, \quad a = 0.$$

Thus one series of circular sections are parallel to the plane of xy.

August 23, 1851.

DEMONSTRATION OF BRIANCHON'S THEOREM, AND OF AN ANALOGOUS PROPERTY IN SPACE.

By Thomas Weddle.

Let $ABCDEF$ be a hexagon circumscribed about a conic and

$$t = 0, \quad u = 0, \quad v = 0,$$

the equations to the straight lines CE, AE, and AC, respectively. Supposing t, u, and v to have been multiplied

by the proper constants, we may denote the equation to the conic by

$$t^2 + u^2 + v^2 + 2\lambda uv + 2\mu tv + 2\nu tu = 0 \ldots\ldots (1).$$

This may be put under the form

$$(t + \mu v + \nu u)^2 + (1 - \nu^2)\, u^2 + (1 - \mu^2)\, v^2 + 2(\lambda - \mu\nu)\, uv = 0,$$

from which we see that

$$(1 - \nu^2)\, u^2 + (1 - \mu^2)\, v^2 + 2(\lambda - \mu\nu)\, uv = 0 \ldots . (2),$$

is the equation to the two tangents drawn from A; that is, of the two straight lines AB and AF. Now by assigning a proper value to l, the expression

$$-\frac{1}{l}\{lv - (1 - \nu^2)\, u\}\,\{lu - (1 - \mu^2)\, v\}$$

can be rendered identical with the left-hand member of (2); hence the equations to AB and AF are $lv = (1 - \nu^2)\, u$ and $lu = (1 - \mu^2)\, v$.

In a similar manner the equations to the other sides of the hexagon may be got; and, collecting the whole, we have the

$$\left.\begin{array}{rr} \text{equation to } CD, & mt = (1 - \lambda^2)\, v \\ DE, & nt = (1 - \lambda^2)\, u \\ AF, & lu = (1 - \mu^2)\, v \\ EF, & nu = (1 - \mu^2)\, t \\ AB, & lv = (1 - \nu^2)\, u \\ BC, & mv = (1 - \nu^2)\, t \end{array}\right\} \ldots\ldots\ldots\ldots (3).$$

Eliminate t from the first and second of these equations, u from the third and fourth, and v from the fifth and sixth; and we get the

$$\left.\begin{array}{rr} \text{equation of } AD, & mu = nv \\ CF, & lt = nv \\ BE, & lt = mu \end{array}\right\} \ldots\ldots\ldots\ldots (4).$$

Hence the diagonals AD, CF, and BE intersect in a point whose equations are

$$lt = mu = nv \ldots\ldots\ldots\ldots\ldots (5);$$

and thus Brianchon's theorem is established.

The analogue referred to in the title is that due to M. Chasles, various forms of which are given in my third memoir on the "Theorems in Space analogous to those of Pascal and Brianchon in a Plane." (See *Journal*, vol. VI.

New Series, theorem (v.) p. 117, (xxii.) p. 131, and (xxiv.) p. 132). In that memoir this theorem is obtained by reciprocal polars, so that perhaps the following independent analytical proof may be interesting to some readers. Before proceeding with the investigation, however, I shall, to avoid unnecessary reference, enunciate the property, and I shall select M. Chasles's own enunciation because the form in which he presents the theorem is really that under which it is here proved.

The twelve tangent planes to a surface of the second degree, drawn through the edges of a tetrahedron, may be considered to intersect, three and three, in four points, each of which is the intersection of three planes drawn through edges in the same face of the tetrahedron; the four straight lines joining these points to the angles of the tetrahedron opposite to these faces will lie in an hyperboloid of one sheet and will belong to the same system of generators.

Let $\qquad t = 0, \quad u = 0, \quad v = 0, \quad w = 0,$

be the equations to the faces of the tetrahedron; then, supposing t, u, v, and w to have been multiplied by the proper constants, we may denote the equation to the surface by

$$t^2 + u^2 + v^2 + w^2 + 2\lambda tu + 2\mu tv + 2\nu tw$$
$$+ 2\rho uv + 2\sigma uw + 2\tau vw = 0 \ldots\ldots\ldots\ldots (6).$$

Put
$$\alpha = 1 - \rho^2 - \sigma^2 - \tau^2 + 2\rho\sigma\tau,$$
$$\beta = 1 - \mu^2 - \nu^2 - \tau^2 + 2\mu\nu\tau,$$
$$\gamma = 1 - \lambda^2 - \nu^2 - \sigma^2 + 2\lambda\nu\sigma,$$
and
$$\delta = 1 - \lambda^2 - \mu^2 - \rho^2 + 2\lambda\mu\rho.$$

Equation (6) may be put under the form

$$(t+\lambda u+\mu v+\nu w)^2+(1-\lambda^2)u^2+(1-\mu^2)v^2+(1-\nu^2)w^2+2(\rho-\lambda\mu)uv$$
$$+ 2(\sigma - \lambda\nu)\,uw + 2(\tau - \mu\nu)\,vw = 0.$$

Multiply this by $1 - \lambda^2$, and it readily takes the form

$$(1-\lambda^2)(t+\lambda u+\mu v+\nu w)^2 + \{(1-\lambda^2)u + (\rho-\lambda\mu)v + (\sigma-\lambda\nu)w\}^2$$
$$+ \delta v^2 + \gamma w^2 + 2kvw = 0,$$

where k has a value which it does not concern us to know.

This form of equation (6) shews us that

$$\delta v^2 + \gamma w^2 + 2kvw = 0 \ldots\ldots\ldots\ldots (7),$$

is the equation to the two tangent planes to the surface drawn through the edge (vw). But (7) can, by attributing a proper

to r, be put under the form

$$-\frac{1}{r}(rv-\gamma w)(rw-\delta v)=0,$$

at the equations to the two tangent planes are

$$rv=\gamma w \text{ and } rw=\delta v;$$

in a similar manner the equations to the other tangent planes may be got. If we collect the whole and place in the same horizontal line the equations to those planes that intersect in one of the points mentioned in the theorem, we obtain

$$\left.\begin{array}{lll} lt=\alpha u, & mt=\alpha v, & nt=\alpha w \\ lu=\beta t, & pu=\beta v, & qu=\beta w \\ mv=\gamma t, & pv=\gamma u, & rv=\gamma w \\ nw=\delta t, & qw=\delta u, & rw=\delta v \end{array}\right\} \ldots\ldots (8).$$

Eliminate t, u, v, and w from the first, second, third, and fourth rows of equations respectively, and we obtain the equations to the four straight lines referred to in the enunciation of the theorem.

$$\left.\begin{array}{l} \frac{u}{l}=\frac{v}{m}=\frac{w}{n} \\ \frac{t}{l}=\frac{v}{p}=\frac{w}{q} \\ \frac{t}{m}=\frac{u}{p}=\frac{w}{r} \\ \frac{t}{n}=\frac{u}{q}=\frac{v}{r} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (9).$$

Now it is easy to see that each of these lines lies in the hyperboloid whose equation is

$$(mq-np)(rtu+lvw)+(np-lr)(qtv+muw)$$
$$+(lr-mq)(ptw+nuv)=0\ldots\ldots\ldots(10),$$

and thus the theorem is established.

Yorktown, near Bagshot,
May 19, 1851.

A GEOMETRICAL PROPERTY OF CURVES OF THE THIRD ORDER.

By Thomas Cotterill, M.A.,
Late Fellow of St. John's College, Cambridge.

The equation to a curve of the third order referred to a triangle by the usual coordinates $\alpha\beta\gamma$, and passing through two of its angular points, can be put under the form

$$\alpha\beta(A\alpha + B\beta) = \gamma f_2,$$

$f_2 = 0$ denoting a conic.

The form of this equation at once points out nine points on the curve of the third order (which for shortness we will call the curve); viz. the six intersections of the lines α, β, and $A\alpha + B\beta$ with the conic f_2, and the three intersections of the same lines with γ.

Now consider the four points in which $\alpha\beta$ cut the conic, and the point in which $A\alpha + B\beta$ meets γ, and observe that the equation to the curve can be put under the form

$$\alpha\beta(A\alpha + B\beta + \lambda\gamma) = \gamma(f_2 + \lambda\alpha\beta).$$

Here we have a conic passing through the first-mentioned four points and a line through the last-mentioned point; and it appears from the form of the equation just given, that the conic and line cut each other in points on the curve. Hence

"Every conic passing through four given points on a curve of the third order will cut it again in two points, such that the straight line joining them will pass through a fixed point on the curve."*

This point, from its characteristic property, we may call the focus of the inscribed quadrangle with respect to the curve.

If we except the case of a double point, this theorem is general. The points may be supposed to coalesce and the curve of the third order may break up into a conic and line, or three lines; and the conic passing through the four points may be any of the three systems of lines passing through the four points.

There are but few descriptive properties of curves of the third order given by Maclaurin, and subsequent writers, which cannot be proved by this theorem, and I think it will be found a powerful and easy instrument in discovering new

* The theorem may also be shewn as follows:

"If the four fixed points be the points of intersection of the conics $U = 0$ and $V = 0$, then the equation of the curve is $pU + qV = 0$, and that of an indeterminate conic is $aU + bV = 0$; the line forming the remaining two points of intersection of the curve and conic is $bq - ap = 0$, which through the point $p = 0$, $q = 0$.

relations. Its simplest corollaries point out the properties of the points of intersection of the common chords of the conic and curve with the curve,—the means of drawing a tangent to the curve at any point,—the conditions of contact of conics (drawn according to given laws) with the curve.

Let the curve of the third order consist of a conic and line. We may take three points on the conic and one on the line, and then apply the theorem.

Again, take two points on the conic and two on the straight line. Draw two lines joining the points on the conic with the points on the line, the remaining chord of intersection will cut the line in the focus; and similarly for the other two lines joining the points. This gives us a converse of Pascal's Theorem, but I am not aware that the more general property of the chord of intersection of the fixed conic and the variable conic passing through the fixed points has been observed.

Now, taking the general case, the conic passing through four points and their focus meets the curve again at the point where the tangent at the focus meets it. Hence if curves of the third order be drawn through five points so that one of them is always the focus of the rest with respect to all the curves, the tangent at this point to each curve will meet the curve on the conic passing through the five points.

Take two sets of four points and their foci, and consider the point where the line joining the foci again meets the curve. The conic through the first set of points and the second focus will pass through this point, as well as the conic through the last set of points and the first focus. But Plucker has shewn, in his "Theorie der Curven," (and it follows easily from Mr. Weddle's paper in the last number of this *Journal*), that it is also the point in which curves of the third order passing through the eight points intersect.

Hence we have the following remarkable theorem:

Through eight points draw a curve of the third order; we shall in general have 70 foci, and any one focus of four points has a corresponding focus of the remaining four. Then the 35 lines joining the corresponding foci pass through the same point on the curve, viz. the point in which all curves of the third order through the eight points intersect.

I think this property in the hands of abler geometers than myself will give a construction for determining the ninth point by purely descriptive linear methods. The analytical expressions for determining its position are not intricate, but involve some functions which require geometrical interpreta-
I may however mention that one resul if the

points 123, 456 are such that each of the three sets of points 162, 243, 351 is in a straight line, then the curves of the third order passing through these points and two other points 78 will again intersect in the common point of intersection of the conics (16578), (24678), and (35478).

The equations on which these remarks are founded were supposed to contain point-coordinates and conics passing through points; but as these can be changed into line-coordinates and conics touching lines, we have the following reciprocal theorem:

"Every conic touching four tangents to a curve of the third class will be touched again by two of its tangents, and their point of intersection will describe another tangent to the curve."

One of the most interesting curves of this class is that touched by the line joining the feet of the perpendiculars from any point in a circle on the sides of an inscribed triangle.

The tangents at its cusps meet in the intersection of the perpendiculars from the angular points of the triangle on the opposite sides. If a, β, γ be the perpendiculars from the angular points of the inscribed triangle, whose angles are A, B, C, on a tangent to the curve, the tangential equation to the curve will be

$$a(\beta-\gamma)^2 \cot A + \beta(\gamma-a)^2 \cot B + \gamma(a-\beta)^2 \cot C = 0.$$

London, *March* 12, 1851.

GEOMETRICAL PROPOSITIONS RELATING TO FOCAL PROPERTIES OF SURFACES AND CURVES OF THE SECOND ORDER.

By John Walker.

Many geometers have at different times endeavoured to obtain from the cone itself a construction for the foci and directrices of Apollonius's sections of an oblique cone which should serve for studying their properties from elementary principles. The following general property of an oblique cone leads to a simple method of effecting this object.

If a plane cut a cone in a conic, the planes of two sub-contrary sections of the cone in two right lines, and the sphere passing through these circular sections in a circle, the rectangle under the segments of a chord (or secant) of the circle drawn from any point in the conic is to the recta
under the perpendiculars let fall from the same poi

two right lines in the constant ratio of the products of the sines of the angles which the plane of the conic and any side of the cone respectively make with the planes of circular section.

To prove this, from any point in the conic and from the vertex of the cone let fall perpendiculars on the planes of circular section; the four perpendiculars will lie in two planes passing through the side of the cone drawn through the point in the conic, and cutting each one of the two planes of circular section in right lines, the segments of which, together with the segments of the side of the cone and the four perpendiculars, form the sides of four right-angle triangles which are similar two and two: from compounding the proportions among the sides of these it easily appears, that the rectangle under the intercepts on the side of the cone between the point and the planes of circular section is to the rectangle under the perpendiculars let fall from the point on those planes, as the rectangle under the segments of the side between the vertex and the same planes is to the rectangle under the perpendiculars let fall from the vertex of the cone. But the latter of these two ratios is equal to the product of the reciprocals of the sines of the angles which any side of the cone makes with the planes of circular section, while the second term of the proportion is equal to the product of the perpendiculars let fall on the lines in which the plane of the conic intersects the same planes multiplied by the product of the sines of the angles it makes with them; also the first term, being the rectangle under the segments of a chord of the sphere, is equal to the rectangle under the segments of a chord (or secant) of the circle, in which the plane of the conic intersects the sphere, drawn from the same point: whence the proportion enunciated in the above theorem is easily inferred. For brevity's sake the circle and pair of right lines may be called a conjugate focal circle and pair of right lines, as it will shortly appear that they are merely a particular case of a pair of conjugate focal conics similarly related to the given conic; the constant ratio (which evidently remains unchanged for any given section of the cone, however the sphere and the circular sections be changed) may be called the modulus of the conic with respect to the cone. When the radius of a focal circle is evanescent it may be called a focal point; and when a pair of focal right lines become coincident, the single right line in which they merge may be called a directrix. The following corollaries from the theorem above will be evident:

(1). If the plane of the conic touch the sphere, the p[oint] of contact will become a focal point, and the square of the line drawn from any point in the conic to it is to the re[ct]angle under perpendiculars let fall on the conjugate fo[cal] pair of right lines in the constant ratio of the modulus of the conic to unity.

(2). When the plane of the conic, passing through the li[ne] of intersection of the circular sections of the cone, cuts the sphere passing through them in a circle, the tangent* dra[wn] from any point on the conic to this focal circle is to the p[er]pendicular let fall on the conjugate directrix in the s[ub]duplicate ratio of the modulus to unity.† (In such a case of course the conic is one of Apollonius's sections.)

(3). And if the plane of the conic touch the sphere, the point of contact and conjugate focal line become a focus and directrix of the conic.

(4). A directrix is the chord of the double contact wh[ich] its conjugate focal circle has with the conic if it meet it;

(5). Or, more generally, a pair of focal right lines will [be] the chords in which the conjugate focal circle intersects the conic if it meets it.

(6). If there be two focal circles described, conjug[ate] respectively each to one of a pair of directrices, the sum [or] difference of tangents drawn to them from any point in the conic will be constant; and the cosines of the angles wh[ich] lines drawn from the centres of these circles to any point [of] the conic make with the tangent at the point, are as the cosines of the angles which the same lines make with the tangents to the circles drawn from that point; also the

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* To the length of a tangent drawn from an external point to the ci[rcle] corresponds of course the half of the chord of the circle bisected at the point in case it fall within the circle.

† If the plane of the conic be perpendicular to the plane of the cen[tres] of circular sections, and be intersected by it in the line ACA', w[hile] any circular section is intersected by the same plane in the line B[CB'] (AA', BB' being the points in which these lines meet the cone, and C the point of intersection), and, D being taken on AA' so that rect. $ACD = B$[CB'] the line DB be drawn, it will easily appear that the angles ACB, A[DB] are respectively equal to the angles which the plane of the conic ma[kes] with the planes of circular section, and ABC, ADB to the angles wh[ich] the side of the cone AB makes with the same planes, and hence that the modulus of the conic is equal to the ratio $AD : AC$, or $AD \cdot AC : AC^2$, or $ACA' \pm BCB' : AC^2$, or $a^2 \pm b^2 : a^2$, if a be half the axis AA', an[d b] the semi-axis perpendicular to AA', the upper sign being used if the co[nic] be an ellipse, the lower if a hyperbola. If the conic be a parabola the modulus is in this case obviously equal to unity.

ormal at any point divides the line joining the centres of e circles into segments whose ratio is equal to that of ngents to the circles drawn from the point; and when the adii of the circles become evanescent, in which case each oincides with a focus of the curve, these obviously reduce the well-known properties of the lines joining any point the conic with its foci: the radical axis of two such circles ill obviously be parallel to, and equally distant from, their espective directrices, if these lines lie on opposite sides of e conic; and the points in the conic at which the sum of e lines drawn to the centres of the circles is a minimum re determined by describing the circle whose diameter is e line joining their centres of similitude.

(7). Generally, if the chord of a conic section passing rough the centre of a focal circle be bisected at that point, e portion of it intercepted between the conjugate focal pair f right lines will be bisected at the same point; and if the oint be on the conic the tangent at that point drawn between he focal right lines will be bisected at the point of contact.

(8). The polar of the point of intersection of a pair of ocal lines with respect to their conjugate focal circle is also he polar of the same point with respect to the conic: hence f any point in the plane of the conic be given, the locus of he centres of focal circles (considering the conic as a section f different cones) conjugate to pairs of focal right lines lrawn through that point is the perpendicular let fall from t on its polar with respect to the conic.

(9). And the focal point conjugate to a focal pair of right ines drawn through any given point in the plane of the onic is the foot of the perpendicular let fall from the given oint on its polar with respect to the conic.

From the third of these corollaries is deduced the following onstruction for the foci of one of Apollonius's sections of an blique cone: Let the plane drawn through V, the vertex of the cone, perpendicular to the planes of circular section, ut the cone in the sides VAB, $VA'B'$, the conic in the line AA', and the plane of any circular section in BB', and let AA', BB' intersect in C; from either end of AA', as A, take on that line lengths AE, AF, such that E, F may be harmonic conjugates with respect to A, A', and AE may be to AF in the subduplicate ratio of $ACA' \mp BCB'$ to ACA'; a problem of elementary geometry: through F draw a plane cutting the cone in a circle; it may easily be shewn that the square of EF is equal to the rectangle under the se

ments of a secant of the circle drawn from F, and hence that a sphere passing through this circle and the point E will touch the plane of the conic in that point and intersect the cone in a second circle, the plane of which will pass through the point F; the planes of the two circles, in which the sphere, which touches the plane of the conic in the point E, cuts the cone, will therefore intersect in a line lying in the plane of the conic, passing through the point F and perpendicular to the axis AA'; thus this line and the point E will be a directrix and focus of the conic; and if from A' lengths $A'E' = AE$, and $A'F' = AF$ be taken on AA', a line through F' perpendicular to AA' and the point E' will be the other directrix and its focus respectively. The upper sign in the above construction is to be used for an ellipse, (and for its applicability it is evidently necessary that AA' should be the major axis,) the lower for a hyperbola. If the conic be a parabola, the above construction evidently becomes indeterminate; but the points E, F may in this case be thus found: let A be the vertex of the parabola, and AB the diameter in which a circular section drawn through A is cut by the plane VAA'; join V with C, the middle point of AB, and draw from C a line making with CB an angle equal to the angle CVB, and meeting the side of the cone VB in D; the line DC produced, and a line from D parallel to AB, will meet the axis AA' in the points E, F: it is easy to shew that, if a circular section be drawn through F, a sphere described passing through this circle and through the point E will touch the plane of the parabola in that point, and will cut the cone in a second circle, the plane of which will pass through F; and therefore that a line through F in the plane of the parabola, perpendicular to AA', and the point E, will be the directrix and focus respectively of the parabola: it is also evident that this line and point are equally distant from the vertex A.

From the same corollaries it may be shewn that the focus is the only point in a conic possessing the property that all chords drawn through it have their poles lying on right lines drawn perpendicular to them from the point. Taking any point in the plane of the conic it may be considered as a focal point; if a chord drawn through it have its pole on a perpendicular line at the point, this pole must be the point of intersection of the focal pair of lines conjugate to the point; in order that this should be the case for all chords therefore the point of intersection of the focal lines must be indeterminate, *i.e.* the lines must coincide; but the focus, as has been

shewn, is the only point for which the conjugate focal pair of right lines merge in a single directrix.*

But the theorem at the commencement of this paper is only a particular case of the following general theorem, which appears to contain the ultimate generalization of the whole theory of foci and directrices: to abbreviate the enunciation of it, it will be convenient to employ the term *conjugate focal conics* with respect to any given conic for two conics so related to it, that the rectangles under the segments of chords (or secants) of them respectively, drawn from any point in the given conic parallel to two fixed lines, have a constant ratio one to the other; and two surfaces of the second order similarly related to a given surface of the same order may be called conjugate focal surfaces with respect to it: these definitions being premised, the theorem may be thus stated: If three surfaces of the second degree have the same pair of curves of intersection (*i.e.* if any one of the three pass through the pair of curves in which the other two intersect), any plane meeting the three surfaces will cut them in three conics so related that any two are conjugate focal conics with respect to the third; and this is a consequence

* It may be objected to this proof, so elementary and general in its nature, that it necessitates the consideration of any given conic as cut from a variety of cones; for it is not of course possible to describe a sphere cutting a given cone in two plane circular sections and touching the plane of any given section of the cone in *any* point. It is easily seen geometrically that if spheres be described cutting a cone in circular sections, and cutting the plane of a given section in a circle of constant radius, the locus of the centre of this circle in the plane of the conic is, $P^2 \tan^2\theta + r^2 = mx^2 + ny^2$; x and y being coordinates with respect to certain axes whose origin is the foot of the perpendicular let fall from the vertex of the cone on the plane of the conic, P the perpendicular let fall on the line in which that plane intersects the plane of centres of circular sections, θ the angle between these planes, and r the constant radius of the circle, m and n being certain constants, depending on the angle θ, and other constant angles in the cone. The locus of points of contact, if the sphere touch the plane of the section, is of course found by putting $r = 0$, and the curve thus determined divides the plane of the conic into two regions; no sphere cutting the cone in circular sections can meet the plane of the conic in any point in one region, but every point in the other region is the centre of a circle in which the plane of the conic is intersected by some such sphere. It may possibly be not difficult to discuss this bounding curve (which of course is different for the same conic according as it is cut out of various cones) geometrically, as it is confocal with the conic; it is a hyperbola, parabola, or ellipse, according as the conic is an ellipse, parabola, or hyperbola; when the focal lines are at right angles it has also its semiaxes proportional to those of the given conic; and when the focal lines are parallel, it becomes that axis of the given conic which is perpendicular to them. The locus of the point of intersection of the focal pair of lines conjugate to the variable focal point is the curve which is reciprocal-polar to the locus of that point with respect to the given conic; the two points are *corresponding points* on their respective loci, and the line joining them is normal to the first locus.

of the analogous property of the surfaces themselves, vi[z.] that any two will be conjugate focal surfaces with respect t[o] the third. The geometrical proof of this theorem is as eas[y] and elementary as the analytical, depending only on the well known corollary from Desargues' theorem (itself, as has bee[n] often shewn, capable of the most elementary geometric[al] proof from the fundamental property of plane sections o[f] a cone, viz. that the rectangles under the segments of chor[ds] (or secants) of a given conic drawn through any point in i[ts] plane, parallel to two fixed lines in that plane, bear a con[-] stant ratio one to the other), that if a transversal meet eac[h] of three conics, passing through the same four points, the si[x] points of intersection will form an *involution*. The onl[y] definition of surfaces of the second order employed in th[e] following proof is, that every section of them made by [a] plane is a conic; and the only property of conics employe[d] is the fundamental one above alluded to, and the theorem o[f] involution.

To fix the ideas, let the three surfaces be called S, S', S'', and L, L' the curves of their intersection; let a be a fix[ed] point in S, abb' a fixed line meeting S' in the points bb', an[d] acc' a fixed line meeting S'' in cc'; p being any variable poi[nt] in S, let the line ap meet S' in the points rr' and S in th[e] points $p'a'$; lastly let a line drawn from p parallel to abb' me[et] S' in qq', and a line drawn from the same point parallel to $ac[c']$ meet S'' in the points kk'. Then $pq.pq' : ab.ab' = pr.pr' : ar.a[r']$; but the latter ratio (by the corollary from Desargues' theore[m] stated above, applied to the system of three conics in whi[ch] the plane of the lines pqq', abb' meets the three surfac[es], which system of course pass through each of the four poin[ts] in which the same plane meets the two curves of intersectio[n] of the three surfaces) is equal to the ratio $pp'.pa' : ap.a[a']$; this last ratio is obviously equal to the ratio $pk.pk' : ac.a[c']$; therefore, finally, $pq.pq' : pk.pk' = ab.ab' : ac.ac'$.

If now the fixed lines abb' and acc' be supposed to [be] drawn in any plane cutting the three surfaces in thr[ee] conics C, C', C'', and if from any point p on C a line [be] drawn parallel to abb' meeting C' in qq', and from th[e] same point p a line parallel to acc', meeting C'' in $k[k']$, the rectangles $pq.pq'$ and $pk.pk'$ will bear a constant rat[io] one to the other.

A few particular cases of this theorem are easily deduc[ed] from its general form: (1). If three surfaces of the secon[d] order pass through the same curves of intersection, and [a] a ... touching one of the three at an umbilic cut th[e]

others in two conics, the point of contact and either conic will be a conjugate focal point and conic to the second conic. (2). If two surfaces of the second order intersect in two plane curves, and any plane cut one of the two in a circle (or touch it at an umbilic), the planes of intersection in two right lines and the second surface in a conic, the circle (or point of contact) and the pair of right lines will be a conjugate focal circle (or point) and pair of right lines to the conic. (3). Or, if through the line in which the planes of the curves of intersection meet, a plane be drawn cutting the surface in two conics, either of these and the right line will be a conjugate focal conic and directrix to the other conic. (4). And if the tangent plane to one of the surfaces, drawn through the same line, should meet it at an umbilic, and cut the other surface in a conic, the point of contact and the right line will be a focus and directrix of the conic; and since every point on a sphere is an umbilic, (5) If through the line of intersection of two planes, which cut a surface of the second order in circles, a plane be drawn touching the sphere which passes through these circles and cutting the surface in a conic, the point of contact and right line will be a focus and directrix of the conic. (6). Also, if two surfaces of the second order envelope one the other along a plane curve, and any plane cut the surfaces in two conics, and the plane of their curve of contact in a right line, either conic and this right line are a conjugate focal conic and directrix to the other conic. (7). And the plane which touches one surface at an umbilic cuts the other surface in a conic, and the plane of the curve of contact in a right line, such that the point of contact and the right line are a focus and directrix of the conic; a theorem given by M. Chasles, in the *Annales de Gergonne*, tom. XIX. p. 167.

It is obvious that if three conics be so related that two of them are conjugate focal conics to the third, any transversal will meet the three in six points forming an involution; also that if three conics be so related that any transversal meets them in six points which form an involution, then any two of the conics will be conjugate focal conics with respect to the third: hence it appears that if three conics be so related that two of them are conjugate focal conics to the third, then this third and either of the two will be conjugate focal conics to the remaining one of the three.

It is also evident from the nature of the proof of the theorem respecting the surfaces S, S', S'', that if (instead of the hypothesis that the third passes through the curves of

intersection of the other two) they had been supposed to so related, that any transversal met them in six points forming an involution, it would equally have followed that a two were conjugate focal surfaces with respect to the third and conversely, &c. &c.

The theorem respecting the surfaces S, S', S'' may ea be thrown into the following form, when S', S'' are cent surfaces: Let t, t' be the lengths of tangents drawn from a point in S to S', S'' respectively; let r be a semi-diameter a certain surface M, similar to and similarly placed as parallel to t, and r' a semi-diameter of a certain surface similar to and similarly placed as S'', parallel to t', th $t : t' = r : r'$. In this form the theorem leads synthetica to the following method of generating a surface S of t second order, from two given central surfaces of the seco order* S' and S''. Let M and M' be two surfaces respectiv similar to and similarly placed as these, then the locus a point such that, tangents t, t' being drawn from it to $S'S''$, semi-diameters r, r' of M, M', respectively parallel to th tangents, t is to t' as r to r', is a third surface of the seco order S; and the three surfaces $SS'S''$ will be so related t any two will be conjugate focal surfaces to the third. shew that the *modular* and *umbilicar* methods of generati surfaces of the second order are particular cases of this general method, it will be necessary in the present state the geometry of such surfaces to call in the aid of conceptio arising from analytical rather than geometrical methods.

It may be shewn that if a sphere S' and a finite surface the second order S'' intersect a given surface S of the seco order in the same pair of curves (real or imaginary), and if and S'', remaining constant in species, diminish in dimensio without limit, then the points s', s'', into which they degen rate, become ultimately any *corresponding points* on a cert surface confocal with S, and its reciprocal polar with respe to S, respectively; if S'' be supposed to be a hyperbolo then, S remaining constant, the same will be true of s', co sidered as the limit of the sphere, and s'' the ultimate positi of the vertex of the cone into which S'' degenerates. shew these things, let

$$Ax^2 + By^2 + Cz^2 = 1$$

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* It is obvious that if $f(xyz)$, $f_1(xyz)$ be any two rational and inte functions of the n^{th} degree, the geometrical interpretation of the equati $f(xyz) = 0$, $f_1(xyz) = 0$, and $f(xyz) - \mu f_1(xyz) = 0$, suggests an analog generation fo[illegible] of any order.

ers in two conics, the point of contact and either conic be a conjugate focal point and conic to the second conic. If two surfaces of the second order intersect in two ne curves, and any plane cut one of the two in a circle touch it at an umbilic), the planes of intersection in two t lines and the second surface in a conic, the circle point of contact) and the pair of right lines will be a conjate focal circle (or point) and pair of right lines to the ic. (3). Or, if through the line in which the planes of curves of intersection meet, a plane be drawn cutting the face in two conics, either of these and the right line will a conjugate focal conic and directrix to the other conic. And if the tangent plane to one of the surfaces, drawn ough the same line, should meet it at an umbilic, and cut other surface in a conic, the point of contact and the ht line will be a focus and directrix of the conic; and ce every point on a sphere is an umbilic, (5) If through line of intersection of two planes, which cut a surface the second order in circles, a plane be drawn touching sphere which passes through these circles and cutting surface in a conic, the point of contact and right line l be a focus and directrix of the conic (6). Also, if surfaces of the second order envelope one the other ng a plane curve, and any plane cut the surfaces in two ics, and the plane of their curve of contact in a right line, her conic and this right line are a conjugate focal conic directrix to the other conic. (7). And the plane which ches one surface at an umbilic cuts the other surface in nic, and the plane of the curve of contact in a right line, h that the point of contact and the right line are a focus directrix of the conic; a theorem given by M. Chasles, the *Annales de Gergonne*, tom. XIX p. 167.

It is obvious that if three conics be so related that two of m are conjugate focal conics to the third, any transversal l meet the three in six points forming an involution; also t if three conics be so related that any transversal meets m in six points which form an involution, then any two the conics will be conjugate focal conics with respect to third: hence it appears that if three conics be so related t two of them are conjugate focal conics to the third, then third and either of the two will be conjugate focal conics the remaining one of the three.

It is also evident from the nature of the proof of the orem respecting the surfaces S, [illegible] t if (instead of hypothesis that the third pass[illegible] he curves of

which is the ultimate position of S', and the point in which the line of intersection of the ultimate position of the planes LL' pierces the plane of the greatest and least axis of S, will be corresponding points on the *excentric conic* of S in that plane, and its reciprocal polar with respect to the section of S by the same plane. The readers of the *Journal* are well aware of the many applications of this latter principle which Mr. Townsend has made in its pages.

Returning, after these preliminary Lemmas, to the proportion $t:t'=r:r'$, it will at once appear from the manner in which this is deduced that, the surface S remaining constant, S' and S'' will diminish in the same ratio, and hence that the surfaces M, M' remain constant; so that, if S' be supposed to be a sphere, and K a certain constant, and l, l' lines drawn from any point in S to the points s', s'', which are the ultimate positions of the sphere S' and the surface S'' respectively, and if r' be a semi-diameter of M parallel to l', then $l:l'=K:r'$, which is the form of Mr. Willock's generalization of MacCullagh's *modular* generation of surfaces of the second order.

If, again, in the same general theorem respecting $SS'S''$, S' be supposed to be a sphere and S'' the system of two planes L, L' in which it intersects S, and if from any point p in S perpendiculars pq, pq' be let fall on the planes L, L' respectively, and a tangent pt be drawn to S', then pt^2 is to $pq.pq'$ in a constant ratio, and, proceeding to limits, this theorem leads synthetically to Mr. Salmon's *umbilicar* method of generation. It is not uninteresting, perhaps, to observe how the *modular* and *umbilicar* methods flow synthetically from one general theorem expressing the conjugate focal relation of any two with respect to the third of three surfaces $SS'S''$ connected by the relation of the involution of the six points in which they are met by any transversal, the geometrical relation which answers to the analytical one of the three surfaces having the same real or imaginary curves of intersection. Perhaps had the ideas suggested by the relation of involution been pursued farther, it might have been unnecessary to introduce the analytical method of proof to establish the connecting link between the general theorem of the three surfaces $SS'S''$ and the *modular* and *umbilicar* methods of generating surfaces of the second order.

To these methods may evidently be added the following one, from what has been established above: If s' be a point, and S'' a cone of the second order, and if a point p be taken such that, joining ps' and drawing from p a line pqq', parallel to a fixed line and meeting the cone in qq', ps'^2 always bear

a constant ratio to the rectangle $pq.pq'$, p will generate a surface S of the second order, and the ratio will remain constant for the same surface S if s' move in a surface confocal with S, and the vertex of the cone (the cone moving parallel to itself) move so as always to be at the point *corresponding* to s' of the surface which is the reciprocal polar of the locus of s' with respect to the same surface S. The *umbilicar* method is obviously the particular case of this in which the cone breaks up into two planes.

From the same general theorem of the three co-intersecting surfaces $SS'S''$ many other theorems may be obviously deduced, such as that, *e.g.*: If two planes touch a surface of the second order in two points, the rectangle under perpendiculars let fall from any point on the surface on the planes is to the square of the perpendicular let fall on the chord of contact in a constant ratio; or, more generally, if two surfaces have double contact, and if through any point of one a chord (or secant) of the other be drawn parallel to a fixed line, and from the same point a perpendicular be let fall on the chord of contact, its square will bear a constant ratio to the rectangle under the segments of the chord (or secant).

The value of the ratio which has been called in the commencement of this paper the *modulus* of a section of a cone, may in the general case be expressed in terms of half the angle (i) between the right lines in which the plane of the section is intersected by the planes of circular section, and of the semi-diameters (a, b) parallel to the internal and external bisectors of that angle, by the formula

$$a^2 \mp b^2 : a^2 \sin^2 i \pm b^2 \cos^2 i,$$

if the conic be an ellipse or hyperbola; and by the formula $1 : \sin^2 i$, if it be a parabola. This value of the modulus may be thus obtained: Let the plane drawn through V, the vertex of the cone, perpendicular to the planes of circular section, intersect the plane of the conic in the line AA', and the planes of two circular sections, drawn through O, any point in AA', in the lines BB', CC'; and let the planes of the circles intersect the plane of the conic in the lines mOm', nOn', the points AA', BB', CC', mm', nn', being the points in which these lines meet the cone; and let any sphere described with the point O as centre be met by the lines Om, On, OA, OB, OC, and by the line of intersection of the planes of the circular sections drawn through O, in the six points θ, θ', f, g, h, ... $\theta\theta\theta'$ are the angles of the spherical triangle fo... ...mes of the conic and

of the two circular sections, and the arcs $\theta''f$, $\theta''g$, $\theta''h$, are quadrants. If ϕ, ϕ' be the angles which the side of the cone $VABC$ makes with the planes of circular section (*i.e.* the angles VBB', VCC'), we shall obviously get, as in note p. 18,

$$\sin AOB . \sin AOC : \sin\phi \sin\phi'$$
$$= AOA' \mp COC' : AOA' = AOA' \mp mOm' : AOA' :$$

but from the spherical triangles,

$$\sin AOB = \sin fg = \sin \theta\theta''f = \sin\theta \sin f\theta = \sin\theta . \sin mOA;$$

similarly,

$$\sin AOC = \sin fh = \sin \theta'\theta''f = \sin\theta' \sin f\theta' = \sin\theta' \sin nOA;$$

therefore the proportion above becomes

$$\sin\theta . \sin\theta' . \sin mOA . \sin nOA : \sin\phi \sin\phi' = AOA' \mp mOm' : AOA';$$

but $$AOA' \mp mOm' : AOA'$$
$$= \sin mOA . \sin nOA \, (a^2 \mp b^2) : a^2 \sin^2 i \pm b^2 \cos^2 i,$$

by the properties of the conic; therefore finally

$$\sin\theta \sin\theta' : \sin\phi \sin\phi' = a^2 \mp b^2 : a^2 \sin^2 i \pm b^2 \cos^2 i.$$

Kingstown, near Dublin,
April 9, 1851.

ON PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

By Professor De Morgan.

The reasons why I think it worth while to present the following investigation, will appear from the historical remarks which follow it.

As usual, let $$dz = pdx + qdy \quad \ldots\ldots\ldots\ldots\ldots (1),$$

and let $$\phi(x, y, z, p, q) = 0 \quad \ldots\ldots\ldots\ldots (2),$$

be a partial differential equation of which the complete solution is required. We must determine the form of ψ in

$$\psi(x, y, z, p, q) = 0 \quad \ldots\ldots\ldots\ldots (3),$$

in such manner that p and q, obtained in terms of x, y, z, from (2) and (3), and substituted in (1), will render (1) integrable by means of a factor.

Let this factor be m. We have then

$$\frac{dm}{dx} = -\frac{d(mp)}{dz}, \quad \frac{dm}{dy} = -\frac{d(mq)}{dz}, \quad \frac{d(mp)}{dy} = \frac{d(mq)}{dx}.$$

[redacted]

$$q\frac{dp}{dz}-p\frac{dq}{dz}+\frac{dp}{dy}-\frac{dq}{dx}=0\ldots\ldots\ldots\ldots(5).$$

Though the single equation (5) is not coextensive with the system (4), nevertheless, (5) being satisfied, (1) can be integrated. To shew this, first let z be constant, and let λ be the factor which *then* makes $pdx + qdy$ integrable: let

$$\lambda pdx + \lambda qdy = dt.$$

Then, making z again variable, we have

$$dt = \left(\lambda + \frac{dt}{dz}\right)dz\ldots\ldots\ldots\ldots\ldots(6).$$

This equation will contain only t and z, if

$$\frac{dt}{dy}\cdot\frac{d}{dx}\left(\lambda+\frac{dt}{dz}\right)-\frac{dt}{dx}\cdot\frac{d}{dy}\left(\lambda+\frac{dt}{dz}\right)\ldots\ldots(7)$$

vanish. By aid of $\frac{dt}{dx}=\lambda p$, $\frac{dt}{dy}=\lambda q$, $\frac{d(\lambda p)}{dy}=\frac{d(\lambda q)}{dx}$, we shall find (7) reducible to λ^2 multiplied by the first side of (5). Consequently, when (5) is satisfied, (6) is a common differential equation, which gives t in terms of z. But t is already known in terms of x, y, z, so that a relation can be found between x, y, z, without t, which is an integral of (1). That is, if p and q be any functions of x, y, z, which render (5) identical, they also render (1) integrable: and the converse.

It follows, then, that to determine every integral of (2), we may determine every form of ψ in (3) which, with the given form of ϕ in (2), implies such values of p and q in terms of x, y, z, as have just been described. Now from (2) and (3) we easily obtain (denoting partial differentiation by variables suffixed)

$$\frac{dp}{dx}=\frac{\phi_q\psi_x-\phi_x\psi_q}{\phi_p\psi_q-\phi_q\psi_p}\quad\frac{dq}{dx}=\frac{\phi_x\psi_p-\phi_p\psi_x}{\phi_p\psi_q-\phi_q\psi_p}\ldots\ldots(8),$$

in which x may be changed into y or into z. Substitution in (5) gives

$$(\phi_x+\phi_z p)\,\psi_p+(\phi_y+\phi_z q)\,\psi_q=(\psi_x+\psi_z p)\,\phi_p+(\psi_y+\psi_z q)\,\phi_q\ldots(9),$$

from which, ϕ being given, ψ is to be determined, or, ψ being given, ϕ is to be determined. Making the first supposition, let $d\phi = Xdx + Ydy + Zdz + Pdp + Qdq$: we have then

$$P\frac{d\psi}{dx}+Q\frac{d\psi}{dy}+(Pp+Qq)\frac{d\psi}{dz}=(X+Zp)\frac{d\psi}{dp}+(Y+Zq)\frac{d\psi}{dq}\ldots(9)'.$$

From (9)′ ψ must be determined by help of the simultaneo
system

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{Pp + Qq} = - \frac{dp}{X + Zp} = - \frac{dq}{Y + Zq} = \frac{d\psi}{0} \;\ldots(10)$$

Of this system ϕ = const. and ψ = const. are both primitive
If we can determine three others, independent of the tw
just named, A = const., B = const., C = const., then the com-
plete integral of (9)′ is $\psi = f(A, B, C, \phi)$, where f is any for
whatever. That is, the complete solution of $\phi = 0$ is to b
found by integrating $dz = pdx + qdy$, where p and q a
determined from

$$\phi = 0, \quad f(A, B, C) = 0 \ldots\ldots\ldots\ldots(11).$$

The above is substantially the procedure of Lagrange a
Charpit, as presently noticed, made perhaps clearer by th
direct use of the criterion in (7), and shorter by that of th
equations in (8). The great difficulty of the result, on whic
I cannot get a satisfactory view from anything which h
been written, is that which arises from its apparently t
great generality. We know that $\phi = 0$ can only introdu
into its solution an arbitrary relation between *two* giv
forms, while (11) exhibits an arbitrary relation betwe
three. Undoubtedly, in any case which might be propos
the actual attainment of the final result from $dz = pdx + q$
would shew that this excess of generality is only appare
but this does not satisfy the inquirer.

The difficulty appears to me to have its source in a negl
of the conditions of the problem. When we demand t
solution of the equation $\phi(x, y, z, p, q) = 0$, we require n
only that $\phi = 0$ shall exist, but that $dz = pdx + qdy$ sh
exist, and not only exist, but exist in a particular mode,
as to admit of integration. The system (11) satisfies only t
first and third conditions: for (1) and (9)′ neither expr
nor imply $dz = pdx + qdy$, but only provide the equation
be integrated, and the means of satisfying the conditio
without which $dz = pdx + qdy$ cannot exist *in the man
proposed*. Consequently, the express introduction of t
last equation is really a new condition, and a limitation
generality. But as we cannot verify this last assertio
directly, at least while f is perfectly general, we must e
deavour to succeed by other than direct means. That i
instead of applying (11) to (1), we must apply (1) to (11).

Assuming $dz = pdx + qdy$, we have

$$dA = (A_x + A_z p)\,dx + (A_y + A_z q)\,dy + A_p dp + A_q dq,$$

e end by a process which produces a solution out of the tisfaction of the criterion, without any apparent need of ttempting the main investigation, to which the satisfaction f the criterion was preliminary. At this end we arrive henever we can completely solve the system (10). But in any cases it will happen that we can only find a partial olution, such as $A = a$. All we can then do is to determine and q from $\phi = 0$, $A = a$, and to integrate $dz = pdx + qdy$: y which we produce a *primary solution*, having the two onstants, a, and the one introduced in integration. This must be treated in the usual way, and its result is often more convenient than that derived from a more complete olution of (10). Thus, in the above example, if we take $q = a^2p$, $z = pq$, we find

$$dz = \sqrt{z}\left(\frac{dx}{a} + ady\right), \quad \sqrt{z} = \tfrac{1}{2}\left(\frac{x}{a} + ay\right) + b,$$

in which we make $b = \psi a$, &c.

Lagrange, in 1772, published in the *Berlin Memoirs* the root of the above method, depending directly upon (5), and using for the integration of (5) such methods as Euler and himself then had at command. A year or two afterwards* he gave, by itself, the now common method by which (9)′ is made to depend upon (10).

But Lagrange did not carry his method so far as to augment the number of independent variables, as done in the transformation of (5) into (9)′, by which the complete dependence upon linear form is established. This "rapprochement," as Lacroix calls it, or connecting step between two methods of Lagrange, was made by Charpit, a young man of high promise, whose death (Lacroix seems to insinuate) prevented his memoir from being printed (*Calc. Diff.*, vol. II. p. 548). The complete method thus bears the name of Charpit in Lacroix's work, which may have caused it to attract less attention than it would have done under that of Lagrange: nothing more than the original method, as it stood previously to the *rapprochement*, appears in Peacock's or Gregory's examples. When it was resumed by Lagrange, it was only with reference to an incident of the solution, as follows. Up to this time, nothing had been contemplated beyond using $\phi = 0$, and $f(A, B, C) = 0$, to determine p and q for substitution in (1), to be followed by actual integration.

* I cannot refer to this paper: Monge and Lacroix give it the date 1779, D. F. Gregory gives 1774 as well.

But this gave too much generality for the solution of equation of the first order with *two* independent variables there being *three* definite forms in the arbitrary function Lagrange (Lacroix, vol. II. p. 564) set himself to remove the difficulty, which he succeeded in doing, and thereby, which he hardly appears to have noticed, he completed his solution as well as removed an anomaly. But his reasoning is not altogether correct. Having shewn on general grounds that such an equation as (13), in which $d\phi = 0$, must exist, he then reasons (according to Lacroix) as follows, symbols and equation-references being altered to suit this paper :—"puisque les équations $A = a$, $B = b$, $C = c$, desquelles résulte $dA = 0$, $dB = 0$, $dC = 0$, vérifient l'équation (13), il faut que la partie multipliée par dp dans l'équation (13) s'anéantisse d'elle-même, et que l'équation (1), transformée par les variables A, B, C, se réduise à la forme (15), θ, λ, μ étant des coefficiens dependans de F_1, F_2, F_3." These last are the expressions for x, y, z, in terms of A, B, C, p: and dp is then declared to disappear from (15) because the criterion already requires that there should be a relation (11) between A, B, C. This argument, if good for anything, would go so far as to prove that (15) must be the very equation which results from that relation; which it is not, and cannot be, (15) being unintegrable. The mode of satisfying an equation which cannot be integrated by help of a factor, and the geometrical meaning of the solution, was given by Monge in 1784, and up to that time such an equation, for instance as $dz = dx + xdy$ was held to be a contradiction, as Monge distinctly states.

The last-mentioned analyst, who, of all that ever lived, had the greatest power of eliciting the geometrical meaning of a formula, deduced the whole method from the properties of surfaces. Not having seen the *rapprochement* of Charpit, which was first printed by Lacroix, he may easily have thought that geometry had given him a method beyond analysis, or at least beyond easy attainment. The *ten* equations which arise from all modes of equating the first five quantities in (10) are deduced in their *separate* meanings, as belonging to his celebrated *characteristics* of a surface, or to the other curves or to the developable surfaces which then arise. This method first appeared either in the second or third (1807) edition of the *Application de l'Analyse, &c.*

M. Cauchy, in a memoir which (having only a separate copy) I can but conjecture to belong to the *Bull. de la Soc. Philomath.*, gives the above method complete, and exte-

it (as indeed did Charpit) to the case of three independent variables; but does not seem to be aware of what Charpit and Monge had done. Poisson (*Bull. de la Soc. Philomath.* 1815, p. 183) has treated of the difficulty in overcoming which Lagrange completed his solution: but he appears to me to shew nothing more than that the apparently too general form fully contains the ordinary; one of which there could be no doubt: the difficulty lies in shewing that it does not contain more.

October 10, 1851.

ON THE LOGARITHMIC PARABOLA.

By the Rev. J. Booth, LL.D., F.R.S., etc.

It has long been known that plane curves of the second order may be rectified by circular arcs, by logarithms, or by elliptic integrals of the first and second orders. It is only very recently however that these geometrical types have been extended so as to embrace elliptic integrals of every order, and that it has been shewn that all elliptic integrals—the parameter ranging from infinity positive to infinity negative—represent the symmetrical intersections of surfaces of the second order. These curves, to distinguish them by an appropriate name, may be called Hyperconic Sections.

Some years ago, however, an extension of this theory was given, both in this country and on the continent. In the Mathematical Journals of that time,* demonstrations were given of the theorem, now well known, that an elliptic integral, of the third order and circular form, represents a spherical conic section, or a particular class of hyperconics.

In a work published in the early part of last year† the author has shewn that an elliptic integral of the third order and logarithmic form, represents the symmetrical intersection of a paraboloid of revolution with an elliptic cylinder, when the parameter is negative and of the form $i^2 \sin^2\theta$, i being the modulus; and that when the parameter is negative and greater than 1, or of the form $\operatorname{cosec}^2\theta$, the integral represents the intersection of a paraboloid of revolution with an hyperbolic cylinder. Thus the ordinary formulæ for the rectification

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* *Liouville's Journal*, 1841. *Philosophical Magazine*, 1842.

† The Theory of Elliptic Integrals, and the Properties of Surfaces of the Second Order applied to the investigation of the Motion of a Body round a Fixed Point. London: G. Bell. 1851.

of the conic sections are merely particular cases of those more general expressions known as elliptic integrals. More than this, all, or nearly all, the forms of comparison of those functions may simply and easily be derived from the properties of those curves or hyperconic sections, as they may with propriety be termed, described as they are, not on a plane, but on a sphere or a paraboloid.

1. We propose here to treat of the rectification of one of those hyperconics, the logarithmic parabola; or the curve of intersection of a parabolic cylinder and a paraboloid of revolution; the vertex of this surface being supposed to touch at its focus the plane of the parabola, the base of the parabolic cylinder.

Let the equation of the paraboloid be

$$x^2 + y^2 = 2kz \ldots\ldots\ldots\ldots\ldots (1),$$

and $y^2 = 4g^2 + 4gx$, that of the parabolic base of the cylinder, the origin being at the focus. k is the semiparameter of the paraboloid, and g is one-fourth of the parameter of the base.

$$\because\ y^2 + x^2 = = (2g + x)^2 = 2kz \ldots\ldots\ldots\ldots (2),$$

hence, x being the independent variable,

$$\frac{dz^2}{dx^2} = \frac{(2g+x)^2}{k^2},\quad \frac{dy^2}{dx^2} = \frac{g}{g+x} \ldots\ldots\ldots (3),$$

therefore

$$\frac{d\Sigma}{dx} = \frac{(2g+x)\,[k^2 + (g+x)(2g+x)]}{\sqrt{\{k^2(g+x)(2g+x)\,[k^2+(g+x)(2g+x)]\}}} \ldots (4).$$

Now the expression under the radical being a quadrinomial in x, must be reducible to the usual form of an elliptic integral. We must choose a suitable transformation. Let

$$\tan^2\mu = \frac{dz^2}{dx^2 + dy^2} = \frac{(2g+x)(g+x)}{k^2} \ldots\ldots (5),$$

deriving this value from (3). Substituting this value in (4) and reducing, we obtain the simple expression

$$\frac{d\Sigma}{dx} = \frac{2g + x}{k\sin\mu} \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

μ is evidently the inclination to the plane of xy, of a tangent drawn to the curve.

We must now eliminate x. Since

$$k^2\tan^2\mu = 2g^2 + 3gx + x^2,$$

dding and subtracting $2g^2 - gx$, we shall have

$$k^2 \tan^2\mu = (2g + x)^2 - g\,(2x + x).$$

Completing the square, and taking the square root,

$$2\,(2g + x) = g + \sqrt{(4k^2 \tan^2\mu + g^2)} \,\ldots\ldots\ldots (7).$$

The positive sign only must be taken, for when $x = -g$, $\tan\mu = 0$. Substituting this value of $2g + x$ in the expression for the arc,

$$\frac{d\Sigma}{dx} = \frac{g + \sqrt{(4k^2 \tan^2\mu + g^2)}}{2k \sin\mu} \,\ldots\ldots\ldots\ldots (8).$$

If now we differentiate (5), we shall obtain

$$\frac{dx}{d\mu} = \frac{2k^2 \sin\mu}{\cos^3\mu\,\sqrt{(4k^2 \tan^2\mu + g^2)}} \,\ldots\ldots\ldots\ldots (9).$$

Multiplying the last equation by this expression,

$$\frac{d\Sigma}{d\mu} = \frac{d\Sigma}{dx}\,\frac{dx}{d\mu} = \frac{gk}{\cos^3\mu\,\sqrt{(4k^2 \tan^2\mu + g^2)}} + \frac{k}{\cos^3\mu},$$

or
$$\Sigma = k\int\frac{d\mu}{\cos^3\mu} + gk \times \int\frac{d\mu}{\cos^3\mu\,\sqrt{(g^2 + 4k^2 \tan^2\mu)}} \,\ldots (10)$$

There are now three cases to be considered:

$$2k = g, \quad 2k < g, \quad 2k > g.$$

Case I. Let $g = 2k$, and the last equation will become

$$\Sigma = k\int\frac{d\mu}{\cos^3\mu} + k\int\frac{d\mu}{\cos^2\mu} = k\int\frac{d\mu}{\cos^3\mu} + k\tan\mu \,\ldots (11).$$

Now $k \tan\mu$ is the ordinate of a parabola, and $k\int\frac{d\mu}{\cos^3\mu}$ is the length of an arc of this parabola from the vertex to a point where a tangent to it makes the angle μ with the ordinate. Hence if we assume on the logarithmic parabola a point M, and through this point draw a plane touching the parabolic cylinder, this plane will be vertical, and will cut the vertical paraboloid in a parabola, whose semiparameter will be k. This parabola will touch the logarithmic parabola at the point M. Hence in this case the length of the logarithmic parabola to the point M, will be equal to the arc of the plane parabola from its vertex to the point M, plus the ordinate of this parabola at the point M.

Case II. Let $g > 2k$.

The general expression may be written

$$\Sigma = k\int\frac{d\mu}{\cos^2\mu\sqrt{\left\{1-\left(\frac{g^2-4k^2}{g^2}\right)\sin^2\mu\right\}}}+k\int\frac{d\mu}{\cos^3\mu}\ \ldots(12).$$

Let
$$\frac{g^2-4k^2}{g^2}=i^2\ \ldots\ldots\ldots\ldots\ldots(13),$$

and the last equation becomes

$$\Sigma = k\int\frac{d\mu}{\cos^2\mu\sqrt{(1-i^2\sin^2\mu)}}+k\int\frac{d\mu}{\cos^3\mu}\ \ldots\ (14).$$

Now U being the arc of an hyperbola, a the transverse axis, and $i^2=\dfrac{a^2}{a^2+b^2}$, we know that

$$\frac{iU}{a(1-i^2)}=\int\frac{d\mu}{\cos^2\mu\sqrt{(1-i^2\sin^2\mu)}}\ \ldots\ldots(15);$$

hence if $k=\dfrac{a(1-i^2)}{i}$, we shall have

Logarithmic parabola = plane hyperbola + plane parabola ....(16).

The semiaxes a, b of this hyperbola may easily be determined by the equations

$$k=\frac{a(1-i^2)}{i},\quad i^2=\frac{a^2}{a^2+b^2};\quad \text{or } a^2=\frac{g^2(g^2-4k^2)}{16k^2},\quad b=\frac{g}{2}\ldots(17).$$

We may eliminate the arc of the hyperbola and introduce instead, elliptic integrals of the first and second orders.

Let $\sqrt{I}=1-i^2\sin^2\mu$, then

$$\frac{\Sigma}{k}=\int\frac{d\mu}{\cos^2\mu\sqrt{I}}+\int\frac{d\mu}{\cos^3\mu}$$

and the formula, for comparing elliptic integrals with reciprocal parameters, gives

$$\int\frac{d\mu}{\cos^2\mu\sqrt{I}}+\int\frac{d\mu}{I\sqrt{I}}=\int\frac{d\mu}{\sqrt{I}}+\frac{\tan\mu}{\sqrt{I}}\ \ldots\ldots(18).$$

We have also

$$-\int\frac{d\mu}{I\sqrt{I}}=\frac{-1}{1-i^2}\int d\mu\sqrt{I}+\frac{i^2}{1-i^2}\,\frac{\sin\mu\cos\mu}{\sqrt{I}}\,.$$

Adding and reducing,

$$\Sigma = k\left[\int\frac{d\mu}{\sqrt{I}} + \int\frac{d\mu}{\cos^3\mu}\right] + \frac{g^2}{4k}\left[\tan\mu\,\sqrt{I} - \int d\mu\,\sqrt{I}\right] \ldots (19).$$

Case III. Let $2k > g$.

To integrate in this case, we must transform the second member of the equation (8). Assume

$$2k\tan\mu = g\tan\nu \ldots\ldots\ldots\ldots (20).$$

Then if we make $\frac{4k^2 - g^2}{4k^2} = j'^2$, we shall have

$$2(2g + x) = g + g\sec\nu, \quad \text{and} \quad \frac{dx}{d\nu} = \frac{g}{2}\,\frac{\sin\nu}{\cos^2\nu}.$$

But
$$\sin^2\mu = \frac{g^2\sin^2\nu}{4k^2(1 - j'^2\sin^2\nu)},$$

hence
$$\frac{d\Sigma'}{d\nu} = \frac{g}{2}\,\frac{\sqrt{(1 - j'^2\sin^2\nu)}}{\cos^2\nu} + \frac{g}{2}\,\frac{\sqrt{(1 - j'^2\sin^2\nu)}}{\cos^3\nu} \ldots (21).$$

Now since
$$\frac{k\,d\mu}{\cos^2\mu} = \frac{g}{2}\,\frac{d\nu}{\cos^2\nu},$$

and
$$\cos\mu = \frac{\cos\nu}{\sqrt{(1 - j'^2\sin^2\nu)}} = \frac{\cos\nu}{\sqrt{J}}.$$

Writing J for $(1 - j'^2\sin^2\nu)$, we shall have

$$k\int\frac{d\mu}{\cos^3\mu} = \frac{g}{2}\int\frac{d\nu\,\sqrt{(1 - j'^2\sin^2\nu)}}{\cos^3\nu},$$

or
$$\Sigma' = \frac{gj'^2}{2}\int\frac{d\nu}{\sqrt{(1 - j'^2\sin^2\nu)}} + \frac{g}{2}(1 - j'^2)\int\frac{d\nu}{\cos^2\nu\sqrt{(1 - j'^2\sin^2\nu)}} + k\int\frac{d\mu}{\cos^3\mu} \ldots\ldots (22).$$

Now the second term of the right-hand member of this equation is the expression for an arc of a hyperbola, the distance between whose foci is g. Hence

$$\Sigma' = \frac{gj'^2}{2}\int\frac{d\nu}{\sqrt{(1 - j'^2\sin^2\nu)}} + U + \Pi \ldots\ldots (23),$$

Π being an arc of the parabola.

We may elimina ion of the first order and represent in this e logarithmic parabola by the arcs of an e d a parabola.

Let U be the arc of an hyperbola whose semi-transverse axis is $\frac{1}{j}$, and putting E and Π for the elliptic and parabolic arcs,

$$\Sigma' = \frac{g}{2}\,\frac{j^2}{(1-j^2)}\left[U_{(\nu)} + E_j(\nu) - \tan\nu\sqrt{J}\right] + \Pi(\mu) \ldots (24);$$

or, as the equation may be written,

$$\Sigma' = \frac{g}{2}\left[\int\frac{d\nu}{\sqrt{J}} - \int d\nu\sqrt{J} + \tan\nu\sqrt{J}\right] + k\int\frac{d\mu}{\cos^2\mu} \ldots (25).$$

[*To be continued.*]

ON THE THEORY OF PERMUTANTS.

By Arthur Cayley.

A form may be considered as composed of blanks which are to be filled up by inserting in them specializing characters, and a form the blanks of which are so filled up becomes a symbol. We may for brevity speak of the blanks of a symbol in the sense of the blanks of the form from which such symbol is derived. Suppose the characters are 1, 2, 3, 4,... the symbol may always be represented in the first instance and without reference to the nature of the form, by V_{1234}.. And it will be proper to consider the blanks as having an invariable order to which reference will implicitly be made; thus, in speaking of the characters 2, 1, 3, 4,. . . . instead of as before 1, 2, 4,. . . . the symbol will be V_{2134}.. instead of V_{1234}... When the form is given we shall have an equation such as

$$V_{1234} = P_{12}Q_3R_4 \ldots \quad \text{or} = P_{12}P_{34} \ldots \;\&\text{c.},$$

according to the particular nature of the form.

Consider now the characters 1, 2, 3, 4,. . . . and let the primitive arrangement and every arrangement derivable from it by means of an even number of inversions or interchanges of two characters be considered as positive, and the arrangements derived from the primitive arrangement by an odd number of inversions or interchanges of two characters be considered as negative; a rule which may be termed "the rule of signs." The aggregate of the symbols which correspond to every possible arrangement of the characters,

giving to each symbol the sign of the arrangement, may be termed a "Permutant;" or, in distinction from the more general functions which will presently be considered, a simple permutant, and may be represented by enclosing the symbol in brackets, thus (V_{1234}). And by using an expression still more elliptical than the blanks of a symbol, we may speak of the blanks of a permutant, or the characters of a permutant.

As an instance of a simple permutant, we may take

$$(V_{123}) = V_{123} + V_{231} + V_{312} - V_{132} - V_{213} - V_{321}.$$

And if in particular $V_{123} = a_1 b_2 c_3$, then

$$(V_{123}) = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_1 b_3 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1.$$

It follows at once that a simple permutant remains unaltered, to the sign *près* according to the rule of signs, by any permutations of the characters entering into the permutant. For instance,

$$(V_{123}) = (V_{231}) = (V_{312}) = -(V_{132}) = -(V_{213}) = -(V_{321}).$$

Consequently also when two or more of the characters are identical, the permutant vanishes, thus

$$V_{113} = 0.$$

The form of the symbol may be such that the symbol remains unaltered, to the sign *près* according to the rule of signs, for any permutations of the characters in certain particular blanks. Such a system of blanks may be termed a quote. Thus, if the first and second blanks are a quote,

$$V_{123} = -V_{213}, \quad V_{132} = -V_{312}, \quad V_{231} = -V_{321},$$

and consequently

$$(V_{123}) = 2(V_{123} + V_{231} + V_{312}).$$

And if the blanks constitute one single quote,

$$(V_{123\ldots}) = NV_{123\ldots},$$

where $N = 1.2.3\ldots n$, n being the number of characters. An important case, which will be noticed in the sequel, is that in which the whole series of blanks divide themselves into quotes, each of them containing the same number of blanks. Thus, if the first and second blanks, and the third and fourth blanks, form quotes respectively,

$$\tfrac{1}{6}(V_{1234}) = V_{12}\ [illegible]\ V_{1423} + V_{34\,2} + V_{4312} + V_{2314}.$$

It is easy now t [illegible] eneral definition of a "Permutant." We h [illegible] er the blanks as forming

not as heretofore a single set, but any number of distinct sets, and to consider the characters in each set of blanks as permutable *inter se* and not otherwise, giving to the symbol the sign compounded of the signs corresponding to the arrangements of the characters in the different sets of blanks. Thus, if the first and second blanks form a set, and the third and fourth blanks form a set,

$$((V_{1234})) = V_{1234} - V_{2134} - V_{1243} + V_{2143}.$$

The word 'set' will be used throughout in the above technical sense. The particular mode in which the blanks are divided into sets may be indicated either in words or by some superadded notation. It is clear that the theory of permutants depends ultimately on that of simple permutants; for if in a compound permutant we first write down all the terms which can be obtained, leaving unpermuted the characters of a particular set, and replace each of the terms so obtained by a simple permutant having for its characters the characters of the previously unpermuted set, the result is obviously the original compound permutant. Thus, in the above-mentioned case, where the first and second blanks form a set and the third and fourth blanks form a set,

$$((V_{1234})) = (V_{1234}) - (V_{1243}),$$

or

$$((V_{1234})) = (V_{1234}) - (V_{2134}),$$

in the former of which equations the first and second blanks in each of the permutants on the second side form a set, and in the latter the third and fourth blanks in each of the permutants on the second side form a set, the remaining blanks being simply supernumerary and the characters in them unpermutable. It should be noted that the term quote, as previously defined, is only applicable to a system of blanks belonging to the same set, and it does not appear that anything would be gained by removing this restriction.

The following rule for the expansion of a simple permutant (and which may be at once extended to compound permutants) is obvious. Write down all the distinct terms that can be obtained, on the supposition that the blanks group themselves in any manner into quotes, and replace each of the terms so obtained by a compound permutant having for a distinct set the blanks of each assumed quote; the result is the original simple permutant. Thus in the simple permutant (V_{1234}), supposing for the moment that the first and second blanks form a quote, and that the third and fourth blanks form a quote, this leads to the equation

$$(V_{1234}) = +((V_{1234})) + ((V_{1342})) + ((V_{1423})) + ((V_{3412})) + ((V_{4213})) + ((V_{2314})),$$

where in each of the permutants on the second side the first and second blanks form a set, and also the third and fourth blanks.

The blanks of a simple or compound permutant may of course, without either gain or loss of generality, be considered as having any particular arrangement in space, for instance, in the form of a rectangle: thus $V_{\substack{12\\34}}$ is neither more nor less general than V_{1234}. The idea of some such arrangement naturally presents itself as affording a means of shewing in what manner the blanks are grouped into sets. But, considering the blanks as so arranged in a rectangular form or in lines and columns, suppose in the first instance that this arrangement is independent of the grouping of the blanks into sets, or that the blanks of each set or of any of them are distributed at random in the different lines and columns. Assume that the form is such that a symbol

$$V_{\substack{\alpha\beta\gamma\\ \alpha'\beta'\gamma\,..}}$$

is a function of symbols $V_{\alpha\beta\gamma\,.}$, $V_{\alpha'\beta'\gamma\,..}$, &c. Or, passing over this general case and the case of intermediate generality of the function being a symmetrical function, assume that

$$V_{\substack{\alpha\beta\gamma\\ \alpha\beta'\gamma\,..}}$$

is the product of symbols $V_{\alpha\beta\gamma}$, $V_{\alpha\beta'\gamma}$, &c. Upon this assumption it becomes important to distinguish the different ways in which the blanks of a set are distributed in the different lines and columns. The cases to be considered are: (*A*). The blanks of a single set or of single sets are situated in more than one column. (*B*). The blanks of each single set are situated in the same column. (*C*). The blanks of each single set form a separate column. The case (*B*) (which includes the case (*C*)) and the case (*C*) merit particular consideration. In fact the case (*B*) is that of the functions which I have, in my memoir on Linear Transformations in the *Journal*, called hyperdeterminants, and the case (*C*) is that of the particular class of hyperdeterminants previously treated of by me in the *Cambridge Philosophical Transactions*, and also particularly noticed in the memoir on Linear Tr[illegible]ns. The functions of the case (*B*) I now pro[illegible] 'Intermutants,' and those in the case (*C*) "C[illegible]mmutants include

as a particular case "Determinants," which term will be used in its ordinary signification. The case (A) I shall not at present discuss in its generality, but only with the further assumption that the blanks form a single set (this, if nothing further were added, would render the arrangement of the blanks into lines and columns valueless), and moreover that the blanks of each line form a quote: the permutants of this class (from their connexion with the researches of Pfaff on differential equations) I shall term "Pfaffians." And first of commutants, which, as before remarked, include determinants.

The general expression of a commutant is

$$\left(V_{\begin{array}{l}11\ldots\\22\\\vdots\\nn\end{array}}\right). \qquad \text{Or} \left(\begin{array}{l}11\ldots\\22\\\vdots\\nn\end{array}\right)$$

and (stating again for this particular case the general rule for the formation of a permutant) if, permuting the characters in the same column in every possible way, considering these permutations as positive or negative according to the rule of signs, one system be represented by

$$\begin{array}{l}r_1 s_1 \ldots\\r_2 s_2\\\vdots\\r_n s_n\end{array}$$

the commutant is the sum of all the different terms

$$\pm\pm\ldots V_{r_1 s_1\ldots}\; V_{r_2 s_2\ldots}\; V_{r_n s_n\ldots}$$

The different permutations may be formed as follows: first permute the characters in all the columns except a single column, and in each of the arrangements so obtained permute entire lines of characters. It is obvious that, considering any one of the arrangements obtained by permutations of the characters in all the columns but one, the permutations of entire lines and the addition of the proper sign will only reproduce the same symbol—in the case of an even number of columns constantly with the positive sign, but in the case of an odd number of columns with the positive or negative sign, according to the rule of signs. For the inversion or interchange of two entire lines is equivalent to as many inversions or interchanges of two characters as there are characters in a line, *i.e.* as there are columns, and consequently introduces a sign compounded of as many negative signs as there are columns. Hence

Theorem. A commutant of an even number of columns may be calculated by considering the characters of any one column (no matter which) as supernumerary unpermutable characters, and multiplying the result by the number of permutations of as many things as there are lines in the commutant.

The mark † added to a commutant of an even number of columns will be employed to shew that the numerical multiplier is to be omitted. The same mark placed over any one of the columns of the commutant will shew that the characters of that particular column are considered as non-permutable.

A determinant is consequently represented indifferently by the notations

$$\begin{pmatrix} 11 \\ 22 \\ \vdots \\ nn \end{pmatrix}^{\dagger} \quad \begin{pmatrix} \overset{\dagger}{1}1 \\ 22 \\ \vdots \\ nn \end{pmatrix}, \quad \begin{pmatrix} 1\overset{\dagger}{1} \\ 22 \\ \vdots \\ nn \end{pmatrix}.$$

And a commutant of an odd number of columns vanishes identically.

By considering, however, a commutant of an odd number of columns, having the characters of some one column non-permutable, we obtain what will in the sequel be spoken of as commutants of an odd number of columns. This non-permutability will be denoted, as before, by means of the mark † placed over the column in question, and it is to be noticed that it is not, as in the case of a commutant of an even number of columns, indifferent over which of the columns the mark in question is placed; and consequently there would be no meaning in simply adding the mark † to a commutant of an odd number of columns.

A commutant is said to be symmetrical when the symbols $V_{\alpha\beta\gamma}$. are such as to remain unaltered by any permutations *inter se* of the characters $\alpha, \beta, \gamma \ldots$ A commutant is said to be skew when each symbol $V_{\alpha\beta\gamma}$ is such as to be altered in sign only according to the rule of signs for any permutation *inter se* of the characters $\alpha, \beta, \gamma \ldots$, which of course implies that the symbol $V_{\alpha\beta\gamma}$ vanishes when any two of the characters $\alpha, \beta, \gamma \ldots$ are identical. The commutant is said to be demiskew when $V_{\alpha_1\beta_1 \cdot}$ is altered in sign only according to the rule of signs fo[illegible]tation *inter se* of non-identical characters $\alpha, \beta, \gamma,$

formations, and there is no difficulty in forming directly the intermutant or commutant on the first side of the on the symbol of derivation (in the sense of the memoir inear Transformations) from which the hyperdeterminant is obtained. Thus

$$\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix} \text{ is } \overline{12}^2.UU, \qquad \begin{bmatrix}0 & 0 & 0 & 0\\ 1 & 1 & 1 & 1\end{bmatrix} \text{ is } \overline{12}^4.UU,$$

$$\begin{bmatrix}\overset{\dagger}{0} & 0\\ 1 & 1\end{bmatrix} \text{ is } \overline{12}\,U^0U^1, \qquad \begin{bmatrix}0 & 0 & \overset{\dagger}{0} & 0\\ 1 & 1 & 1 & 1\end{bmatrix} \text{ is } \overline{12}^3U^0U^1,$$

permutable column $\begin{matrix}0\\1\end{matrix}$ corresponding to a $\overline{12}^*$ and a non-utable column $\begin{matrix}\overset{\dagger}{0}\\1\end{matrix}$ changing UU into U^0U^1. Similarly

$$\begin{matrix}0 & 0\\ 1 & 1\\ 2 & 2\end{matrix} \text{ becomes } (\overline{12}\,.\,\overline{13}\,.\,\overline{23})^2.UUU,$$

$$\begin{matrix}\overset{\dagger}{0} & 0\\ 1 & 1\\ 2 & 2\end{matrix} \text{ becomes } \overline{12}\,.\,\overline{13}\,.\,\overline{23}\;U^0U^1U^2,$$

$$\begin{Bmatrix}0 & 0\\ 1 & 1\\ 2 & 2\\ 3 & 3\end{Bmatrix} \text{ becomes } (\overline{12}.\overline{13}.\overline{14}.\overline{23}.\overline{24}.\overline{34})^2UUUU, \&c.$$

analogy would be closer if in the memoir on Linear [illegible]sformations, just as $\overline{12}$ is used to signify $\begin{vmatrix}\xi_1 & \eta_1\\ \xi_2 & \eta_2\end{vmatrix}$, $\overline{123}$ had [illegible] used to signify $\begin{vmatrix}\xi_1^2 & \xi_1\eta_1 & \eta_1^2\\ \xi_2^2 & \xi_2\eta_2 & \eta_2^2\\ \xi_3^2 & \xi_3\eta_3 & \eta_3^2\end{vmatrix}$ &c., for then [illegible] would [illegible] corresponded to $\overline{123}^2.UUU$, [illegible] $U^0U^1U^2$.

Viz. $\begin{matrix}0\\1\end{matrix}$ corresponds to $\overline{12}$ because [illegible] first and *second* blanks of a [illegible] [illegible]pying the *second* and third [illegible] $\overline{23}$ and so on. It will be [illegible] .... in the hyperdeterminant [illegible] [illegible]uish from each other [illegible] rentiations are performed.

and this would not only have been an addition of
importance to the theory, but would in some instances
facilitated the calculation of hyperdeterminants. The
ceding remarks shew that the intermutant $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & \bar{1} \\ 0 & 0 & 0 \\ 1 & 1 & \bar{1} \end{pmatrix}$ (w

the first and fourth blanks in the last column are t
considered as belonging to the same set) is in the hy
determinant notation $(12.34)^2.(14.23)UUUU$.

It will, I think, illustrate the general theory to perform
development of the last-mentioned intermutant. We hav

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & \bar{1} \\ \bar{0} & \bar{0} & 0 \\ 1 & 1 & \bar{1} \end{pmatrix} = \begin{array}{c} \quad\quad\;\; \dagger \\ \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ \bar{0} & \bar{0} & 0 \\ 1 & 1 & 1 \end{pmatrix} \end{array} - \begin{array}{c} \quad\quad\;\; \dagger \\ \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \bar{0} & \bar{0} & 1 \\ 1 & 1 & 1 \end{pmatrix} \end{array} - \begin{array}{c} \quad\quad\;\; \dagger \\ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \end{array} + \begin{array}{c} \quad\quad\;\; \dagger \\ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ \bar{0} & \bar{0} & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{array}$$

$$= 2\left\{ \begin{array}{c} \quad\quad\;\; \dagger \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \quad\quad\;\; \dagger \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array} - \begin{array}{c} \quad\quad\;\; \dagger \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \quad\quad\;\; \dagger \\ \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array} \right\}$$

$$= 2\{(ad - bc)^2 - 4(ac - b^2)(bd - c^2)\},$$

$$= 2(a^2d^2 + 4ac^3 + 4b^3d - 3b^2c^2 - 6abcd),$$

the different steps of which may be easily verified.

The following important theorem (which is, I beli
the same as a theorem of Mr. Sylvester's, published in
Philosophical Magazine) is perhaps best exhibited by m
of a simple example. Consider the intermutant

$$\begin{pmatrix} x & 1 \\ \underline{y} & 4 \\ x & 3 \\ y & 2 \end{pmatrix}$$

where in the first column the sets are distinguished as be
by the horizontal bar, but in the second column the 1, 2
to be considered as forming a set, and the 3, 4 as form
a second set. Then, partially expanding, the intermutan

$$\begin{array}{c} \dagger\;\;\; \\ \begin{pmatrix} x & 1 \\ y & 4 \\ x & 3 \\ y & 2 \end{pmatrix} \end{array} - \begin{array}{c} \dagger\;\;\; \\ \begin{pmatrix} y & 1 \\ x & 4 \\ x & 3 \\ y & 2 \end{pmatrix} \end{array} - \begin{array}{c} \dagger\;\;\; \\ \begin{pmatrix} x & 1 \\ y & 4 \\ y & 3 \\ x & 2 \end{pmatrix} \end{array} + \begin{array}{c} \dagger\;\;\; \\ \begin{pmatrix} y & 1 \\ x & 1 \\ y & 3 \\ x & 2 \end{pmatrix} \end{array}$$

[redacted]

e entire horizontal lines [illegible]

$$
\begin{array}{cccc}
+ & - & - & - \\
x\,1 & y\,1 & z\,1 & y\,1 \\
y\,2 & y\,2 & z\,2 & z\,2 \\
z\,3 & x\,3 & y\,3 & y\,3 \\
y\,4 & x\,4 & y\,4 & z\,4
\end{array}
$$

serving that the 1, 2 form a [illegible] as [illegible] 3, 4, the second and third [illegible] while the [illegible] fourth terms are equivalent to each other, we may e write

$$
2\begin{array}{c} x\,1 \\ \underline{y\,2} \\ x\,3 \\ y\,4 \end{array} = \begin{array}{c} x\,1 \\ y\,4 \\ x\,3 \\ y\,2 \end{array}
$$

n the first side of the equation the [illegible] ed into the second column [illegible] ut the equation the 1, 2 and the 3, 4 are to be [illegible] as forming distinct sets.

der in like manner the expression

$$
\begin{array}{c} x\,1 \\ y\,2 \\ \underline{z\,3} \\ x\,4 \\ y\,5 \\ \underline{z\,6} \\ x\,7 \\ y\,8 \\ z\,9 \end{array}
$$

n the first column the sets are distinguished by the al bars and in the second column the characters nd 4, 5, 6 and 7, 8, 9 are to be considered as forming distinct sets. The same reasoning as in the former l shew that this is a multiple of

$$
\begin{array}{c} x\,1 \\ y\,2 \\ \underline{z\,3} \\ x\,4 \\ y\,5 \\ \underline{z\,6} \\ x\,7 \\ y\,8 \\ z\,9 \end{array} .
$$

And to find the numerical multiplier it is only necessary to inquire in how many ways, in the expression first written down, the characters of the first column can be permuted so that x, y, z may go with $1, 2, 3$ and with $4, 5, 6$ and with $7, 8, 9$. The order of the x, y, z in the second triad may be considered as arbitrary; but once assumed, it determines the place of one of the letters in the first triad; for instance, $x8$ and $z9$ determine $y7$. The first triad must therefore contain $x1$ and $z6$ or $x6$ and $z1$. Suppose the former, then the third triad must contain $z3$, but the remaining two combinations may be either $x4, y5$, or $x5, y4$. Similarly, if the first triad contained $x6, z1$, there would be two forms of the third triad, or a given form of the second triad gives four different forms. There are therefore in all 24 forms, or

$$24\overset{\dagger}{\left\{\begin{array}{c|c} x & 1 \\ y & 2 \\ z & 3 \\ \hline x & 4 \\ y & 5 \\ z & 6 \\ \hline x & 7 \\ y & 8 \\ z & 9 \end{array}\right\}} = \left\{\begin{array}{cc} x & 1 \\ y & 7 \\ z & 6 \\ \hline x & 8 \\ y & 2 \\ z & 9 \\ \hline x & 4 \\ y & 5 \\ z & 3 \end{array}\right\}$$

where the bars in the second column on the first side shew that *throughout* the equation $1, 2, 3$ and $4, 5, 6$ and $7, 8, 9$ are to be considered as forming distinct sets. The above proof is in reality perfectly general, and it seems hardly necessary to render it so in terms.

To perceive the significance of the above equation it should be noticed that the first side is a product of determinants, viz.

$$24\left\{\begin{matrix} x & 1 \\ y & 2 \\ z & 3 \end{matrix}\right\}^{\dagger}\left\{\begin{matrix} x & 5 \\ y & 6 \\ z & 7 \end{matrix}\right\}^{\dagger}\left\{\begin{matrix} x & 7 \\ y & 8 \\ z & 9 \end{matrix}\right\}^{\dagger}.$$

And if the second side be partially expanded by permuting the characters of the second column, each of the terms so obtained is in like manner a product of determinants, so that

$$24\left\{\begin{matrix} x & 1 \\ y & 2 \\ z & 3 \end{matrix}\right\}^{\dagger}\left\{\begin{matrix} x & 4 \\ y & 5 \\ z & 6 \end{matrix}\right\}^{\dagger}\left\{\begin{matrix} x & 7 \\ y & 8 \\ z & 9 \end{matrix}\right\}^{\dagger} = \left\{\begin{matrix} x & 1 \\ y & 7 \\ z & 6 \end{matrix}\right\}^{\dagger}\left\{\begin{matrix} x & 8 \\ y & 2 \\ z & 9 \end{matrix}\right\}^{\dagger}\left\{\begin{matrix} x & 4 \\ y & 5 \\ z & 3 \end{matrix}\right\}^{\dagger} \pm \text{\&c.},$$

the permutations on the second side being the permut
inter se of $1, 2, 3$, of $4, 5, 6$, and of $7, 8, 9$.

It is obvious that the preceding theorem is not confined to intermutants of two columns.

[*To be continued.*]

POSTSCRIPT.

I wish to explain as accurately as I am able, the extent of my obligations to Mr Sylvester in respect of the subject of the present memoir. The term permutant is due to him—intermutant and commutant are merely terms framed between us in analogy with permutant, and the names date from the present year (1851). The theory of commutants is given in my memoir in the *Cambridge Philosophical Transactions*, and is presupposed in the memoir on Linear Transformations. It will appear by the last-mentioned memoir that it was by representing the coefficients of a biquadratic function by $a = 1111$, $b = 1112 = 1121 = \&c.$, $c = 1122 = \&c.$, $d = 1222 = \&c.$, $e = 2222$, and forming the commutant $\left.\begin{matrix}1111\\2222\end{matrix}\right\}$ that I was led to the function $ae - 4bd + 3c^2$. The function $ace + 2bcd - ad^2 - b^2e - c^3$ or $\begin{vmatrix} a, & b, & c \\ b & c, & d \\ c, & d, & e \end{vmatrix}$ is mentioned in the memoir on Linear Transformations, as brought into notice by Mr. Boole. From the particular mode in which the coefficients $a, b, \ldots$ were represented by symbols such as 1111, &c., I did not perceive that the last-mentioned function could be expressed in the commutant notation. The notion of a permutant, in its most general sense, is explained by me in my memoir (*Crelle*, tom. XXXVIII. p. 93) "Sur les determinants gauches." See the paragraph (p. 94) commencing "On obtient ces fonctions, &c." and which should run as follows: "On obtient ces fonctions (dont je reprends ici la théorie) par les propriétés générales d'un determinant défini comme suit. En exprimant &c.;" the sentence as printed being ". ... défini. Car en exprimant &c." which confuses the sense. Some time in the present year (1851) Mr. Sylvester, in conversation, made to me the very important remark, that as one of a class the above-mentioned function, $ace + 2bcd - ad^2 - b^2e - c^3$, could be expressed in the commutant notation $\left(\begin{matrix}0 & 0\\1 & 1\\2 & 2\end{matrix}\right)$, viz. by considering $00 = a$, $01 = 10 = b$, $02 = 11 = 20 = c$, $12 = 21 = d$, $22 = e$; and the subject being thereby recalled to my notice, I found shortly afterwards the expression for the function

$$a^2d^2 + 4ac^3 + 4b^3d - 3b^2c^2 - 6abcd$$

(which cannot be expressed as a commutant) in the form of an intermutant, and I was thence led to see the identity, so to say, of the theory of hyperdeterminants, as given in the memoir on Linear Transformations, with the present theory of intermutants. It is understood between Mr. Sylvester and myself, that the publication of the present memoir is not to affect Mr. Sylvester's right to claim the origination, and to be considered as the first publisher of such part as may belong to him of the theory here sketched out.

ON THE PRINCIPLES OF THE CALCULUS OF FORMS.

By J. J. Sylvester, M.A., F.R.S.

PART I.—GENERATION OF FORMS.*

Sect. I.—*On Simple Concomitance.*

The primary object of the Calculus of Forms is the determination of the properties of Rational Integral Homogeneous Functions or systems of functions: this is effected by means of transformation; but to effect such transformation experience has shewn that forms or form-systems must be contemplated not merely as they are in themselves, but with reference to the ensemble of forms capable of being derived from them, and which constitute as it were an unseen atmosphere around them. The first part of this essay will therefore be devoted to the theory of the external relations of forms or form-systems; the second part to the analysis of forms: that is to say, the first part will treat of the Generation and affinities, and the second part of the Reduction and equivalences of forms.

In its most crude and absolute, or, so to speak, archetypal condition a Rational Integral Homogeneous Function may be regarded as a linear function of several distinct and perfectly independent classes of variables. The first step towards the limitation of this very general but necessary conception consists in imagining the total number of classes to become segregated into groups, and certain correspondencies to obtain between the variables of a class in any group with some the variables in each other class of the same group. The investigations in this and the subsequent section will be confined exclusively to the theory of functions where the several classes of variables, if more than one, all belong to a single group, so that the variables in one class have each their respective correspon-

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* It may be well at the outset to give notice to my readers of the exact meaning to be attached to the following terms:

1. The linear-transformations are supposed to be always taken such that the modulus, *i.e.* the determinant of the coefficients of transformation, is unity; or, as it may be phrased, the transformations are uni-modular.

2. The word Determinant is restricted in all cases to signify the alternate function formed in the usual manner from a group of quantities arranged in square order.

3. The word *Discriminant* (typified by the prefix-symbol $\square$) is used to denote the determinant (usually but most perplexingly so called) of a homogeneous function of variables.

4. The resultant of two or more homogeneous functions of as variables is the left-hand side of the final equation (in its *complete f*
free from extraneous factors) which results from eliminating the v
between the equations obtained by making each of the functions zer.

dents in the remaining classes. Such a group may again be conceived to become subdivided into sets each of the same number of variables, and the corresponding variables in the different sets to become absolutely identical. This leads to the conception of a homogeneous function of related classes of variables of various degrees of exponency in respect to the several classes. The relation of the different classes, if containing the same number of variables (in which case the relation may be termed Simple) will be understood to be defined by their being simultaneously subject to similar or contrary operations of linear substitution; so that, for example, if x, y, z; ξ, η, ζ are two such classes when x, y, z are replaced by $ax + by + cz$, $a'x + b'y + c'z$, $a''x + b''y + c''z$, respectively, ξ, η, ζ will be, according to the *species* of the relation, subject to be at the same time replaced either by $a\xi + b\eta + c\zeta$, $a'\xi + b'\eta + c'\zeta$, $a''\xi + b''\eta + c''\zeta$, or otherwise by $\alpha\xi + \beta\eta + \gamma\zeta$, $\alpha'\xi + \beta'\eta + \gamma'\zeta$, $\alpha''\xi + \beta''\eta + \gamma''\zeta$, where

$$\alpha = \begin{matrix} 1 & 0 & 0, \\ 0 & b' & c', \\ 0 & b'' & c'', \\ & \&c. & \end{matrix} \qquad \beta = \begin{matrix} 0 & 1 & 0, \\ a' & 0 & c', \\ a & 0 & c'', \\ & \&c. & \end{matrix} \qquad \gamma = \begin{matrix} 0 & 0 & 1, \\ a' & b' & 0, \\ a'' & b'' & 0, \\ & \&c.^* & \end{matrix}$$

On the former supposition the related classes x, y, z, ξ, η, ζ will be said to be cogredient, and on the latter supposition contragredient.† If now we have one or more functions of classes of variables so related,‡ such function or system of functions may have associated with it a concomitant, also made up of distinct but related classes of variables, such classes being capable of being either greater or fewer in number than the classes of the given function or system of functions.

In the primitive function or system, as also in the concomitant, the related classes may be all of the same species, or some of one and the others of the contrary species. Even if we limit ourselves to the conception of a primitive function or system of functions with only one class of variables, its concomitant may be composed of various classes of variables, in respect to some of which it will be covariant with, and in respect to the others contravariant to, the primitive function

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* See my paper in the preceding number of this *Journal*.

† The germ of the notion of contragredience will be found in the immortal Arithmetic of the great and venerable Gauss.

‡ ... here spoken of will be observed to be of a *dynamical* ... as they are in themselves, but to the ... ously subject.

or system.* This is an immense and most important extension of the conception of a concomitant given in my preceding paper in this *Journal*, and will be shewn to have the effect of reducing the whole existing theory under subjection to certain simple abstract and universal laws of operation.

The relation of concomitance is purely of form. A being a given form, B is its concomitant, when A' being derived from A by simultaneous substitutions impressed upon the class of variables or upon each of the classes (if there be more than one) in A, and B' from B by corresponding (coincident or contrary) substitutions impressed upon the class or classes of variables in B, B' is capable of being derived from A' after the same law as B from A; or, as it may be otherwise expressed, "functions are concomitant when their correlated linear derivatives are homogeneous in point of form."†

This definition implies that one at least of the forms must be the most general possible of its kind: in a secondary but very important sense, however, functions obtained by impressing particular values or relations upon the quantities entering into the primitive and its associate form, will still be called concomitant. Thus $x^3 - y^3$ will be termed a concomitant to $x^3 + y^3$, not that we can affirm that $(ax + by)^3 - (cx + dy)^3$:

$$\text{i.e. } (a^3 - c^3)x^3 + 3(a^2b - c^2d)x^2y + 3(ab^2 - cd^2)z^2 + (b^3 - d^3)y^3,$$

treated as a function of x and y, can be derived from $(ax + by)^3 + (cx + dy)^3$,

$$\text{i.e. } (a^3 + c^3)x^3 + 3(a^2b + c^2d)x^2y + 3(ab^2 + cd^2)xy^2 + (b^3 + d^3)y^3,$$

when $ad - bc = 1$ by the same law as $(x^3 - y^3)$ from $(x^3 + y^3)$, for the elements for forming such comparison are wanting, but because $x^3 + y^3$ and $x^3 - y^3$ are the correspondent particular values respectively assumed by $fx^3 + 3gx^2y + 3hxy^2 + ky^3$, and its concomitant

$$(ad^2 - 2c^3 - 3bcd)\,x^3 + (6b^2d - 3c^2b - 3acd)\,x^2y$$
$$+ (6ac^2 - 3cb^2 - 3cba)\,xy^2 + (a^2d + 2b^3 - 3bca)\,y^3,$$

when $\qquad a = 1,\ b = 0,\ c = 0,\ d = 1.$

With the aid of this extended signification of the term concomitant (whether it be a covariant or contravariant) we can in all cases speak (as otherwise we in general could not) of

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* And of course the concomitant may be an invariant to its originant in respect of one or more systems of variables entering into the former.

† Or, more generally, it may be said that concomitance consists persistence of morphological affinity.

concomitant of a concomitant. The relation between stems of variables has been stated to be Simple whether ey be congredient or contragredient when each variable one system corresponds with some one in each other. ompound relation arises as follows:—Suppose $x, y : \xi, \eta$ ro independent systems of two variables each, and that e system of four variables u, v, w, t is subject to linear riations imitating, in the way of congredience or contraredience, those to which $x\xi, x\eta, y\xi, y\eta$ are subject: then v, w, t may be said to be congredient or contragredient to e continued systems $x, y ; \xi, \eta$. If $x, y : \xi, \eta$ be themselves gredient, then a system of only three variables u, v, w may e cogredient or contragredient in respect to $x\xi, x\eta + y\xi, y\eta$. nd if $x, y ; \xi, \eta$ be coincident, u, v, w may be similarly lated to x^2, xy, y^2. The illustration may easily be generalized, nd it will be seen in the sequel that this conception of comound-relation between systems of a differing number of ariables will greatly extend the power and [illegible] e methods about to be developed. Without having [illegible] a formal definition, it is obvious that the notion of a [illegible] omitant conveyed in my former paper in the [illegible] self without difficulty [illegible] hich can be made of [illegible] f systems of related variables are [illegible] elation be, whether simple or compound [illegible] ogredience or of contragredience. [illegible] y last paper relative to a [illegible] function being a concomitant of the [illegible] oncomitants in the [illegible] and that term, and the [illegible] the second concomitant [illegible] ither species if there [illegible] rimitive, will be [illegible] hich determines the [illegible] ing made to operate each upon [illegible] on the other.

The highest law and [illegible] hich I have yet [illegible] ay be expressed [illegible] sses of variables [illegible] ncomitant, [illegible] *mber of* [illegible] ll still be a [illegible] reafter as the La[illegible] en led up [illegible]

and a rigorous proof. It is the keystone of the subject, and any one who should suppose that it is a self-evident proposition (as from the simplicity of the enunciation it might be supposed to be) will commit no slight error.

If $\phi(x, y \ldots . z)$ be any homogeneous form of function of $x, y \ldots z$, every homogeneous sum in the expansion by Taylor's theorem of $\phi(u + u', v + v' \ldots w + w')$, which in fact, on making $u' = x, v' = y \ldots w' = z$, becomes identical (to a numerical factor près) with $\left(u \dfrac{d}{dx} + v \dfrac{d}{dy} + w \dfrac{d}{dz}\right)^{i} \phi$, is what I have elsewhere termed an Emanant, and by a partial method I had demonstrated that every invariant of such an emanant in respect to $u, v \ldots w$, in which $x, y \ldots z$ are treated as constants, or *vice versâ*, would give a covariant of ϕ. The reason of this is now apparent. For it may easily be shewn* that every emanant is in fact itself a covariant of the function to which it belongs with respect to each of the related classes of variables which enter into it, or is as it may be termed a double covariant. The law of Succession shews therefore that a concomitant to an emanant from which one of the classes has disappeared will be a covariant of the primitive in respect to the remaining class.

In applying the law of Succession, great use can be made of a function of two classes of letters which may be termed a Universal Mixed Concomitant; this is $x\xi + y\eta + \ldots + z\zeta$, which has the property of remaining unaltered when any linear substitution (for which the modulus is unity) is impressed upon $x, y \ldots z$, and the contrary one upon $\xi, \eta \ldots \zeta$.†

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* To demonstrate this it is only necessary to observe that if $u, v, \ldots w$, $u', v', \ldots w'$ be cogredient with themselves and with $x, y, \ldots z$,

$$\phi\,(u + \lambda u',\ v + \lambda v', \ldots\ w + \lambda w')$$

will evidently be a concomitant of $\phi\,(x, y, \ldots z)$; and, λ being arbitrary, the coefficients of the different powers of λ must be separately concomitants of $\phi\,(x, y, \ldots z)$, but these coefficients are the emanants of ϕ. Q. E. D.

† Thus, if

$$\begin{aligned} x &= ax' + by' + cz', & \xi &= (gn - hm)\,\xi' + (hl - fn)\,\eta' + (fm - gl)\,\zeta, \\ y &= fx' + gy' + hz', & \eta &= (-nb + mc)\,\xi' + (-lc + na)\,\eta' + (-ma + lb)\,\zeta, \\ z &= lx' + my' + nz', & \zeta &= (bh - cg)\,\xi' + (cf - ah)\,\eta' + (ag - bf)\,\zeta, \end{aligned}$$

then
$$x\xi + y\eta + z\zeta = \begin{pmatrix} a & b & c \\ f & g & h \\ l & m & n \end{pmatrix} \times (x'\xi' + y'\eta' + z'\zeta')$$
$$= x'\xi' + y'\eta' + z'\zeta' :$$

when the coefficients of transformation correspond to the direction-cosines between one system of rectangular axes and another, the reciprocal system

[redacted]

Let $f(x, y)$ be any function of x, y, of the degree m, [illegible] $\lambda(x\xi + y\eta)^m$ will be a mixed concomitant of f, it being [illegible]dent that every function of concomitants of a function is [illegible]self a concomitant of the same.

Suppose now

$$f = ax^m + mbx^{m-1}y + m\,.\,\frac{m-1}{2}\,cx^{m-2}y^2 + \&c.,$$

[illegible] concomitant becomes

$$[illegible] + \lambda\xi^m)\,x^m + m(b + \lambda\xi^{m-1}\eta)\,x^{m-1}y + m\,\frac{m-1}{2}\,(c + \lambda\xi^{m-2}\eta^2)\,x^{m-2}y^2 + \&c.$$

[illegible]nsequently if P be any concomitant of f, P' obtained [illegible]om P by writing $a + \lambda\xi^m$, $b + \lambda\xi^{m-1}\eta$, &c. for a, b, &c., [illegible]ll still be a concomitant of f; and by Taylor's theorem P' [illegible]vidently equals

$$P + \left(\xi^m\frac{d}{da} + \xi^{m-1}\eta\,\frac{d}{db} + \&c.\right)P$$

$$+ \frac{1}{1.2}\left(\xi^m\frac{d}{da} + \xi^{m-1}\eta\,\frac{d}{db} + \&c.\right)^2 P$$

$$+ \&c.$$

If we take P an invariant of f, we have M. Hermite's [illegible]heorem* for $f(x, y)$, and precisely the same [illegible] applies to the general case of $f(x, y \ldots z)$. P [illegible] the general rule, a contravariant of f in respect to ξ, [illegible] if P be taken a function containing one single system, and is [illegible]

is identical with the direct system: [illegible] particular supposition, may be regarded [illegible] cogredient; accordingly they may be [illegible] remains invariable, which is the [illegible] transformation. It may be observed here that [illegible] theory of concomitance limited to such [illegible] which may be termed Conditional [illegible] several cases that the invariants of [illegible] absolute invariants of the primitive. [illegible] that there exists a theory of universal [illegible] instead of merely two systems of variables [illegible] sequel it will be seen that the [illegible] (like the touch of an enchanter's wand, [illegible] into contravariants, and back again, [illegible] nate and fructify into complete [illegible]

* This theorem was first stated [illegible] Mr. Cayley [illegible] derived it from M. Eisenstein, under [illegible] which of course it becomes on interchanging x, y [illegible] theorem of covariants it could not be [illegible] two variables. M. Hermite appears to have [illegible] its more eligible form, subsequently to [illegible]

also a contravariant to f in respect to that system, P' will be a double contravariant; and if we make the two systems in P' identical, we have the extension of M. Hermite's theorem alluded to by me in one of the notes to my last paper, wherein I have stated that "*I* may be taken any *covariant* of the function:" as regards the purpose of that statement, the word covariant was used in error for contravariant.

The preceding method may be viewed as a particular application of the general principle, that if U_1, $U_2 \ldots U_m$ be any m functions (whether concomitants any of them of the others or not), then any concomitant of $\lambda_1 . U_1 + \lambda_2 . U_2 + \ldots + \lambda_m . U_m$ being expressed as a function of $\lambda_1, \lambda_2 \ldots \lambda_m$, every coefficient in such expression will be a concomitant of the system $U_1, U_2 \ldots \ldots U_m$. Thus, for example, if U and V be two quadratic functions of n variables $x, y \ldots z$, the *discriminant* $\square(\lambda U + \mu V)$ will contain $n+1$ terms, of which the coefficients of the first and last will be $\square U$ and $\square V$; and every one of the $(n+1)$ coefficients will be a concomitant (of course an invariant) of U and V. These $(n+1)$ invariants will in fact constitute the fundamental scale of invariants to the system U and V, and every other invariant of U and V will be an explicit rational function of the $(n+1)$ terms of the scale. In connexion with this principle may be stated another relative to any system of homogeneous functions of a greater number of variables of the same class, viz. that if any set of the variables one less in number than the number of the functions be selected at will, and any invariant of a given kind be taken of the resultant of the functions in respect to the variables selected, all such invariants so formed will have an integral factor in common, and this common factor will be an invariant of the given system of functions.

It will be convenient to speak hereafter of systems for which the march of the linear substitutions is coincident as congredient, and those for which the march is contrary as contragredient systems.

Suppose m congredient classes of m variables, the determinant formed by writing the $m \times m$ quantities in square order will evidently be a universal covariant. Thus, take the two systems x, y; ξ, η. $x\eta - y\xi$ is an universal covariant, and evidently therefore F, which I use to denote $\phi(x, y) \times \phi(\xi, \eta) + \lambda(x\eta - y\xi)^m$, will be a covariant to $\phi(x, y)$. Let $\phi(x, y)$ be of m dimensions, any invariant of F will be an invariant of ϕ: thus, let the two systems x, y; ξ, η be treated as perfectly independent, and take the discriminant of F (viewed as a function of x, y; ξ, η), *i.e.* the resultan

the four functions $\frac{dF}{dx}, \frac{dF}{dy}, \frac{dF}{d\xi}, \frac{dF}{d\eta}$; this resultant will be an invariant of ϕ; and λ being arbitrary, all the coefficients of its different powers will be invariants of ϕ. We thus fall upon another theorem of M. Hermite, viz. that if $F = \frac{\phi(x, y) \times \phi(\xi, \eta)}{(x\xi - y\eta)^n}$, the coefficients of the equation which will give the minimum values of λ are invariants of ϕ. So more generally, any invariant of $f(x, y, \xi, \eta) - \lambda(x\xi - y\eta)^m$, f being of the degree m in x, y and in ξ, η, will be an invariant of f; and among other invariants may be taken the discriminant obtained by treating x, ξ, y, η as absolutely unrelated.

If f be a function of various classes each containing n covariables, and if not less than n of these classes be covariable classes, and after, by selecting at will any n of such systems as $x_1, y_1 \ldots z_1$; $x_2, y_2 \ldots z_2$; ...... $x_n, y_n \ldots z_n$; the symbolical determinant

$$\begin{matrix} \frac{d}{dx_1}, & \frac{d}{dy_1} & \cdots\cdots & \frac{d}{dz_1}, \\ \frac{d}{dx_2}, & \frac{d}{dy_2} & \cdots\cdots & \frac{d}{dz_2}, \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{d}{dx_n}, & \frac{d}{dy_n} & \cdots\cdots & \frac{d}{dz_n}, \end{matrix}$$

be expanded and written equal to D, $D^{\iota}.f$ will be a concomitant of f; and, more generally, by selecting different combinations of the covariable systems n and n together in every way possible, and forming corresponding symbols of operation $E, F \ldots H$, we shall have $D^{\iota}.E^{\iota'} \ldots H^{(\iota)}.f$ for all values of $\iota, \iota' \ldots\ldots (\iota)$, a covariant of f in respect to the classes so combined. This explains and contains the whole pith and marrow of Mr. Cayley's simple but admirable method of obtaining covariants and invariants (or, as termed by their author, hyperdeterminants) to a function ϕ_1 of a single system $x_1, y_1 \ldots\ldots z_1$; he forms similar functions $\phi_2 \ldots\ldots \phi_\mu$ of $x_2, y_2 \ldots z_2$; ... $x_\mu, y_\mu \ldots z_\mu$, and uses the product $\phi_1 \times \phi_2 \ldots \times \phi_\mu$ as a function f of μ systems: the multiple covariant obtained by operating thereupon becomes a simple covariant on identifying the different classes of covariables introduced in the procedure.

Section II.—*On Complex Concomitance.*

We have hitherto been engaged in considering only a particular case of concomitance, the true idea of which relates not to an individual associated form (as such), but to a complex of forms capable of degenerating into an individual form. Such a complex may be called a Plexus. A plexus of forms is concomitant to a given form or combination of forms under the following circumstances.

If (O) be the originant, meaning thereby the primitive form or system of forms, and P the concomitant plexus made up of the μ forms P_1, P_2...$'P_\mu$, and if, when by duly related linear substitutions, O becomes O', the plexus P becomes P', made up of the forms P_1, P_2. . . .P'_μ, and if the plexus $'P$ formed from O' after the same law as P from O be made up of the forms $'P_1$, $'P_2$...P_μ, then will each form in either of the plexuses P, P' be a linear function of all the forms in the other plexus, and the connecting constants in every such linear function will be functions of the coefficients of the substitution whereby O and P have become transformed into O' and P'.

A function forming part of a concomitant plexus may be termed a concomitantive. Concomitantives therefore usually have a joint relation to a common plexus and a concomitant is only another name for an unique concomitantive. Every plexus contains a definite number of concomitantives ; in place of any one of these may be substituted an arbitrary linear function of all the rest, but the total number of independent forms *sufficient and necessary* to make the complete plexus respond to the requirements of the definition will remain constant.

If now we combine together the whole number of functions contained in one or more plexuses concomitant to any given originant, all of the same degree relative to any given selected system or systems of variables, and if the number of the concomitantives so combined be exactly equal to the number of terms in each, arranged as a function of the selected class or classes of variables, then the dialytic resultant (obtained by treating each combination of the selected variables as an independent variable, and forming a determinant in the usual manner), will be a concomitant to the given originant. This, which is only the partial expansion of some much higher law, may be termed the "Law of Synthesis."

Let f be any function of a single class of variables $x_1, x_2 \ldots x_n$. Let χ represent any product of these variables or of their

powers of any given degree r; the number of different
f χ will be μ, where

$$\mu = \frac{n(n+1)\ldots(n+r-1)}{1.2\ldots2},$$

f, $\chi_2.f \ldots\ldots \chi_\mu.f$ will form a covariantive plexus to f.
s, let ϑ represent any product of the symbols

$$\frac{d}{dx_1}, \ \frac{d}{dx_2} \ldots\ldots \frac{d}{dx_n} \text{ of the degree } r.$$

$f \ldots\ldots \vartheta_\mu.f$ will also form a covariant plexus to f.
coefficients of connexion between the forms of either
depend in an analogous manner upon the coefficients
ubstitution supposed to be impressed upon the varia-
ith the sole difference that every coefficient taken from
e r and column s of the determinant of substitution
appears in any coefficient of connexion of the one
is replaced by the coefficient taken from the line s and
umn r in the corresponding coefficient of connexion
other plexus.
$f(x, y)$ be any function of x, y of the degree $2m$

$$\left(\frac{d}{dx}\right)^m, \ \left(\frac{d}{dx}\right)^{m-1}\frac{d}{dy}, \ \&c.\ldots \left(\frac{d}{dy}\right)^m$$

m a covariantive plexus; thus, suppose

$$f(x, y) = a_1 x^{2m} + 2mb_2 x^{2m-1} y + \&c.\ldots + a_{2m+1} y^{2m}.$$

omitting numerical factors, the plexus will be composed
$(m+1)$ lines following:

$$\begin{aligned}
&a_1.x^m + ma_2.x^{m-1}y + \ldots + a_{m+1}.y^m,\\
&a_2.x^m + ma_3 x^{m-1}.y + \ldots + a_{m+2}.y^m,\\
&\&c.\\
&a_{m+1}.x^m + ma_{m+2}.x^{m-1} + \ldots + a_{2m+1}.y^m,
\end{aligned}$$

sequently, by the law of synthesis, the determinant

$$\left.\begin{array}{l} a_1, a_2 \ldots\ldots a_{m+1} \\ a_2, a_3 \ldots\ldots a_{m+2} \\ \ldots\ldots\ldots\ldots \\ a_{m+1}, a_{m+2} \ldots a_{2m+1} \end{array}\right\} \text{ is an invariant of } f.$$

this determinant is zero, I have proved in my paper
onical Forms, in the *Philosophical Magazine* for No-
r last, that f is resoluble into the sum of $(m+1)$ powers

of linear functions of x and y. I shall hereafter refer to a determinant formed in this manner from the coefficients of f as its catalecticant. Mr. Cayley was, I believe, the first to observe that all catalecticants* are invariants.

Again, more generally, let $f(x, y, \xi, \eta)$ be a function of the m^{th} degree of x, y, and of a like degree in respect of ξ, η, which are supposed to be congredient with x and y,

$$f(x, y, \xi, \eta) + \lambda(x\eta - y\xi)^m$$

(say F) will be a concomitant of f; and therefore if we take the system

$$\left(\frac{d}{dx}\right)^m F, \ \left(\frac{d}{dx}\right)^{m-1} F, \ldots . \left(\frac{d}{dy}\right)^m F,$$

which will be functions of ξ and η alone, and take their resultant, this resultant will be an invariant of f. As a particular case of this theorem, let

$$f = \left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right)^m \phi,$$

whence ϕ is supposed to be a function of x and y only, and $2m$ dimensions f is a concomitant of ϕ, and therefore the invariant of f, obtained in the manner just explained, will be an invariant of ϕ. Thus then we have an instantaneous demonstration of the theorem given by me in the paper of the *Philosophical Magazine* before named, viz. if

$$\phi(x, y) = a_1.x^{2m} + 2ma_2.x^{2m-1}.y + \ldots\ldots + a_{2m+1}.y^{2m},$$

say, in order to fix the ideas, $= ax^6 + bx^5y + cx^4y^2 + \ldots gy^6$; then the determinant

$$\begin{matrix} a & b & c & d+\lambda \\ b & c & d-\frac{1}{3}\lambda & e \\ c & d+\frac{1}{3}\lambda & e & f \\ d-\lambda & e & f & g, \end{matrix}$$

and (the analogously formed determinant for the general case) will be an invariant of ϕ. The general determinant so formed is peculiarly interesting, because it furnishes when equated to zero the one sole equation necessary to be solved in order

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* But the catalecticant of the biquadratic function of x, y was first brought into notice as an invariant by Mr. Boole ; and the discriminant of the quadratic function of x, y is identical with its catalecticant, as also with its hessian. Meicatalecticizant would more completely express the meaning of that which, for the sake of brevity, I denominate the catalecticant.

able to effect the reduction of $\phi(x, y)$ to its canonical , and gives the means, irrespective of any other view e theory of invariants, of determining completely and utely the condition of the possibility of two given func- of the same degree of x, y being linearly transformable into the other. This theorem will be obtained in a more ral manner in the following section. I only pause now ake the very important observation, that not only is the rminant an invariant, but every minor system* of determi- s that can be formed from it (there are of course m such ems) is an invariantive plexus to the given function (ϕ).

he form under which this theorem presents itself suggests eorem vastly more general and of peculiar interest, as ing a connexion between the theory of functions of a ain degree and of a certain number of variables with r functions of a lower degree but of a greater number riables. Here again, under a different aspect, is repro- ed the great principle of dialysis, which, originally dis- ed in the theory of elimination, in one shape or another ades the whole theory of concomitance and invariants.

et ϕ represent any function of the degree pq (of any ber, or, to fix the idea, say of three variables x, y, z); he general term of ϕ be represented by

$$\frac{pq(pq-1)\ldots 1}{(1.2\ldots\alpha)(1.2\ldots\beta)(1.2\ldots\gamma)}(\alpha, \beta, \gamma)\, x^{\alpha}.y^{\beta}.z^{\gamma},$$

* hese minor systems mean as follows:—the system of r^{th} minors com- all the distinct determinants that can be got by striking out from the e array (which I call the Matrix) from which the complete deter- nt is formed, any r lines and any r columns selected at will. The last, minor, is of course a system consisting of the coefficients of $\phi(x, y)$. t is evident that if $\phi(x, y \ldots z)$ be any function of any number of bles $x, y \ldots z$, that the coefficients will form an invariantive plexus to ϕ. he following remark as to the changes undergone by the coefficients of ϕ the variables undergo any substitution, is not without interest and tance for the theory.

$$\begin{array}{lll} \text{Let } x \text{ become} & fx + f'y + \ldots + (f)z, \\ y & gx + g'y + \ldots + (g)z, \\ \ldots & \ldots \\ z \ldots\ldots\ldots & hx + h'y + \ldots + (h)z. \end{array}$$

the coefficient of the highest power of x becomes

$$\phi(f, g \ldots h),$$

he coefficient of the term containing $y^{r} \ldots z^{s}$ becomes

$$\left(f' \frac{d}{df} + g' \frac{d}{dy} + \ldots + h' \frac{d}{dh}\right)^{r} \times \&c. \times \left\{(f)\frac{d}{d(f)} + (g)\frac{d}{d(g)} + \ldots + (h)\frac{d}{d(h)}\right\}^{s} .\phi(f, g \ldots h).$$

where $\alpha + \beta + \gamma = pq$, and (α, β, γ) represents a portion of the coefficient of $x^\alpha.y^\beta.z^\gamma$.

Let $$\frac{1.2\ldots p}{(1.2\ldots r)(1.2\ldots s)(1.2\ldots t)}\,x^r.y^s.z^t = \theta_{r,s,t},$$

where $r + s + t = p$, so that there are as many θ's as there are modes of subdividing p into three integral parts (zeros being admissible); i.e. $\dfrac{(p+1)(p+2)(p+3)}{1.2.3}$. Then any product such as $x^\alpha.y^\beta.z^\gamma$ may be divided in a variety of ways into the product of q of these θ's, and it may be shewn that the entire quantity

$$\frac{pq.(pq-1)\ldots 1}{(1.2\ldots\alpha)(1.2\ldots\beta)(1.2\ldots\gamma)}\,(x^\alpha.y^\beta.z^\gamma)$$

$$= \Sigma\left\{\frac{1.2\ldots q}{(1.2\ldots m_1)(1.2\ldots m_2)\ldots(1.2\ldots m_\iota)}\,(\theta_{\mu_1}{}^{m_1}.\,\theta_{\mu_2}{}^{m_2}\ldots\theta_{\mu_\iota}{}^{m_\iota})\right\},$$

where $m_1 + m_2 + \ldots + m_\iota = q$. Consequently ϕ may be represented under the form of a function of the degree q of $\dfrac{(p+1)(p+2)(p+3)}{1.2.3}$ (say ι) variables $\theta_1, \theta_2 \ldots\ldots \theta_\iota$, and its general term will be of the form

$$\frac{1.2\ldots q}{(1.2\ldots m_1)(1.2\ldots m_2)\ldots(1.2\ldots m_\iota)}\,(\alpha, \beta, \gamma)\,\{\theta^{m_1}.\theta^{m_2}\ldots\theta^{m_\iota}\},$$

where α, β, γ are the indices respectively of x, y, z, when the last factor is expressed as a function of these variables.* Now if ϑ be used to denote this new representation of ϕ when $\theta_1, \theta_2 \ldots \theta_\iota$ are treated as absolutely independent variables, and if we attach to it any universal concomitant, as $(x\xi + y\eta + z\zeta)^p$ admitting of being written under the form $\omega(\theta_1, \theta_2 \ldots \theta_\iota)$, wherein the coefficients will be functions of ξ, η, ζ; then any invariant to ϑ and ω, treated as two systems of ι variables, will be a concomitant to ϕ, the original function in x, y, z.† ϑ and ω may be termed respectively, for facility

* See Note (1) in Appendix.

† In fact ϑ is a concomitant to ϕ and ω to a power of the universal concomitant; the θ's forming a system of variables congredient with the compound system $x^{r_1}.y^{s_1}.z^{t_1}$, $x^{r_2}.y^{s_2}.z^{t_2}$, &c.: and it must be well observed that the same substitutions which render ϑ and ω respectively identical with ϕ, and a power of the universal concomitant, would render an infinite number of other functions also coincident with the same; but none of these other functions would be concomitants. Herein we see the importance of the definition and conception of compound relation; the θ system being compound by relation with the x, y, z system, after the manner of cogredience

of reference, the Particular and Absolute functions. Thus, for example, we take ϕ a function of x, y of the degree $4n$ (say $a_1 . x^{4n} + 4n . a_2 . x^{4n-1} + \&c. + a_{4n+1}$), and make $p = 2n$, $q = 2$, so that ϑ becomes a quadratic function of $(2n+1)$ variables obtained by making $x^{2n} = \theta$, $x^{2n-1}y = \theta_2 \ldots y^{2n} = \theta_{2n+1}$,* and the concomitant ω, formed from $(\xi x + \eta y)^{2n}$, becomes

$$\theta_1 \xi^{2n} + 2n . \theta_2 . \xi^{2n-1} . \eta + \ldots . + \theta_{2n+1} . \eta^{2n};$$

and if we take R the quadratic invariant of ω, that is

$$R = \theta_1 . \theta_{2n+1} - 2n\theta_2 . \theta_{2n} \ \&c. \pm \frac{1.2.3\ldots(2n)}{(1.2\ldots n)^2} \tfrac{1}{2}(\theta_{n+1})^2,$$

it will readily be seen the determinant of $\vartheta + \lambda R$, treated as a quadratic function of $(2n+1)$ variables, will give an invariant of ϕ, and this will be the same as that obtained by the particular method above given. Thus, suppose

$$\phi(xy) = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4.$$

Let
$$x^2 = \theta_1 \; 2xy = \theta_1 \theta_2 \; y^2 = \theta_3,$$

$$\vartheta = a\theta_1^2 + 2b\theta_1\theta_2 + c\theta_2 + 2c\theta_1\theta_3 + 2d\theta_2\theta_3 + e\theta_3^2,$$

$$\omega = (x\xi + y\eta)^2 = x^2\theta_1 + xy\theta_2 + y^2\theta_3,$$

$$R = \theta_1\theta_3 - \frac{\theta_2^2}{4},$$

Then Λ the discriminant of $\vartheta + 2\lambda R$ in respect to $\theta_1 \theta_2 \theta_3$

$$= \begin{vmatrix} a & b & c+\lambda \\ b & c - \frac{1}{2}\lambda & d \\ c+\lambda & d & e \end{vmatrix}$$

and I may remark that the relations between the several transformees of the invariantive plexuses formed by the minor determinant systems of λ (in this, and in general for the case of an evenly-even index) may be found by treating $\vartheta + 2\lambda R$ as a quadratic function of the variables (in this case $\theta_1, \theta_2, \theta_3$) and applying the rule given by me in the *Philosophical Mag.* in my paper "On the relation between the Minor Determinants of linearly-equivalent Quadratic

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* A slight variation upon the method as above explained for the general case has been here introduced inadvertently by writing $x^{2n-1}y = \theta_2$, &c., in lieu of $2nx^{2n-1}y = \theta_2$, &c., which, as it does not in any degree affect the reasoning, I have not deemed it worth while to alter.

[illegible]

[illegible]

[illegible]

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* [illegible] and the analytical characteristics of each of them. [illegible] will have their correspondences in the relations of these quadrant[illegible].

[redacted]

$$\begin{array}{cccccc} x^2\phi & y^2\phi & z^2\phi & xy\phi & yz\phi & zx\phi \\ x^2\omega & y^2\omega & z^2\omega & xy\omega & yz\omega & zx\omega \\ x^2\psi & y^2\psi & z^2\psi & xy\psi & yz\psi & zx\psi. \end{array}$$

shall thus have in all $3 + 3 \times 6$, that is, 21 functions into ch the 21 terms x^5, x^4y, x^4z, &c. enter linearly: the linear ultant of these 21 functions is the resultant of ϕ, ψ, ω, x of all extraneousness.

econd process: Write

$$\begin{aligned} \phi &= x^3P + yQ + zR, \\ \psi &= x^3P' + yQ' + zR', \\ \omega &= x^3P'' + yQ'' + zR'', \end{aligned}$$

l, as before, take the linear resultant $H_{2,1,1}$, which will never be of $9 - 5$, that is, of only 4 dimensions.

Again, take

$$\begin{aligned} \phi &= x^2L + y^2M + zN, \\ \psi &= x^2L' + y^2M' + zN', \\ \omega &= x^2L'' + y^2M'' + zN'', \end{aligned}$$

l form the determinant $H_{2,2,1}$; we shall thus have the ciliary system

$$H_{2,1,1},\ H_{1,2,1},\ H_{1,1,2},\ H_{2,2,1},\ H_{2,1,2},\ H_{1,2,2}.$$

Let this be combined with the augmentative system

$$x\omega,\ y\omega,\ z\omega;\quad x\phi,\ y\phi,\ z\phi;\quad x\psi,\ y\psi,\ z\psi.$$

tween these $6 + 9$, that is, 15 functions, the 15 terms x^3y, x^3z, &c. may be linearly eliminated, and the resultant is obtained will be precisely the same as that got by the ceding process.

Here we have 6 auxiliaries and 9 augmentatives; the xiliaries are of three dimensions in respect to the coeffi- nts of ϕ, ψ, ω; the augmentatives of one dimension only; the former process there were 3 auxiliaries and 18 aug- ntatives, $6 \times 3 + 9 = 27 = 3 \times 3 + 18$.

Now let this method be compared with the following:

First process: Take the 18 augmentatives $x^2\phi$, $x^2\omega$, $x^2\psi$, &c. in the first process of the algebraical method above ex- ined; but in place of the 3 auxiliaries therein given, take other system of 9 as follows:

Write the determinant

$$\left.\begin{matrix} \dfrac{d\phi}{dx}, & \dfrac{d\phi}{dy}, & \dfrac{d\phi}{dz} \\[2ex] \dfrac{d\psi}{dx}, & \dfrac{d\psi}{dy}, & \dfrac{d\psi}{dz} \\[2ex] \dfrac{d\omega}{dx}, & \dfrac{d\omega}{dy}, & \dfrac{d\omega}{dz} \end{matrix}\right\} = R;$$

$\dfrac{dR}{dx}, \dfrac{dR}{dy}, \dfrac{dR}{dz}$ form a concomitantive plexus; the 18 mentatives form another; the linear resultant of these plexuses will be an invariant of ϕ, ψ, ω, and of precisely same dimensions as the resultant last found; if they not identical it will be indeed a matter of exceeding wond and even more interesting than if they should be prov so to be.

Second process: Combine the augmentative plexus

$$x\omega, y\omega, z\omega; \quad x\phi, y\phi, z\phi; \quad x\psi, y\psi, z\psi,$$

with the differential plexus

$$\frac{d^2R}{dx^2}, \frac{d^2R}{dx\,dy}, \frac{d^2R}{dy^2}, \frac{d^2R}{dy\,dz}, \frac{d^2R}{dz^2}, \frac{d^2R}{dz\,dx},$$

we thus obtain a linear resultant in a manner precisely similar to that afforded by the second process of our algebraical method.

In general, if ϕ, ψ, ω be of the degrees n, n, n, as there are two algebraical varieties of the linear method for finding the resultant, so are there two varieties of the concomitantive method for finding the resembling invariant. In both methods the augmentatives are identical; the only difference being in the auxiliary system.

In the first process the augmentative system will be got by operating upon each of the functions ϕ, ψ, ω, with the multipliers x^{n-1}, y^{n-1}, z^{n-1}, and the other homogeneous products of x, y, z; the auxiliary system by operating upon R with the symbolical multipliers $\left(\dfrac{d}{dx}\right)^{n-2}, \left(\dfrac{d}{dy}\right)^{n-2}, \left(\dfrac{d}{dz}\right)^{n-2}$, and the other homogeneous products of $\dfrac{d}{dx}, \dfrac{d}{dy}, \dfrac{d}{dz}$ of the degree $n-2$.

In the second process the augmentative system is formed by the aid of the multipliers x^{n-2}, y^{n-2}, z^{n-2}, &c., and the auxiliary system by aid of $\left(\dfrac{d}{dx}\right)^{n-1}, \left(\dfrac{d}{dy}\right)^{n-1}, \left(\dfrac{d}{dz}\right)^{n-1}$, &c.

or the particular case of $x = 2$ the first process of the omitantive method is merely an application under its symmetrical form of the first process of the general braical method. The second process of the concomitan- method for this same case (at least when ϕ, ψ, ω are the al differential coefficients of the same function of the third ee) has been shewn by Dr. Hesse to give the resultant, hat for this case, at all events, we know that each con- itantive auxiliary must be a linear function of the aug- tatives and the algebraical auxiliaries.

gain, if we go to the system where ϕ, ψ, ω are of the ective degrees n, n, $n+1$. In the algebraical method applying which there are no longer two, but one only ess), the augmentative system is obtained by multiplying the homogeneous products of x^{n-1}, $x^{n-1}y$, $x^{n-1}z$, &c., ψ by like procduts, and ω by the homogeneous products x^{n-2}, , &c.

he auxiliary system is made up of functions of the eral form $H_{p,q,r}$ where $p+q+r=n+2$,

$_{q,r}$ being the determinant obtained by writing

$$\phi = Lx^p + My^q + Nz^r,$$
$$\psi = L'x^p + M'y^q + N'z^r,$$
$$\omega = L''x^p + M''y^q + N''z^r;$$

in like manner for the case of ϕ, ψ, ω, being of the ective degrees n, n, $n-1$, the augmentative system is ined by affecting ϕ, ψ each with multipliers x^{n-2}, $x^{n-2}y$, &c., ω with the multipliers x^{n-1}, $x^{n-1}y$, &c.

he number of functions (for either case) in the augmenta- and auxiliary plexuses thus obtained will be found to be tly equal to the number of terms in each such function, hewn by me in the paper alluded to. Let this be com- d with the transcendental method (I use this word at this t in preference to concomitantive, because in fact the braical and differential auxiliary systems are both alike comitantive plexuses to ϕ). For the case of n, n, $n+1$, e Jacobian determinant of ϕ, ψ, ω will be of the degree -2, and the system $\left(\frac{d}{dx}\right)^{n-1} R$, $\left(\frac{d}{dx}\right)^{n-2}.\left(\frac{d}{dy}\right) R$, &c. com- d with the augmentative systems

$$x^{n-2}\omega,\ x^{n-1}y\omega,\ \&c.$$
$$x^{n-1}\phi,\ x^{n-2}y\phi,\ \&c.$$
$$x^{n-1}\psi,\ x^{n-2}y\psi,\ \&c.$$

will give an invariant resembling (at least in generation and form) if not identical with the resultant of ϕ, ψ, ω. For the case of ϕ, ψ, ω being of the degrees n, n, $n-1$, the Jacobian R is of the degree $3n-4$ and $\left(\frac{d}{dx}\right)^{n-2} R$, $\left(\frac{d}{dx}\right)^{n-3} \frac{d}{dy} R$, &c. is the system which, combined with the augmentative system

$$x^{n-2}\phi,\ x^{n-3}y\phi,\ \&c.$$
$$x^{n-2}\psi,\ x^{n-3}y\psi,\ \&c.$$
$$x^{n-1}\omega,\ x^{n-2}y\omega,\ \&c.$$

will produce the resembling invariant.

Finally, for the last and more special case which the algebraical method applies to, viz. of ϕ, ψ, ω, θ, four quadratic functions of x, y, z, t, there can be here little doubt (upon the first impression) that in place of the algebraically obtained plexus

$$H_{2,1,1,1},\ H_{1,2,1,1},\ H_{1,1,2,1},\ H_{1,1,1,2},$$

may be substituted the differential plexus

$$\frac{dR}{dx},\ \frac{dR}{dy},\ \frac{dR}{dz},\ \frac{dR}{dt},$$

which, combined with the augmentatives

$$x\phi, x\psi, x\omega, x\theta;\ y\phi, y\psi, y\omega, y\theta;\ z\phi, z\psi, z\omega, z\theta;\ t\phi, t\psi, t\omega, t\theta,$$

will render possible the dialytic elimination of the 20 homogeneous products

$$x^3,\ x^2y,\ x^2z,\ x^2t,\ xyz,\ y^3,\ \&c.\ \&c.^*$$

Upon precisely the same principles may be verified instantaneously the method given by Hesse (without demonstration) for finding the polar reciprocal of lines of the third and fourth orders, at least to the extent of seeing that the functions obtained by his methods are contravariants (of the right degree and order) of the function from which they are derived. The polar reciprocal to a *surface* of the third degree may be obtained in the same manner.

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* Subsequent reflection induces me to reject as very improbable the (at first view likely) conjecture of the identity of the resultant with the invariant which simulates its form, except in the proved cases of three quadratic functions and the strongly resembling case of four quadratic functions last adverted to in the text above. Did this identity obtain, analogy would indicate that the catalecticant of the Hessian of two homogeneous functions of the same degree in x, y should be identical with their resultant, which is easily d[illegible] be false, except when the functions are of the third [illegible]

...et $\phi(x, y, z, t)$ be the characteristic of such a surface. ...e form a differential plexus of the first emanant of ϕ ...n together with the concomitant $W (= x\xi + y\eta + z\zeta + t\theta)$, or ...ating with

$$\frac{d}{dy}, \frac{d}{dz}, \frac{d}{dt} \text{ upon } \left(\xi' \frac{d}{dx} + \eta' \frac{d}{dy} + \zeta' \frac{d}{dz} + \theta' \frac{d}{dt}\right)(\phi + \lambda w),$$

... combining this plexus with $x\xi' + y\eta' + z\zeta' + t\theta'$, the resultant ...n in respect to $\xi', \eta', \zeta', \theta'$ (say R) will (according to the ... of synthesis) be a contravariant to the system $\phi + \lambda W$ ...w, and therefore to ϕ, because w is itself a concomitant ...ϕ. R is of the third degree in x, y, z, t, as also in the ...ficients of ϕ. If we form a differential plexus of $R + \mu w$ ...logous to that formed above with $\phi + \lambda w$, and combine these ... plexuses with the augmentative system xw, yw, zw, tw, ...e will be $4 + 4 + 4$, that is, 12 functions containing the 12 ...ms $x^2, y^2, z^2, t^2, xy, xz, xt, yz, yt, zt, \lambda, \mu$, and the dialytic ...ultant, which will be found to be a contravariant of the ...lfth degree in ξ, η, ζ, θ, and of the twelfth order in respect ...the coefficients of ϕ, will be (there can be little doubt) the ...ar reciprocal to the characteristic ϕ. A few remarks upon ... analytical character of a polar reciprocal may be not here ... of place. If ϕ be any homogeneous function of the ...ree m of any number (n) of variables $(x, y \ldots z)$, the object ...the theory of polar reciprocals is to discover what is the ...tion between $\xi, \eta \ldots \zeta$ expressed in the simplest terms such ... when this equation is satisfied $\xi x + \eta y + \ldots + \zeta z = 0$ will ...tangential to $\phi = 0$. In order for this to take effect it is ...essary that when any one of the variables z is expressed ...terms of the others $\ldots y, x$, and this value established in ϕ, ...n the discriminant of ϕ, so transformed, must be zero. ...nsequently the characteristic of the polar reciprocal to ϕ ...that rational integral function which is common to all ... discriminants obtained by expressing ϕ (by aid of the ...ation $\xi x + \eta y + \ldots + z\zeta$) as a function of any $(n - 1)$ of the ...iables. Let I_x be any invariant whatever of the order r ...ϕ_x (meaning by this last symbol what ϕ becomes when x is ...inated), and $I_y \ldots I_z$ the corresponding invariants when $y \ldots z$ ...ectively are eliminated; I_x will evidently be of the form

..., the numerator being an integer of r dimensions in the ...cients of ϕ and of mr dimensions in respect of $\xi, \eta \ldots \zeta$;

and by the fundamental definition of invariants it may easily be shewn that

$$\underset{x}{I} : \underset{y}{I} : \ldots : \underset{z}{I} :: \frac{1}{\xi^{\frac{mr}{n-1}}} : \frac{1}{\eta^{\frac{mr}{n-1}}} : \ldots : \frac{1}{\zeta^{\frac{mr}{n-1}}},^{*}$$

and therefore

$$\frac{\underset{x}{E}}{\xi^p} = \frac{\underset{y}{E}}{\eta^p} = \ldots. = \frac{\underset{z}{E}}{\zeta^p}, \quad \text{where } p = \frac{m(n-2)\,r}{n-1}.$$

Consequently all these quotients must be essentially integer, and any one of them will be of the order r in respect of the coefficients of ϕ and of the degree $\frac{mr}{n-1}$ in respect of $\xi, \eta \ldots \zeta$. Consequently the polar characteristic of ϕ, which is the common factor of the *discriminants* of $\underset{x}{I}, \underset{y}{I} \ldots \underset{z}{I}$ (for which species of invariant r evidently is equal to $(n-1)(m-1)^{n-2}$, being in fact the discriminant of a function of the m^{th} degree of $(n-1)$ variables) will be of the order $(n-1)(m-1)^{n-2}$ in respect of the coefficients of ϕ and of the degree $m(m-1)^{n-2}$ in respect of the contragredients $\xi, \eta \ldots \zeta$.

As to what relates to the reciprocity which exists between ϕ and its polar reciprocal ψ, this is included in a much higher theory of elimination, one proposition of which may be enunciated somewhat to the effect following, viz. that if ϕ be a homogeneous function of $x, y \ldots z$, and ω of $x, y \ldots z$, $u, v \ldots w$, and if, by aid of the equations

$$\phi = 0,$$

$$\frac{d\phi}{dx} + \lambda \frac{d\omega}{dx} = 0,$$

$$\frac{d\phi}{dy} + \lambda \frac{d\omega}{dy} = 0,$$

&c.

$$\frac{d\phi}{dz} + \lambda \frac{d\omega}{dz} = 0,$$

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* We see indirectly from this, that for a function of $(n-1)$, say γ variables of the degree m, an invariant of the order r must be subject to the condition that $\frac{mr}{\gamma}$ = an integer. This is easily shewn upon independent grounds; when $\gamma = 2$ $\frac{mr}{\gamma}$ must be not merely an integer but an *even integer*, and doubtless some analogo[illegible] to the general case.

...z be eliminated and the resultant be called ψ, then the ...ct of performing a similar operation upon ψ, ω, with ...pect to $u, v \ldots w$, as that just above indicated for the system ...ω, with respect to $x, y \ldots z$, will be to give a resultant, one ...or of which will be the primitive function ϕ over again. ...ere is some reason for supposing that polar reciprocals, ...ich are scarcely ever (if ever, except indeed for quadratic ...ctions) the simplest contravariants to a given function, ...y be expressed algebraically by means of the simpler con...variants, in the same way as discriminants admit in many, ...ot in all cases, with the same exception as above) of being ...resented as algebraical functions of invariants of a lower ...er or simpler form.

I close this section with the remark that every complete ... unambiguous system of functions of the constants in ...iven form or set of forms *characteristic** of any singularity ...olute or relative in such form or forms must constitute an ...ariantive plexus or set of invariantive plexuses. The ...tem unambiguously characteristic of a singularity of an ...der n will (except when $n = 1$) almost universally consist of ... more than n functions, subject of course to the existence ...syzygetic† relations between any $(n + 1)$ of such functions. ...e existence of multiple roots of a function of two variables ... specific, but by no means a peculiar case of singularity, ...d requires, for its complete and systematic elucidation, to ... treated in connexion with the general theory of the ...bject.

SECT. III.—*On Commutants.*

The simplest species of commutant is the well-known com...on determinant.

If we combine each of the n letters $a, b \ldots l$ with each of ...e other n, $\alpha, \beta \ldots \lambda$, we obtain n^2 combinations which may ... used to denote the terms of a determinant of n lines and

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* I repeat here that a function or system of functions which severally ...ated to zero express unequivocally and completely the existence of any ...ition or negation, is termed the characteristic of such position or negation. ...s (*ex. gr.*) the resultant of a group of equations is the characteristic of ... possibility of their coexistence. The discriminant of a function of two ...iables is the characteristic of its possession of two equal factors; the ...alectieant is the characteristic of its decomposability into the sum of a ...ned number of powers of linear functions of the variables, &c.

† Rational integer functions which admit of being multiplied severally ...other rational integer functions such that the sum of the products is ...tically zero, are said to be 'syzygetically related.'

Let ϕ be a function homogeneous and linear in respect of an even number r of any systems whatever of variables,

$$x_1, y_1 \ldots t_1 ; \quad x_2, y_2 \ldots t_2 ; \quad x_r, y_r \ldots t_r .$$

Form the determinant

$$\begin{array}{cccc} \frac{d}{dx_1}, & \frac{d}{dy_1} & \cdots\cdots & \frac{d}{dt_1}, \\ \frac{d}{dx_2}, & \frac{d}{dy_2} & \cdots\cdots & \frac{d}{dt_2}, \\ \cdots & \cdots & \cdots & \cdots \\ \frac{d}{dx_r}, & \frac{d}{dy_r} & \cdots\cdots & \frac{d}{dt_r}. \end{array}$$

Let the general term of this determinant, expanded, be called

$$F_1 \times F_2 \times \ldots \times F_r \text{ then is } \Sigma \pm F_1\phi \times F_2\phi \times \ldots \times F_r\phi$$

a covariant or invariant, as the case may be, of ϕ.

Be it observed that the march of the substitution for the different sets of variables in the above proposition is supposed to be perfectly independent. All the systems but one may undergo linear transformation, or they may all undergo distinct and disconnected transformations at the same time, and the proposition still continue applicable. It will however evidently be no less applicable should the march of substitution for any of the systems become congredient or contragredient to that of any other systems.

If we suppose ϕ to be a function of an even degree r of a single system of n variables $x, y \ldots t$, so that the r systems x_1, y_1, &c. x_2, y_2, &c.... x_r, y_r, &c. become identical, we can at once infer from the above scheme the existence and mode of forming an invariant to ϕ of the order n. This last appears for the case $n = 2$, and ought, for all other values of n, to have been known* to the author of the immortal discovery

* That this was not known explicitly to and should have escaped the penetration of the sagacious author of the theory, and those who had studied his papers, must be attributed to the imperfection of the notation heretofore employed for denoting the coefficients of a homogeneous polynomial function. The umbral method of denoting such a function ϕ of the degree r under the form of $(ax + by + \ldots + cz)^r$, which is equivalent to, but a more compendious and independent mode of mentally conceiving and handling the representation

$$\left(x\frac{d}{dx} + y\frac{d}{dy} + \ldots\ldots + z\frac{d}{dz}\right)\phi$$

exhibits the true internal constitution of such functions, and necessarily leads to the discovery of their essential properties and attributes.

nts, termed by him hyperdeterminants, in the sense cording to the nomenclature here adopted, would ed by the term hyperdiscriminants.

proceeding to discuss the theory of compound total s, or enlarging upon that of partial commutants, ake an interesting application of the preceding roposition to the discovery of Aronhold's S and T, variants respectively of the fourth and sixth order ng to a homogeneous cubic function (say F) of ables x, y, z. These may be termed respectively . As to H_6 a theoretically possible but eminently l ungraceful method immediately presents itself, e $F^2 = G$, and after forming the commutant with

$$\begin{array}{ccc}
\frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\
\frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\
\frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\
\frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\
\frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\
\frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz}
\end{array}$$

with the 6^3 ternary products of which this is made G: the result being an invariant of G, will be so being of the third degree in respect to the co- of G, will be of the sixth in respect to those of F. idently therefore be H_6, or at least a numerical of H_6, the form of which, inasmuch as the only riant is H_6, we know in form to be unique. But al theorem affords another and probably the most compendious* solution as regards H_6, of which on admits.

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since this was printed been favored with a view of some of the of Mr. Salmon's most valuable Second Part of his System of Geometry (about to appear, and which is calculated, in my waken a higher idea of and excite a new taste for geometrical this country), I find that I am mistaken in this point; the ical method operated with by Mr. Salmon being decidedly the ractically obtaining S and T in the general case. Symmetry, of an eastern robe, has not unfrequently to be purchased at the ne sacrifice of freedom and rapidity of action.

Let G* represent the mixed concomitant to F formed the bordered determinant

$$\begin{array}{cccc} \frac{d^2F}{dx^2} & \frac{d^2F}{dx\,dy} & \frac{d^2F}{dx\,dz} & \xi \\ \frac{d^2F}{dy\,dx} & \frac{d^2F}{dy^2} & \frac{d^2F}{dy\,dz} & \eta \\ \frac{d^2F}{dz\,dx} & \frac{d^2F}{dz\,dy} & \frac{d^2F}{dz^2} & \theta \\ \xi & \eta & \theta & 0 \end{array}$$

G is a function of the second order as to x, y, z, and of like order in respect to ξ, η, θ, which two systems wil respectively congredient and contragredient in respect tc x, y, z system in F. In other words, which is all we nee look to, G is a concomitant of F, and so also will be

$$G + \lambda(x\xi + y\eta + z\zeta)^2,$$

which may be termed H. Form now the commutant

$$\begin{array}{ccc} \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ \frac{d}{d\xi} & \frac{d}{d\eta} & \frac{d}{d\zeta} \\ \frac{d}{d\xi} & \frac{d}{d\eta} & \frac{d}{d\theta} \end{array}$$

this being applied to H will give an invariant (the fact the march of the substitutions for the systems x, y, z; ξ is contrary, being completely immaterial to the applical of the general theorem above given); the commutant so fo: will be a cubic function of λ, in which the coefficient is a numerical quantity, that of λ^2 is zero, that of λ is H_4 the constant term is H_6.

Thus (*ex. gr.*) let $F = x^3 + y^3 + z^3 + 6mxyz$, then

$$G = \begin{vmatrix} x & mz & my & \xi \\ mz & y & mx & \eta \\ my & mx & z & \zeta \\ \xi & \eta & \zeta & \theta \end{vmatrix}$$

---

* G is the mixed concomitant to the given cubic function, which is way (so to speak) between it and its polar reciprocal. In fact, whe operation is repeated upon G, which was executed upon the given fu to obtain G (that is, when ... Hessian of G in respect to a vertically and horiz... line ξ, η, ζ) the determ thereby represent... al to the given functi

and therefore

$$H = \Sigma \left\{ (\lambda - m^2)\, x^2\xi^2 + (\lambda + m^2)\, 2yz\eta\zeta + yz\xi^2 - 2mx^2\eta\zeta \right\},$$

the Σ implying the sum of similar terms with reference to the interchanges between x, ξ; y, η; z, ζ.

In developing the commutant above, the first line may be kept in a fixed position; for the sake of brevity, $(x), (y), (z)$; $(\xi), (\eta), (\zeta)$ may be written in the place of

$$\frac{d}{dx}, \ \frac{d}{dy}, \ \frac{d}{dz}; \quad \frac{d}{d\xi}, \ \frac{d}{d\eta}, \ \frac{d}{d\zeta},$$

and it will readily be seen that the only effective arrangements will be as underwritten:

$$\left.\begin{array}{l} (x)(y)(z) \\ (x)(y)(z) \\ (\xi)(\eta)(\zeta) \\ (\xi)(\eta)(\zeta) \end{array}\right| \quad \left.\begin{array}{ll} (x)(y)(z) & (x)(y)(z) \\ (x)(y)(z) & (x)(y)(z) \\ (\eta)(\zeta)(\xi) & (\zeta)(\xi)(\eta) \\ (\zeta)(\xi)(\eta) & (\eta)(\zeta)(\xi) \end{array}\right\}$$

$$\begin{array}{llllll} (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) \\ (x)(z)(y) & (x)(z)(y) & (\zeta)(\eta)(\xi) & (z)(y)(x) & (y)(x)(z) & (y)(x)(z) \\ (\xi)(\eta)(\zeta) & (\xi)(\zeta)(\eta) & (\xi)(\eta)(\zeta) & (\zeta)(\eta)(\xi) & (\xi)(\eta)(\zeta) & (\eta)(\xi)(\zeta) \\ (\xi)(\zeta)(\eta) & (\xi)(\eta)(\zeta) & (\zeta)(\eta)(\xi) & (\xi)(\eta)(\zeta) & (\eta)(\xi)(\zeta) & (\xi)(\eta)(\zeta) \end{array}$$

$$\begin{array}{llllll} (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) \\ (x)(z)(y) & (x)(z)(y) & (z)(x)(y) & (z)(y)(x) & (y)(x)(z) & (y)(x)(z) \\ (\eta)(\zeta)(\xi) & (\zeta)(\eta)(\xi) & (\xi)(\zeta)(\eta) & (\zeta)(\xi)(\eta) & (\eta)(\zeta)(\xi) & (\xi)(\zeta)(\eta) \\ (\zeta)(\eta)(\xi) & (\eta)(\zeta)(\xi) & (\zeta)(\xi)(\eta) & (\xi)(\zeta)(\eta) & (\xi)(\zeta)(\eta) & (\eta)(\zeta)(\xi) \end{array}$$

$$\left.\begin{array}{ll} (x)(y)(z) & (x)(y)(z) \\ (y)(z)(x) & (z)(x)(y) \\ (\zeta)(\xi)(\eta) & (\eta)(\zeta)(\xi) \\ (\xi)(\zeta)(\eta) & (\eta)(\zeta)(\xi) \end{array}\right\} \begin{array}{llll} (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) \\ (y)(z)(x) & (y)(z)(x) & (z)(x)(y) & (z)(x)(y) \\ (\xi)(\eta)(\zeta) & (\eta)(\zeta)(\xi) & (\xi)(\eta)(\zeta) & (\zeta)(\xi)(\eta) \\ (\eta)(\zeta)(\xi) & (\xi)(\eta)(\zeta) & (\zeta)(\xi)(\eta) & (\xi)(\eta)(\zeta) \end{array}$$

The signs of the four lines in each of these arrangements are two alike, and two contrary to the signs of the correspondent lines in the first arrangement; hence the effective sign is the same for all, and the result, after rejecting from each term the common factor -16, is seen, from inspection, to be

$$4(\lambda - m^2)^3 - 8m^3 + 6(\lambda - m^2)(\lambda + m^2)^2 - 12m(\lambda + m^2) + 2(\lambda + m^2)^3 + 1,$$

which is equal to

$$12\lambda^3 + 0.\lambda^2 - 12(m - m^4)\lambda + 1 - 20m^3 - 8m^6,$$

$m - m^4$ and $1 - 20m^3 - 8m^6$ are the two invariants (Aronhold's S and T) for the canonical form operated upon; and it will be observed that

$$(1 - 20m^3 - 8m^6)^2 + 64(m - m^4)^3 = (1 + 8m^3)^4,$$

which is easily proved to be the discriminant of

$$x^3 + y^3 + z^3 + 6mxyz.$$

It may however be observed, that this is not criminant of the function in λ just found, as re analogy* might have suggested it probably would order that this might be the case, the coefficient of λ be 4 instead of 12, and of λ, m m^4 instead of There is ground for supposing that another functi may be found by a different method, in which this will take effect.

The theorem above given for simple total com admits of an interesting application to the general a function F of the n^{th} degree, in respect to each independent systems of two variables x, y; ξ, η. be symbolically represented by $(ax + by)^n . (a\xi + \beta\eta)^n$, $a^n.a^n$ represents the coefficient of $x^n\xi^n$, $na^{n-1}.ba^n$ of &c. &c.; then the commutant

$$\begin{array}{l} a\ b\ \ldots.\ (1), \\ a\ b\ \ldots.\ (2), \\ \ldots \\ a\ b\ \ldots.\ (n), \\ a\ \beta\ \ldots.\ (1), \\ a\ \beta\ \ldots.\ (2), \\ \ldots \\ a\ \beta\ \ldots.\ (n), \end{array}$$

will represent a quadratic invariant of F, which will $(n+1)^2$ coefficients. By expanding this commutant we a general expression for the invariant under a very i ing form.

I now proceed to give the general theorem for co total commutants as applicable to the discovery of inv

Let there be a function of m disconnected cl systems of variables: let the systems in the same c supposed all distinct but congredient with one anothe function is supposed to be linear in respect to each in each class, and the number of systems is the all the classes, and the number of variables the same

* The biquadratic function of x, y having only one para therefore two invariants, its theory possesses striking analog theory of the cubic function of three letters. The function in gives these invariants for the first-named function, according to th given in the first section, has the same discriminant as the functio

system. This function may then be represented symbolically under the form

$$({}^1a_1.{}^1x_1 + {}^1b_1.{}^1y_1 + \dots + {}^1l_1.{}^1t_1)({}^1a_2.{}^1x_2 + {}^1b_2.{}^1y_2 + \dots + {}^1l_2.{}^1t_2)$$
$$\dots\dots ({}^1a_n.{}^1x_n + {}^1b_n.{}^1y_n + \dots {}^1l_n.{}^1t_n)$$
$$\times ({}^2a_1.{}^2x_1 + {}^2b_1.{}^2y_1 + \dots + {}^2l_1.{}^2t_1)({}^2a_2.{}^1x_2 + {}^2b_2.{}^1y_2 + \dots + {}^2l_2.{}^1t_2)$$
$$\dots\dots ({}^2a_n.{}^1x_n + {}^2b_n.{}^1y_n + \dots {}^2l_n.{}^1t_n)$$
$$\times \text{\&c.}$$
$$\times ({}^pa_1.{}^px_1 + {}^pb_1.{}^py_1 + \dots + {}^pl_1.{}^pt_1)({}^pa_2.{}^px_2 + {}^pb_2.{}^py_2 + \dots + {}^pl_2.{}^pt_2)$$
$$\dots\dots ({}^pa_n.{}^px_n + {}^pb_n.{}^py_n + \dots {}^pl_n.{}^pt_n).$$

In this expression the $x, y, \dots\, t$'s are all real, but the $a, b, \dots\, l$'s all umbral; in fact, ${}^fa_g$, ${}^fb_g$, &c. may be understood to denote $\dfrac{d}{d^fx_g}$, $\dfrac{d}{d^fy_g}$, &c.

The n systems in each of variables in each of the lines above written are supposed to be congredient *inter se.*

Take the symbolical product of the first line, first making *for the moment*

$${}^1x_1 = {}^1x_2 = \dots {}^1x_n = x, \text{ \&c. \&c.}, \quad {}^1t_1 = {}^1t_2 = \dots = {}^1t_n = t;$$

and let the coefficients of the several terms

$$x^n, \; x^{n-1}.y \dots \text{\&c.},$$

be called $\qquad {}^1A_1 \quad {}^1A_2 \; \dots\, {}^1A_\mu,$

where μ is the number of terms contained in a homogeneous function of the n^{th} degree of the m variables $x, y \dots\, t$. In like manner proceed with each of the lines, and then write down the commutant

$$\begin{matrix} {}^1A_1 & {}^1A_2 & \dots & {}^1A_\mu \\ {}^2A_1 & {}^2A_2 & \dots & {}^2A_\mu \\ \dots & \dots & \dots & \dots \\ {}^pA_1 & {}^pA_2 & \dots & {}^pA_\mu. \end{matrix}$$

This commutant is an invariant of F: it will of course be remembered that, unless p is even, the commutant vanishes.

Thus, for example, take two sets of two systems of two variables: in all four systems,

$$x, y; \quad \xi, \eta \; : \; p, q; \quad \phi, \psi,$$

each couple of systems on either side of the colon (:) being congredient *inter se:* and let F be symbolically represented by

$$(ax + by)(a\xi + \beta\eta)(lp + mq)(\lambda\phi + \mu\psi);$$

then the invariant given by the theorem will be the commutant

$$\begin{array}{l} a\alpha;\ a\beta + ab;\ b\beta, \\ l\lambda;\ l\mu + \lambda m;\ m\mu. \end{array}$$

The six positions of which are as below written (the first three being positive and the second three negative)

$$\begin{array}{lll} a\alpha;\ a\beta+ab;\ b\beta & a\alpha;\ a\beta+ab;\ b\beta & a\alpha;\ a\beta+ab;\ b\beta \\ l\lambda;\ l\mu+\lambda m;\ m\mu & l\mu+\lambda m;\ m\mu;\ l\lambda & m\mu;\ l\lambda;\ l\mu+\lambda m \\ a\alpha;\ a\beta+ab;\ b\beta & a\alpha;\ a\beta+ab;\ b\beta & a\alpha;\ a\beta+ab;\ b\beta \\ l\mu+\lambda m;\ l\lambda;\ m\mu & l\lambda;\ m\mu;\ l\mu+\lambda m & m\mu;\ l\mu+\lambda m;\ l\lambda \end{array}$$

If we write F under its explicit form,

$$\begin{array}{l} Ax\xi p\phi + Bx\xi p\psi + Cx\xi q\phi + Dx\xi q\psi \\ + A'x\eta p\phi + B'x\eta p\psi + C'x\eta q\phi + D'x\eta q\psi \\ + A''y\xi p\phi + B''y\xi p\psi + C''y\xi q\phi + D''y\xi q\psi \\ + A'''\eta\xi p\phi + B'''y\eta p\psi + C'''y\eta q\phi + D''y\eta q\psi, \end{array}$$

we have identically the relations following,

$$\begin{array}{llll} a\alpha l\lambda = A & a\alpha l\mu = B & a\alpha m\lambda = C & a\alpha m\mu = D \\ a\beta l\lambda = A' & a\beta l\mu = B' & a\beta m\lambda = C' & a\beta m\mu = D' \\ ba l\lambda = A'' & ba\lambda\mu = B'' & bam\lambda = C'' & bam\mu = D'' \\ b\beta l\lambda = A''' & b\beta l\mu = B''' & b\beta m\lambda = C''' & b\beta m\mu = D''', \end{array}$$

and the commutant expanded becomes

$$\begin{array}{l} A(B + C'' + C' + B'')\,D''' + (B + C)\,(D' + D'')\,A''' + D(A' + A'')\,(B''' + C''') \\ \quad - (B + C)\,(A' + A'')\,D''' - A(D' + D'')(B''' + C''') - D(B' + C'' + C' + B'')\,A''. \end{array}$$

In the foregoing the x's in the several lines were *for the moment* taken identical, in order the more easily to explain the law of formation of the quantities A.

But suppose that they become actually identical for the same line, F then becomes a function of the n^{th} degree in respect to each of p systems of variables, and may be represented symbolically under the form

$$\begin{array}{l} ({}^1a^1x + {}^1b^1y + \ldots\ldots {}^1l^1t)^n \times ({}^2a^2x + {}^2b^2x + \ldots\ldots + {}^2l^2t)^n \\ \qquad \ldots\ldots \times ({}^pa^px + {}^pb^px + \ldots\ldots + {}^pl^pt)^n. \end{array}$$

We may still further limit the generality of the theorem by supposing

$$\begin{array}{l} {}^1x = {}^2x = \ldots\ldots {}^px = x, \\ {}^1y = {}^2y = \ldots\ldots {}^py = y, \\ \quad \&\text{c.} \\ {}^1t = {}^2t = \ldots\ldots {}^pt = t; \end{array}$$

[illegible] becomes $(ax + bu + \&\text{c.} \ldots\ldots + lt)^{pn}$.

ordingly, as many different factors as can be found tained an even number of times in the exponent of the ction, so many invariants can be formed immediately a function of any number of variables m by the method otal commutation.

f one of these factors be called n, the commutant corponding thereto will be of the order

$$\frac{(n+1)(n+2)\ldots\ldots(n+m-1)}{1.2\ldots\ldots(m-1)}$$

espect to the coefficients. Thus $m = 2$, so that

$$F = (ax + by)^{np}.$$

The general form of such a commutant will be found taking $A_1 A_2 \ldots\ldots A_{n-1}$, the coefficients of the several binations of x, y in $(ax + by)^n$, from which the numerical fficients $n, n.\frac{n-1}{2}$, &c. may be rejected, as only introing a numerical factor into the result; the commutant l therefore be expressed by means of the form

$$\begin{array}{lr} a^n;\ a^{n-1}.b;\ a^{n-2}.b^2\ldots\ldots;\ b^n & (1) \\ a^n;\ a^{n-1}.b;\ a^{n-2}.b^2\ldots\ldots;\ b^n & (2) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a^n;\ a^{n-1}.b;\ a^{n-2}.b^2\ldots\ldots;\ b^n & (p). \end{array}$$

f $p = 2$, the compound commutant

$$\begin{array}{l} a^n;\ a^{n-1}.b;\ \ldots\ldots;\ b^n \\ a^n;\ a^{n-1}.b;\ \ldots\ldots;\ b^n \end{array}$$

l easily be seen to be only another form for the catalecticant $(ax + by)^{2n}$. Thus, let $n = 2$,

$$(ax + by)^4 = Ax^4 + 4Bx^3y + 6Cx^2y^2 + 4Dxy^3 + Ey^4;$$

that $a^4 = A,\ a^3b = B,\ a^2b^2 = C,\ ab^3 = D,\ b^4 = E.$

The commutant (which is of the form of the matrix to ordinary determinant, with the exception that the umbræ r compoundly instead of simply into the several terms rated by the marks of punctuation,) will be

$$\begin{array}{l} a^2;\ ab;\ b^2 \\ a^2;\ ab;\ b^2: \end{array}$$

this, written in the six forms

$$\left.\begin{matrix} a^2; & ab; & b^2 \\ a^2; & ab; & b^2 \end{matrix}\right\} \quad \left.\begin{matrix} a^2; & ab; & b^2 \\ a^2; & b^2; & ab \end{matrix}\right\} \quad \left.\begin{matrix} a^2: & ab; & b^2 \\ ab; & a^2; & b^2 \end{matrix}\right\}$$

$$\left.\begin{matrix} a^2; & ab; & b^2 \\ b^2; & ab; & a^2 \end{matrix}\right\} \quad \left.\begin{matrix} a^2; & ab; & b^2 \\ ab; & b^2; & a^2 \end{matrix}\right\} \quad \left.\begin{matrix} a^2; & ab: & b^2 \\ b^2; & a^2: & ab \end{matrix}\right\}$$

gives the expression

$$a^4 \times a^2b^2 \times b^4 \quad a^4 \times (ab^3)^2 - b^4 \times (a^3b)^2 - (a^2b^2)^3 + 2a^3b \times ab^3 \times$$

$$\text{i.e. } ACE - AD^2 - EB^2 - C^3 + 2BCD.$$

One important observation may here be made of which otherwise might easily escape attention, which commutants, where the same terms simple or compou found in all or several of the lines, in general give products, some of them equal and with the same sig others equal but with the *contrary* sign.

This last phenomenon does not manifest itself in mutants appertaining to functions of two variables of t particular and different species which first and most na present themselves, viz where there are only two li only two columns*—I believe that it displays itself in other case of commutantives to functions of two vari Thus it is that algebraical expressions derived from functions disguise their symmetry; to make which co light it becomes necessary to add terms of contrary s such expressions. As an example, the reader is invi develope the cubic invariant of a function of x and y bolically expressed by $(ax + by)^8$, where

$$a^8 = A, \quad a^7.b = B \; .... \; ab^7 = Hb^8 = I,$$

by means of the commutant

$$\begin{matrix} a^2 & ab & b^2 \\ a^2 & ab & b^2 \\ a^2 & ab & b^2 \\ a^2 & ab & b^2.\dagger \end{matrix}$$

* These commutants give respectively the quadrinvariant and the lecticant, the former of which alone was formerly recognised by Mr. as a commutant.

† The number of terms resulting from the independent permu each of the 3 linear lines is 6^3, that is 216, but the actual result i small letters instead of large) $P - Q$, where

$$P = aei + 3ag^2 + 12bch + 3c^2i + 24cf^2 + 24d^2g + 15e^3,$$

$$Q = 4afh \quad 4bid + 8bgf + 22ceg + 8chd + 36def,$$

so that the effective number of permutations is only 164. The di

Suppose F to be the general even-degreed function of two variables of the degree $2np$.

Let $$H = \left(\xi\frac{d}{dy} - \eta\frac{d}{dx}\right)^{np} . F + \lambda(x\xi + y\eta)^{np},$$

and express H umbrally under the form

$$(ax + by)^{np}.(\alpha\xi + \beta\eta)^{np}.$$

The commutant

$$\begin{array}{ccccc} a^n & a^{n-1}.b & \ldots. & b^n & (1) \\ a^n & a^{n-1}.b & \ldots. & b^n & (2) \\ \vdots & \vdots & \vdots & \vdots & \\ a^n & a^{n-1}.b & \ldots. & b^n & (p) \end{array}$$

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between this and 216 divided by 216 may be termed the Index of Demolition, which we see in this case is [illegible] or [illegible]; that is, somewhat less than [illegible]. For the cubic invariant of the function of the fourth degree this index is zero, all the permutations being effective. If we take the cubic invariant of the function $ax^{12} + 12bx^{11}y + 66cx^{10}y^2 + \&c. + my^{12}$ under the form $P - Q$, we shall find

$$P = 6ahl + 10agj + 6bfm + 54bhk + 54cfl + 155cii + 10ddm + 430dgj$$
$$+ 155eek + 520chh + 520ffl + 280ggg.$$
$$Q = agm + 15aik + 30bgl + 50bij + 15cem + 4cgk + 150chj + 30del + 210dfk$$
$$+ 250dhi + 230efj + 655egl + 660fgh.$$

The number of terms in P and Q is of course the same, and will be found to be 2200 for each; so that out of the 6^5, that is 7776 permutations of the 5 lower rows, only 4400 are effective, and the index of demolition becomes [illegible], that is [illegible], or rather greater than [illegible]. The Index of Demolition thus goes on constantly increasing as the degree of the function rises, probably (?) it converges either towards ½ or else towards unity. In arranging the terms it will be found most convenient to adopt, as I have done above, the dictionary method of sequence. The computations are greatly facilitated by the circumstance of the effect of any arrangement of each of the 5 lower lines not being altered when these *lines* are permuted with one another, this gives rise to the subdivision of the 7776 permutations into groups as follows: 6 of 120 identical terms, 60 of 60, 36 of 20, 60 of 30, 24 of 20, 30 of 10, 30 of 5, and 6 of 1. So that the total number of permutational arrangements to be constructed is only 252. Other methods of abridging the labour will readily suggest themselves to the practical computer. The total number of the *groups* of terms is of course always known *à priori*, and, for instance, in the case before us, must be equal to the number of ways in which ½(12 × 3), that is the number 18, can be divided into 3 parts, none of which is to exceed the number 12, that is 25; for the cubic invariant of the function of the eighth degree of two variables it is the number of ways in which 12 can be divided into 3 parts, of which none shall exceed 8, and so forth, zeros being always understood to be admissible; and of course in general for an invariant of the order r to a function of the degree n of i variables, the number of distinct terms is in general the number of ways in which $\frac{nr}{i}$ can be divided into r parts, of which none shall exceed n, subject however always to the possibility in particular cases of a diminution in consequence of some of the groups assuming zero for their coefficient.

$$\begin{array}{llll} a^n & a^{n-1}.\beta \ldots\ldots \beta^n & & (1) \\ a^n & a^{n-1}.\beta \ldots\ldots \beta^n & & (2) \\ \cdot & \cdot \quad \cdot & \cdot & \\ \cdot & \cdot \quad \cdot & \cdot & \\ a^n & a^{n-1}.\beta \ldots\ldots \beta^n & & (p) \end{array}$$

will be a function of λ, and all the several coefficients will be invariants of F.*

When $p=1$ we obtain the Λ given in the preceding section, and originally published by me in the *Philosophical Magazine* for the month of November 1851. The Λ obtained on this supposition has for its coefficients a series of independent invariants, commencing with the catalecticant and closing with the quadratic invariant. When p has any other value, we observe a similar series commencing with a commutantive invariant of a lower order than the catalecticant, but always closing with the quadratic invariant. Thus (*ex. gr.*), when $2np=8$, we may obtain by the preceding theorem three different quadratic functions; one giving the invariants of the orders 5, 4, 3, 2, the second those of the orders 3, 2, the third the invariant of the order 2.

In this case the invariants of the same order given by the different Λ's are the same to numerical factors *près*. Whether this is always necessarily the case is a point reserved for further examination.

The commutants applied in the preceding theorems have been called by me total commutants, because the total of each line of umbræ is permuted in every possible manner. If the lines be divided into segments, and the permutation be local for each segment instead of extending itself over the whole line, we then arrive at the notion of partial commutants, to which I have also (in concert with Mr. Cayley) given the distinctive name of Intermutants. In order to find the invariants of functions of odd degrees, the theory of total commutants requires the process of commutation to be applied, not immediately to the coefficients of the proposed function, but to some derived concomitant form. I became early sensible of this imperfection, and stated to the friend

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* By substituting the symbols $\frac{d}{dx}, \frac{d}{dy}$, &c. in place of the *umbræ* a, b, &c., the theorem is easily stated for covariants generally. But in applying the commutantive method to obtain covariants, or rather in the statement of the results flowing from each application, it is never necessary to go beyond the case of invariants, because the commutantive covariants of any given homogeneous function are always identical with commutantive invariants of emanants of the same function.

above named, to whom I had previously imparted my general method of total commutation, my conviction of the existence of a qualified or restricted method of permutation, whereby the invariants of the cubic function, for instance, of two and of three letters would admit, without the aid of a derived form of being represented. Many months ago, when I was engaged in this important research, and had made some considerable steps towards the representation of the invariant, *i.e.* the discriminant of the cubic function of x and y under the form of a single permutant, I was surprised by a note from the friend above alluded to, announcing that he had succeeded in fixing the form of the permutant of which I was at that moment in search. It is with no intention of complaining of this interference on the part of one to whose example and conversation I feel so deeply indebted, (and the undisputed author of the theory of Invariants,) that I may be permitted to say that, independent of the intervention of this communication, I must inevitably have succeeded in shaping my method so as to furnish the form in question; and that with the greater certainty, after my theory of commutants had furnished me with the precedent of permutable forms giving rise to terms identical in value but affected with contrary signs. As I have understood that Mr. Cayley is likely to develope this part of the subject in the present number of the *Journal*, it will be the less necessary for me to enter at any length into the theory of partial commutants on the present occasion.

The method of partial commutation is a simple but most important corollary from that of total commutation hereinbefore explained. To fix the ideas, conceive a class of p congredient systems, and that there are qr such classes perfectly independent. Proceed to divide these qr classes in any manner whatever into r sets, each containing q classes; and form the symbol of the total commutant corresponding to each such set. Now let these commutants be placed side by side against one another, and transpose the terms in each compound line thus formed once for all, but in any arbitrary manner. Then permute in every possible way all those symbols in each line, *inter se*, which belong to the same class, and operate with the symbols thus produced by reading off the vertical columns and attending to the rule of the + and − signs, as in the case of a total commutant; the result will be a commutant of the form operated upon. For instance, let $p = 1$, $q = 3$, $r = 2$, and let the number of variables in each system be 2. Form the commutant operators

$$\begin{array}{cc|cc} \dfrac{d}{dx} & \dfrac{d}{dy} & \dfrac{d}{d\xi} & \dfrac{d}{d\eta} \\ \dfrac{d}{dp} & \dfrac{d}{dt} & \dfrac{d}{d\phi} & \dfrac{d}{d\theta} \\ \dfrac{d}{dr} & \dfrac{d}{ds} & \dfrac{d}{d\rho} & \dfrac{d}{d\sigma} \end{array}$$

Interchange *in any manner* but *once for all* the symbols [of] each line, as thus:

$$\begin{array}{cccc} \dfrac{d}{dx} & \dfrac{d}{dy} & \dfrac{d}{d\xi} & \dfrac{d}{d\eta} \\ \dfrac{d}{d\phi} & \dfrac{d}{dp} & \dfrac{d}{dt} & \dfrac{d}{d\theta} \\ \dfrac{d}{ds} & \dfrac{d}{d\rho} & \dfrac{d}{dr} & \dfrac{d}{d\sigma} \end{array}$$

Now permute, *inter se*, the variables of each system, as

$$\frac{d}{dx},\ \frac{d}{dy};\quad \frac{d}{dp},\ \frac{d}{dt},\ \&c.;$$

the total number of the operative forms resulting will be $(1.2)^6$, and the sum of the $(1.2)^6$ quantities, half positive and half negative, formed after the type of

$$\Sigma\left\{\begin{array}{l} \dfrac{d}{dx}\cdot\dfrac{d}{d\phi}\cdot\dfrac{d}{ds}\,U\times\dfrac{d}{dy}\cdot\dfrac{d}{dp}\cdot\dfrac{d}{d\rho}\cdot U \\ \times\dfrac{d}{d\xi}\,\dfrac{d}{dt}\,\dfrac{d}{dz}\,U\times\dfrac{d}{d\eta}\cdot\dfrac{d}{dt}\cdot\dfrac{d}{d\sigma}\cdot U \end{array}\right\},$$

U being supposed to be a function homogeneous in

$$x, y;\ \xi, \eta;\ p, t;\ \phi, \theta;\ r, s;\ \rho, \sigma,$$

will be a covariant of U.

The proof of the truth of this proposition is contained in what is shewn in the Notes of the Appendix for total commutants, it being only necessary to make the systems which are independent vary consecutively, and then apply the inference to the supposition of their varying simultaneously.

It may be extended to the more general supposition of classes of an unequal number of congredient systems of unequal numbers of variables in each, the only condition apparently required being that the number of distinct terms shall be the same in each line of the final commutantive operator. The important remark to be made is, that in

this theorem there is nothing to prevent any of the ›eing made *identical;* or, in other words, a given of one system of variables may be regarded as a ›f as many different, although coincident, sets as we se to suppose. Thus, suppose

$$U = Ax^2 + 2Bxy + Cy^2,$$

ıke the partial commutant formed of the two total ıt operators

$$\begin{array}{cc} \frac{d}{dx} & \frac{d}{dy} \\ \frac{d}{dx} & \frac{d}{dy} \end{array}$$

with itself. If we write them in the same order,

$$\begin{array}{cccc} \frac{\dot{d}}{dx} & \frac{\dot{d}}{dy} & \frac{\acute{d}}{dx} & \frac{\acute{d}}{dy} \\ \frac{\dot{d}}{dx} & \frac{\dot{d}}{dy} & \frac{\acute{d}}{dx} & \frac{\acute{d}}{dy} \end{array}$$

use the dots and dashes to distinguish those in the : which are considered as belonging to the same therefore as permutable, *inter se,*) we shall evidently $\{AC - B^2\}^2$; if we commence with a permutation, ave the form of operation

$$\begin{array}{cccc} \frac{\dot{d}}{dx} & \frac{\dot{d}}{dy} & \frac{\acute{d}}{dx} & \frac{\acute{d}}{dy} \\ \frac{\dot{d}}{dx} & \frac{\acute{d}}{dy} & \frac{\acute{d}}{dx} & \frac{\dot{d}}{dy} \end{array}$$

found that we obtain $2\{AC - B^2\}^2$.
suppose that we have

$$U = Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3.$$

vrite

$$\begin{array}{cccc} \frac{\dot{d}}{dx} & \frac{\dot{d}}{dy} & \frac{\acute{d}}{dx} & \frac{\acute{d}}{dy} \\ \frac{\dot{d}}{dx} & \frac{\dot{d}}{dy} & \frac{\acute{d}}{dx} & \frac{\acute{d}}{dy} \\ \frac{\dot{d}}{dx} & \frac{\dot{d}}{dy} & \frac{\acute{d}}{dx} & \frac{\acute{d}}{dy} \end{array}$$

the value of the commutant would come out zero; make a permutation, and write

$$\begin{matrix} \frac{\dot{d}}{dx} & \frac{\dot{d}}{dy} & \frac{\acute{d}}{dx} & \frac{\acute{d}}{dy} \\[1em] \frac{\dot{d}}{dx} & \frac{\dot{d}}{dy} & \frac{d}{dx} & \frac{d}{dy} \\[1em] \frac{\dot{d}}{dx} & \frac{d}{dy} & \frac{d}{dx} & \frac{\dot{d}}{dy} \end{matrix}$$

the operation indicated by the above performed upon give a multiple of the discriminant of U.

In like manner we may represent Aronhold's variant of the form $(x, y, z)^3$ by means of the permutant

$$\begin{matrix} \frac{\dot{d}}{dx} & \frac{\dot{d}}{dy} & \frac{\dot{d}}{dz} & \frac{\acute{d}}{dx} & \frac{\acute{d}}{dy} & \frac{d}{dz} \\[1em] \frac{\acute{d}}{dx} & \frac{\dot{d}}{dy} & \frac{\dot{d}}{dz} & \frac{\dot{d}}{dx} & \frac{d}{dy} & \frac{\acute{d}}{dz} \\[1em] \frac{\acute{d}}{dx} & \frac{\dot{d}}{dy} & \frac{\dot{d}}{dz} & \frac{\dot{d}}{dx} & \frac{\dot{d}}{dy} & \frac{\acute{d}}{dz} \end{matrix}$$

If we make

$$V = \left(\xi \frac{d}{d\xi} + \eta' \frac{d}{d\eta} + \zeta' \frac{d}{d\zeta}\right) \cdot \left(\xi \frac{d}{dx} + \eta \frac{d}{dy} + \zeta \frac{d}{d}\right.$$
$$\cdot (x, y, z)^3,$$

and use H to signify the determinant

$$\begin{matrix} x; & y; & z, \\ \xi; & \eta; & \zeta, \\ \xi'; & \eta'; & \zeta' \end{matrix}$$

which is evidently an universal triple covariant, and

$$W = V + \lambda H,$$

and apply to W the partial commutantive symbol

$$\begin{matrix} \frac{\dot{d}}{dx} & \frac{\dot{d}}{dy} & \frac{\dot{d}}{dz} & \frac{\acute{d}}{dx} & \frac{\acute{d}}{dy} & \frac{\acute{d}}{dz} \\[1em] \frac{\acute{d}}{d\xi} & \frac{\dot{d}}{d\eta} & \frac{\dot{d}}{d\zeta} & \frac{\dot{d}}{d\xi} & \frac{\acute{d}}{d\eta} & \frac{\acute{d}}{d\zeta} \\[1em] \frac{\acute{d}}{d\xi'} & \frac{\acute{d}}{d\eta'} & \frac{\dot{d}}{d\zeta'} & \frac{\dot{d}}{d\xi'} & \frac{\dot{d}}{d\eta'} & \frac{\acute{d}}{d\zeta'} \end{matrix}$$

hall obtain a function of λ of which all the odd powers the second power will disappear, and such that the icients of λ^2 and the constant term will be Aronhold's id T, and the discriminant of the entire function in ect to λ^2 (if not for the distribution assigned to the dots dashes in the foregoing, at least for some other distribu-) may not improbably be the discriminant of the given tion $(x, y, z)^3$.

[*To be Continued.*]

NOTES IN APPENDIX.

1) More generally, in as many ways as the number n can be divided parts, in so many ways can a given function of one set of variables it were *unravelled* so as to furnish concomitant forms.

For instance, the form $ax^3 + 3bx^2y + 3cxy^2 + dy^3$ has for its concomitant

$$aux + buy + bvx + cvy + cwx + dwy,$$

re u, v, w are cogredient with x^2, $2xy$, y^2, and

$$auu'x + buu'y + buv'x + bvu'x + cvv'x + cvu'y + cuv'y + dvv'y,$$

re u, v; u', v' are cogredient with each other and with x and y; and proposition in the text may be best derived from this more general orem by dividing the index into equal parts, forming as many systems there are such parts, and then identifying the systems so formed.

2) The following additional example will illustrate the power of this thod.

Let $\phi = (x, y, z)^4$ be the general function of the fourth degree. Form *unravelment* the concomitant form $(u, v, w, p, q, r)^2$ (say P) where v, w, p, q, r are congredient with $x^2, y^2, z^2, 2zy, 2xz, 2yx$.

Again, the universal concomitant $(x\xi + y\eta + z\zeta)^4$ will have for its con-nitant

$$u\xi^2 + v\eta^2 + w\zeta^2 + p\eta\zeta + q\zeta\xi + r\xi\eta,$$

re ξ, η, ζ are contragredient to x, y, z. Now take the reciprocal polar his last form with respect to ξ, η, ζ; that is,

$$\Sigma\,(vw - \tfrac{1}{4}p^2)\,x_1^2 + 2\Sigma\,(\tfrac{1}{4}qr - \tfrac{1}{2}pu)\,y_1z_1 \text{ (say } G),$$

re x_1, y_1, z_1, being contragredient to ξ, η, ζ, will be cogredient with , z. $P + \lambda G$ is a quadratic function of the six variables u, v, w, p, q, r, its discriminant will give a function of λ of the sixth degree, all of se even coefficients will be covariants of ϕ. If we replace x_1, y_1, z_1 x, y, z, these even coefficients will be respectively (understanding *order* refers to the dimensions *quoad* the coefficients of ϕ and degree he dimensions *quoad* x, y, z) as follows:

Of order	6	degree	0,
	5	"	2,
	4	"	4,
	3	"	6,
	2	"	8,
	1	"	10,
	0	"	12.

two last coefficients must evidently be identically zero. It is now

that some of the others may be so too as regards the one of the th[illegible] and sixth degree, this is of the same form as, and may be [illegible] the Hessian of ϕ; as regards the one of the fourth or[illegible] degree, this may be ϕ itself multiplied by the cubic invariant (theory of Section 3 proves to exist) of ϕ. But the covarian[illegible] order and second degree, and of the second order and eigh[illegible] if they are not identically zero, and if the latter is not ϕ^2 [illegible] or two of some very simple cases will easily establish [illegible] other) are probably irreducible forms. The existence of a [illegible] conic section to a curve of the fourth order, if established, [illegible] particularly interesting, and its geometrical meaning would [illegible] being elicited.

(3) If any form (f) of the degree n be written symbolically, [illegible]

$$(a_1x_1 + a_2x_2 + \ldots + a_\iota x_\iota)^n,$$

where $x_1, x_2 \ldots x_\iota$ are real but $a_1, a_2 \ldots a_\iota$ umbral, and if I_r be any [illegible] of the order r in respect of the real coefficients of (f), it is eas[illegible] reason of I_r remaining unaltered when $x_1, x_2 \ldots x_\iota$ become re[illegible] $f_1x_1, f_2x_2 \ldots f_\iota x_\iota$, provided that $f_1 f_2 \ldots f_\iota = 1$, that each term [illegible] pressed by means of the umbræ, must contain an equal number [illegible] $a_1, a_2 \ldots a_\iota$, so that each such term will contain $\frac{nr}{\iota}$ of each of [illegible] course differently subdivided and grouped; hence we have the [illegible] condition that $\frac{nr}{\iota}$ must be an integer; but this is less stringent [illegible] actual condition, which is that $\frac{nr}{\iota}$ must be an integer of a cert[illegible] for instance, as before observed, when $\iota = 2$ $\frac{nr}{\iota}$ must be an *even* i[illegible]

(4) To prove the theorem given in the text for total sim[illegible] mutants it is only necessary to bear in mind that whene[illegible] columns in any total commutant become identical, the co[illegible] vanishes. To fix the ideas, take the commutant formed of lines [illegible] $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$, written under one another; let there be (r) such li[illegible] total number of terms will be $(1.2.3)^r$: the 1.2.3 positions of [illegible] written above will correspond to $(1.2.3)^{r-1}$ several groupings [illegible] remaining lines; now when x, y, z undergo a uni-modular line[illegible] stitution, $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ will undergo a related substitution not co[illegible] with that of x, y, z, but still uni-modular; let x, y, z change all th[illegible] systems remaining fixed, and suppose $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ to become resp[illegible]

$$f\frac{d}{dx} + g\frac{d}{dy} + h\frac{d}{dz},$$

$$f'\frac{d}{dx} + g'\frac{d}{dy} + h'\frac{d}{dz},$$

$$f''\frac{d}{dx} + g''\frac{d}{dy} + h''\frac{d}{dz},$$

ch of the $(1.2.3)^{r-1}$ groups of the terms arising from the permuta-
$\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ will subdivide into 27 groups, of which we may
hose in which any of the terms $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ occurs twice or three
accordingly there will be left only the six effective orders of per-
ons,

$$f\frac{d}{dx},\ g'\frac{d}{dy},\ h''\frac{d}{dz};\ f\frac{d}{dx},\ k'\frac{d}{dz},\ g'\frac{d}{dy};\ \&c.$$

uently each of the $(1.2.3)^{r-1}$ groups gives rise to 6 times 6 products
sum will be $\left.\begin{matrix} f; & g; & h \\ f'; & g'; & h' \\ f''; & g''; & h'' \end{matrix}\right\} \times$ the sum of the 6 products correspond-
the permutations of $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$; and therefore, the transforma-
peing uni-modular, the sum of the products corresponding to the
$(1.2.3)^r$ permutations remains constant when x, y, z change. In
anner, all the systems may change one after the other, and con-
ntly all of them at the same time without affecting the value of the
utant: and in like manner for the general case. Q. E. D.

The truth of the proposition relative to compound commutants
he mode of the demonstration will be apparent from the subjoined
ple.

t the function be supposed to be

$$(ax+by)(a'x'+b'y')(a\xi+\beta\eta)(a\xi'+\beta'\eta'),$$

e $x, y; x', y'$ are cogredient and $\xi, \eta; \xi', \eta'$ cogredient; the a, b, a, β.
re of course mere umbræ. Now take the compound commutant

$$\begin{matrix} ad'; & ab'+a'b; & bb', \\ aa'; & a\beta'+a'\beta; & \beta\beta'. \end{matrix}$$

$x, y; x', y'$ undergo a linear substitution, and, accordingly,

$$\begin{matrix} \text{let } a & \text{become} & fa-gb, \\ a' & \text{"} & fa'-gb', \\ b & \text{"} & ha-kb, \\ b' & \text{"} & ha'-kb', \end{matrix}$$

h, k being of course actual and not umbral; then the above com-
ant will be easily seen to decompose into 6 others, which will be
l to the original commutant multiplied by the determinant

$$\begin{matrix} f^2; & 2fg; & g^2, \\ fh; & fk-gh; & gk, \\ h^2; & 2hk; & k^2, \end{matrix}$$

h is equal to $(fk-gh)^3$, i. e. = 1.
nd so in general, which shews, as in the preceding note, that all the
es of congredient systems may be transformed successively one after
other, and therefore simultaneously, without altering the value of the
nutant.

(6) In the last May Number of the *Journal*, Mr. Boole, to whose modest labours the subject is perhaps at least as much indebted as to any one other writer, has given a theorem,* (14) p. 94, the excellent idea contained in which there is no difficulty in shaping so as to render it generalizable by aid of the theory of contravariants. It may be regarded in some sort a pendant or reciprocal to the Eisenstein-Hermite theorem presented by me under a wider aspect in the First Section of this paper.

Let $\phi(x, y \ldots z)$ have any contravariant $\theta(x, y \ldots z)$; then will

$$\phi\left(\frac{d}{dx}, \frac{d}{dy} \cdots\cdots \frac{d}{dz}\right) . \theta(x, y \ldots z)$$

be a contravariant of ϕ. For orthogonal transformations the terms contravariant and covariant coincide, and the theorem for this case appears to have been known to Mr. Boole, see (15), same page. More generally, if ψ and θ be any two concomitants of ϕ, the algebraical product $\psi.\theta$ will also be a concomitant of ϕ, provided that the systems of variables in ψ and θ have all distinct names, or that those which bear the same names are cogredient with another. If this proviso does not hold good, the product in question will evidently be no longer a concomitant of ϕ. Let however Ψ denote what ψ becomes, and ϑ what θ becomes, when in place of the variables $x, y \ldots\ldots z$ of every two contragredient synonymous systems in ψ and θ we write $\frac{d}{dx}, \frac{d}{dy} \cdots\cdots \frac{d}{dz}$, then will $\vartheta.\psi$ and $\Psi.\theta$ be each of them concomitants of ϕ, the synonymous systems becoming cogredient with ψ in the one case and with θ in the other.

(7) There is one principle of *paramount* importance which has not been touched upon in the preceding pages, which I am very far from supposing to exhaust the fundamental conceptions of the subject, (indeed, not to name other points of enquiry, I have reason to suppose that the idea of contragredience itself admits of indefinite extension through the medium of the reciprocal properties of commutants; the particular kind of contragredience hereinbefore considered having reference to the reciprocal properties of ordinary determinants only).

The principle now in question consists in introducing the idea of *continuous* or *infinitessimal* variation into the theory. To fix the ideas, suppose C to be a function of the coefficients of $\phi(x, y, z)$, such that it remains unaltered when x, y, z become respectively fx, gy, hz, provided that $fgh = 1$. Next, suppose that C does not alter when x becomes $x + ey + \epsilon z$, when e and ϵ are indefinitely small: it is easily and obviously

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* Mr. Boole applied his theorem to obtain the cubic invariant of $(x, y)^4$, say $\phi(x, y)$, by operating upon its Hessian with $\phi\left(\frac{d}{dy} - \frac{d}{dx}\right)$. More generally, when $\phi(x, y) = (x, y)^{4n}$, the catalecticant of the antepenultimate emanant of ϕ is also of the degree $2r$; and this, when operated upon by $\phi\left(\frac{d}{dy}, -\frac{d}{dx}\right)$, will give an invariant of the order $n+1$, which is probably identical with the catalecticant of ϕ itself. There exists a most interesting transformation of the catalecticant of any emanant of a function of any degree in x, y, whether even or odd, under the form of a determinant, some of the lines of which contain combinations only of x and y, without any of the coefficients, and all the rest the coefficients only of the given function without x or y. The Hessian being the catalecticant of the second emanant is of course included within this statement.

monstrable that if this be true for e and ϵ indefinitely small, it must be e for *all* values of e and ϵ. Again, suppose that C alters neither when receives such an infinitesimal increment, y and z remaining constant, when y nor z separately receive corresponding increments, z, x and y in the respective cases remaining constant, it then follows from what been stated above that this remains true for finite increments to x y or z separately, and hence it may easily be shewn that C will remain nstant for any *concurrent* linear transformations of x, y, z, when the odulus is unity. This all-important principle enables us at once to fix e form of the symmetrical functions of the roots of $\phi\left(\frac{x}{y}, 1\right)$ which present invariants of $\phi(x, y)$ when the coefficient of the highest power x is made unity. It also *instantaneously* gives the necessary and *fficient* conditions to which an invariant of any given order of any mogeneous function whatever is subject, and thereby reduces the oblem of discovering invariants to a definite form. But as these conions coincide with those which have been stated to me as derived m other considerations by the gentleman whose labours in this departent are *concomitant* with my own, I feel myself bound to abstain from essing my conclusions until he has given his results to the press.

(8) By aid of the general principle enunciated in Note (6) above, e can easily obtain Aronhold's S and T. Let U be the given cubic nction of x, y, z, and let $G(x, y, z;\ \xi, \eta, \zeta)$ be the polar reciprocal in spect to ξ, η, ζ of $\left(\xi\frac{d}{dx} + \eta\frac{d}{dy} + \zeta\frac{d}{dz}\right)^2 U$, then $G(\xi, \eta, \zeta;\ x, y, z)$ as ell as the former G will be a concomitant to U, but the homonymous stems of variables in the two G's will be contragredient; and, accordngly, $G\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz};\ \xi, \eta, \zeta\right).G(\xi, \eta, \zeta;\ x, y, z)$ will be a concomitant U; this concomitant is readily seen to be an invariant of the fourth rder; that is, Aronhold's S. Again, from S, by means of the Eisenstein-Hermite theorem, we may derive a form $K(x, y, z)$ of the third degree n x, y, z, and whose coefficients will be of three dimensions; and, ccordingly, if the Hessian of U be called $H(U)$,

$$K\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right).H(U)$$

ill be a Sextic Invariant of U, that is, Aronhold's T.

CORRECTION OF THE POSTSCRIPT TO THE PAPER ON PERMUTANTS.

By Arthur Cayley.

Mr. Sylvester has represented to me that the paragraph elating to his communications conveys an erroneous idea f the nature, purport, and extent of such communications; have, in fact, in the paragraph in question, singled out hat immediately suggested to me the expression of the unction $6abcd + 3b^2c^2 - 4ac^3 - 4b^3d - a^2d^2$ as a partial com-

mutant or intermutant, but I agree that a fuller refe... ought to have been made to Mr. Sylvester's results, ... that the paragraph in question would more properly h... stood as follows:

"Under these circumstances Mr. Sylvester communic... to me a series of formal statements, not only oral but in writing, to the effect that he had discovered a permutati... method of obtaining as many invariants—viz. commutantive invariants—by direct inspection from a function of any degree of any number of letters as the index of the degree contains even factors; one of these commutantive invariants being in fact the function $ace + 2bcd - ae^2 - bd^2 - c^3$, expressible, according to Mr. Sylvester's notation, by $\begin{pmatrix} a^2, & ab, & b^2 \\ a^2, & ab, & b^2 \end{pmatrix}$; and, according to the notation of my memoir in the *Camb. Phil. Trans.*, supposing $00 = a$, $01 = 10 = b$, $02 = 11 = 20 = c$, &c. by $\begin{vmatrix} 00 \\ 11 \\ 22 \end{vmatrix}$."

Mr. Sylvester and I shall, I have no doubt, be able to agree to a joint statement of any further correction or explanation which may be required.

Jan. 27, 1852.

ON DEMONSTRATIONS OF THE BINOMIAL THEOREM.

By Homersham Cox, B.A.

The Binomial Theorem fails arithmetically when it expands a finite power of a binomial in an infinite divergent series. For instance, if $(\frac{1}{3})^2$ be expressed by the expansion of $(1-4)^{-2}$ by this theorem,

$$\tfrac{1}{9} = 1 + 2.4 + 3.4^2 + 4.4^3 + \ldots \; \textit{ad infinitum}.$$

The second side of this equation is divergent and its arithmetical sum infinite. We have therefore $\frac{1}{9} = \infty$, an obvious absurdity if the symbol $=$ designate arithmetical equality. But it is said that the symbol $=$ here designates *symbolical equivalence*. The truth of this assertion depends on the definition of this phrase, and without doubt many arbitrary definitions might be given, in accordance with which the Binomial Theorem might be considered to hold for divergent series. But if symbolical equivalence be ever interpreted to include arithmetical equivalence, it seems

ear from the foregoing consideration, that the definition ght to admit that interpretation for convergent, and exude it for divergent series expanded by the Binomial heorem.

In the best known demonstrations of the theorem the eaning of the symbol = is certainly meant to include *ways* arithmetical equality, and yet the reasoning of these monstrations in no way excludes expansion by divergent ries. As they therefore necessarily lead to the absurdity at a finite arithmetical quantity may equal infinity, it ppears certain that they contain incorrect steps.

In Euler's proof $f(m)$ being defined by the equation

$$f(m) = 1 + mx + \frac{m \,.\, m-1}{1.2} x^2 + \frac{m \,.\, m-1 \,.\, m-2}{1.2.3} x^3 + \&c. \textit{ ad inf.}$$

it is inferred from the case when m and n are positive integers that $f(m).f(n) = f(m+n)$ when m and n are positive or negative integral or fractional. Now in the series for $f(m+n)$ the $(r+1)^{th}$ term is

$$\frac{n+m \,.\, n+m-1 \,.\, n+m-2 \ldots n+m-r+1}{1.2.3 \ldots r} x^r,$$

which when r is infinite takes the form

$$\frac{n+m \,.\, n+m-1 \,.\, n+m-2 \ldots n+m-\infty+1}{1.2.3 \ldots \infty} x^\infty,$$

an indeterminate quantity.

But if m or n be fractional or negative, the series in which they are involved are continued *ad infinitum*, and then, therefore, r becomes infinite. In order therefore to find the value of $f(m+n)$ from its expansion in such cases, the sum of terms involving indeterminate quantities has to be evaluated.

This consideration does not appear to be regarded in Euler's demonstration, which, as will be shewn, tacitly assumes that *the sum of those terms of an infinite series which contain a zero factor is necessarily zero*, an assumption frequently erroneous.

The demonstration for negative indices is made to depend on the assumption that $f(n-n) = 1$. But the development of $f(n-n)$ is

$$1 + n - n \,.\, x + \frac{n-n \,.\, n-n-1}{1.2} x^2 + \ldots + \frac{n-n \,.\, n-n-1 \ldots n-n-r+1}{1.2 \ldots r} x^r$$
$$+ \&c. \textit{ ad in}$$

and this series cannot = 1 unless the sum of all the terms in it following 1 be zero. Now though each of these terms is zero when r is finite, they when r is infinite take the form

$$\frac{0. -1. -2. \ldots (-\infty + 1)}{1.2.3. \ldots \infty} x^{\infty},$$

an indeterminate quantity not zero unless x be a fraction.

Similarly the demonstration for fractional indices depends on the assumption that

$$f\left(\frac{h}{k} + \frac{h}{k} + \text{\&c. to } k \text{ terms}\right) = (1+x)^h,$$

h and k being positive integers. The development of the first side of the equation is an infinite series with terms having zero and infinite factors. The objection against neglect of these terms is the same as in the demonstration for negative indices.

It may be observed that the demonstration for fractional indices assumes also that any fraction may be expressed by the ratio of two integers. This assumption excludes incommensurable fractions, but for such indices the proof might probably be completed by considerations somewhat resembling those by which Duchayla extends his proof of the parallelogram of forces to the case of incommensurable components.

In Lund's edition of Wood's *Algebra* (Cambridge, 1848), Euler's proof is dismissed to an Appendix to make way for another by the Rev. J. Griffith, introduced in the following terms: "This proof is new and appears to possess peculiar merit.... Any one who still retains an affection for Euler's proof will find it in Note 3." The question of transference of affection to the new demonstration need not be examined here, as precisely the same objections apply to it as to Euler's, from which it differs in no other respect whatever, than in obtaining laboriously by actual multiplication the form of the series which is the product of two series of the form $f(m)$ above.

In Hutton's *Mathematical Tracts* (London, 1812) an account is given of several demonstrations of the Binomial Theorem. The following is a brief abstract of this account. James Bernouilli, in his *Ars Conjectandi*, demonstrated the theorem for positive integral indices. John Stewart, in his *Commentary on Newton's Quadrature of Curves*, copied Bernouilli's demonstration, and added a demonstration for fractional exponents by the principles of fluxions. De Moivre, in the

phical Transactions. No 230 for 1697, demonstrated ultinomial theorem and the binomial theorem as a lar case of it, by the principles of [illegible] and [illegible]tions. Landen in his *Discourse* [illegible] *al Analysis*, 1758, and *Residual Analysis*, [illegible], the theorem by assuming that a binomial raised to ional power is equal to a series ascending by [illegible] of one term of the binomial, and equating [illegible] mpares this investigation with that by [illegible] which [illegible]ve series is differentiated. Hales, in his *Analysis* *[illegible]um*, gives an account of this investigation. Hutton, above work cited, gives a proof of his own by [illegible] a series similar to that just described, and equating ents. On reference to Stewart's [illegible] s that he first shews that if x be a flowing quantity xion of x^n is $nx^{n-1}\dot{x}$: he then assumes [illegible] ion of the binomial raised to any power, and [illegible]ates.

[illegible] *Penny Cyclopædia* (BINOMIAL THEOREM) is a notice roof by Messrs. Swinburne and Tylecote [illegible] which is stated to be, if the details [illegible] al character, but far above the student."

[illegible] of the proofs mentioned by Dr. Hutton [illegible] ppears to regard the convergency of the series [illegible]. The same neglect of convergency appears in [illegible]strations of the principle of equations [illegible] pposing it admitted that the meaning of the [illegible] ys to include the signification [illegible] ct proof of the Binomial Theorem [illegible] to the convergency of the series [illegible] Now [illegible]estigation, whether the sum of a series [illegible] which cannot be exceeded however great the number ns—what is this but a question of the [illegible] of [illegible] It seems certain therefore that this [illegible] must olved at some stage or other of the [illegible] and that any attempt to escape from this [illegible] by aical artifices merely evades, without [illegible] of the question. If however the principles of [illegible] be sarily involved, they surely ought to be openly avowed distinctly applied. So that the desideratum in demonstrating the Binomial Theorem seems to resolve itself into explanation and application of the parts of the doctrine h the nature of the subject requires, in [illegible] intelligible to the algebraical student. ly be effected by methods [illegible].

better than the following, which is offered merely on account of its brevity and as an illustration of the views here taken.

Proof of the Binomial Theorem.

(1). Let m be any finite positive integer. It may be ascertained by the multiplication of $b+k$ by itself m times that $(b+k)^m = b^m + mb^{m-1}k +$ *terms multiplied by the second and higher powers of* k, and from the manner in which this result is obtained, it will easily be seen to be true for all positive integral values of m.

(2). Let the *limit* of a quantity involving k be defined to be the value from which that quantity differs by quantities which decrease indefinitely as k is diminished. For instance, it appears from the above that $\frac{(b+k)^m - b^m}{k} = mb^{m-1} +$ *terms multiplied by the first and higher powers of* k. Now these terms decrease indefinitely as k is diminished; therefore "limit" of $\frac{(b+k)^m - b^m}{k} = mb^{m-1}$. This result can be extended algebraically to all real values of m.

(3). Let X be any quantity involving x, such that when $x=0$ or h, X is zero but has different values for intermediate values of x. Of these other values of X there must be one at least which is not exceeded in positive magnitude by any other of them. Let this be X_1 corresponding to a value x_1. Then if x have either a large value x_1+k or smaller x_1-k, the corresponding values X_2 and X_3 do not exceed X_1; or X_2-X_1 and X_3-X_1 have the same sign; therefore $\frac{X_2-X_1}{k}$ and $\frac{X_3-X_1}{-k}$ have different signs. Let both these quantities have the same "limit" L, from which, therefore, by the definition, they differ by differences which decrease indefinitely as k is diminished. If then L had any magnitude, these differences might be taken less than it; and then $\frac{X_2-X_1}{k}$ and $\frac{X_3-X_1}{-k}$ would have the same sign as L. But it has been shewn that they must have contrary signs; therefore L must not be of any magnitude or must $=0$.

(4). Let $X =$

$$(a+x)^n - \left(a^n + na^{n-1}x + \frac{n.n-1}{1.2}a^{n-2}x^2 + \frac{n.n-1.n-2}{1.2.3}a^{n-3}x^3 + .. + Ax^r\right) ..(1),$$

r being a positive integer and A such a quantity not involving

...at $X=0$ when $x=h$. Also $X=0$ when $x=0$. When $x=x_1$,

$$\frac{...-X_1}{k}=\frac{(a+x+k)^n-(a+x)^n}{k}-\left\{na^{n-1}+\frac{n.n-1}{1.2}a^{n-2}\frac{(x+k)^2-x^2}{k}\right.$$
$$\left.+\ldots+A\frac{(x+k)^r-x^r}{k}\right\}.$$

...ing m successive values in (2) the limit of this is found ...be

$$...+x)^{n-1}-n\left(a^{n-1}+n-1\,.\,a^{n-2}x+\frac{n-1\,.\,n-2}{1.2}a^{n-3}x^2+\ldots\right)+rAx^{r-1}\ldots\text{(II)}.$$

By writing $-k$ for k, it may easily be seen that $\frac{X_2-X_1}{-k}$ ... the same limit, which by (3) $=0$ when $x=x_1$. But (II) ...also zero when $x=0$. Therefore let the operation by ...ich (II) was obtained from (I) be repeated on (II) and on ... successive results from it, each of which becomes zero ...der like conditions. After r such operations we have ...ally $0=n.n-1\ldots n-r+1\,.\,(a+x)^{n-r}-1.2\ldots r.A$ for some value ... x between 0 and h. Call this value θh where θ is some ...oper fraction. Then $1.2\ldots r.A=n.n-1\ldots n-r+1\,(a+\theta h)^{n-r}$. ...bstituting from this in (1) and putting $X=0$ when $x=h$,

$$...h)^n=a^n+na^{n-1}h+\frac{n.n-1}{1.2}a^{n-2}h^2+\ldots+\frac{n.n-1\ldots n-r+1}{1.2\ldots r}(a+\theta h)^{n-r}h^r.$$

... the last term of this series vanish when r is sufficiently ...ge,

$$(a+h)^n=a^n+na^{n-1}h+\frac{n.n-1}{1.2}a^{n-2}h^2+\ldots\text{ to }0,$$

...ich is the Binomial Theorem.

Cambridge, October 1851.

MATHEMATICAL NOTES.

I.—Solution of a Functional Equation.

To solve the functional equation

$$\int_{-\infty}^{+\infty}\phi_m x\phi_n(a-x)\,dx=\phi_{m+n}a,$$

...ovided $$\phi x=\phi(-x).$$

Let $\int_{-\infty}^{+\infty}\phi a\cos az\,da=\psi z$. Then

$$\psi_{m+n}z=\int_{-\infty}^{\infty}da\int_{-\infty}^{+\infty}dx\,\phi_m x\phi_n(a-n)\cos az.$$

Write for a, $a+x$, then $\cos az$ becomes

$$\cos az \cos xz - \sin az \sin xz,$$

and as the sines change sign with their arcs, we get

$$\psi_{m+n} z = \int_{-\infty}^{+\infty} da \int_{-\infty}^{x} dx\, \phi_m x\, \phi_n a \cos az \cos xz.$$

Or $$\psi_{m+n} z = \psi_m z\, \psi_n z,$$

whence $$\psi_m z = \chi z^m,$$

χ being independent of m, and therefore, by Fourie[r's] theorem,

$$\phi_m x = \frac{1}{\pi}\int_0^{\infty} \chi z^m \cos xz\, dz.$$

The required solution. R. L. E.

II.—Note on a Question in the Senate-House Papers for 18[..]

"The sections of a surface of the second order made [by] parallel planes are similar and similarly situated coni[cs]. Determine the ratio of the magnitudes of the sections ma[de] by two given planes."

The equations of the surface of the second order and [of] the parallel planes being assumed to be

$$lx^2 + my^2 + nz^2 - k = 0,$$

$$\alpha x + \beta y + \gamma z - \delta = 0;$$

then if

$$x = s\xi + a, \quad y = s\eta + b, \quad z = s\zeta + c,$$

and the constants s, a, b, c be properly determined, the t[wo] equations will take the form

$$l\xi^2 + m\eta^2 + n\zeta^2 - k = 0,$$

$$\alpha\xi + \beta\eta + \gamma\zeta = 0;$$

which shews that the section made by the plane $\alpha x + \beta\eta + \gamma$[...] $-\delta = 0$, is a curve similar and similarly situated to the secti[on] made by the parallel plane $\alpha x + \beta y + \gamma z = 0$, and each secti[on] is of course a conic. The values of a, b, c, s are

$$a = \frac{\alpha\delta}{Kl}, \quad b = \frac{\beta\delta}{Km}, \quad c = \frac{\gamma\delta}{Kn}, \quad s^2 = 1 - \frac{\delta^2}{kK},$$

where $$K = \frac{\alpha^2}{l} + \frac{\beta^2}{m} + \frac{\gamma^2}{n}.$$

And the quantity s represents the ratio of the magnitudes [of] the sections made by the two parallel planes. A. C.

[redacted]

ON THE FAMILY OF THE WAVE-SURFACE.

By William Walton.

Vave-surface in biaxal crystals may be regarded in nder which, as far as I am aware, it has not been ed, viz. as a member of a family of surfaces con- ogether by a common principle of generation. I in fact to shew that it may be traced out by the vement of a generator of double curvature (the on of two cylinders of the second order, the axes cut each other at right angles), which is subjected hrough three concentric circular directors in planes angles to each other, and then to investigate the ifferential equation of the family of surfaces which oci of the same generator regulated by any directors .

Co find the locus of the generator

$$\left.\begin{aligned} \frac{y^2}{\mu} - \frac{z^2}{\nu} &= b^2 - c^2 \\ \frac{z^2}{\nu} - \frac{x^2}{\lambda} &= c^2 - a^2 \\ \frac{x^2}{\lambda} - \frac{y^2}{\mu} &= a^2 - b^2 \end{aligned}\right\} \dots\dots\dots\dots\dots (1),$$

being arbitrary parameters, which is constrained pass through the three directors

$$\left.\begin{aligned} x &= 0 \\ y^2 + z^2 &= a^2 \end{aligned}\right\} \dots\dots\dots\dots\dots (2),$$

$$\left.\begin{aligned} y &= 0 \\ z^2 + x^2 &= b^2 \end{aligned}\right\} \dots\dots\dots\dots\dots (3),$$

$$\left.\begin{aligned} z &= 0 \\ x^2 + y^2 &= c^2 \end{aligned}\right\} \dots\dots\dots\dots\dots (4).$$

curve (1) passes through each of the curves (2), (3), asily obtain the three respective equations

$$\begin{aligned} \lambda a^2 + \mu b^2 + \nu c^2 &= (1 + \lambda + \mu + \nu)\, a^2 \\ &= (1 + \lambda + \mu + \nu)\, b^2 \\ &= (1 + \lambda + \mu + \nu)\, c^2, \end{aligned}$$

[redacted]

which are evidently equivalent to the two following:

$$1 + \lambda + \mu + \nu = 0 \text{ . . . . } (5),$$

$$\lambda a^2 + \mu b^2 + \nu c^2 = 0 \text{ . . . . } (6).$$

From the equations (1) it is plain that, r being some quantity,

$$\left.\begin{aligned} \frac{x^2}{\lambda} &= a^2 - r^2 \\ \frac{y^2}{\mu} &= b^2 - r^2 \\ \frac{z^2}{\nu} &= c^2 - r^2 \end{aligned}\right\} \text{ . . . . } (7).$$

Hence $\quad x^2 + y^2 + z^2 = \lambda a^2 + \mu b^2 + \nu c^2 - r^2 (\lambda + \mu + \nu)$

$$= r^2, \text{ by (5) and (6).}$$

From (5) and (7), we see that

$$\frac{x^2}{r^2 - a^2} + \frac{y^2}{r^2 - b^2} + \frac{z^2}{r^2 - c^2} = 1,$$

which is the equation to the wave-surface, the requir locus.

(B). Suppose that $\mu = 0$. Then, from (1), we see that tl equations to the corresponding generator are reduced to

$$y = 0,$$

$$\frac{z^2}{\nu} - \frac{x^2}{\lambda} = c^2 - a^2,$$

the equation connecting the variables λ and ν being

$$1 + \lambda + \nu = 0,$$

$$\lambda a^2 + \nu c^2 = 0.$$

From the last two equations we have

$$\nu (c^2 - a^2) = a^2, \quad \lambda (c^2 - a^2) = - c^2,$$

and therefore the generator is an ellipse defined by t equations

$$y = 0,$$

$$\frac{z^2}{a^2} + \frac{x^2}{c^2} = 1.$$

The intersections of this generator with the director (3) a the singular points of face.

It may easily be proved that this is the only one of the generators which passes through a singular point.

Suppose, in fact, that a generator, for which μ is not equal to zero, passes through such a point. Then, at the point in question, as we see from (1),

$$\frac{z^2}{\nu} = c^2 - b^2, \quad \frac{x^2}{\lambda} = a^2 - b^2,$$

and therefore

$$\lambda (a^2 - b^2) z^2 + \nu (b^2 - c^2) x^2 = 0.$$

But, at a singular point,

$$c^2 (a^2 - b^2) z^2 - a^2 (b^2 - c^2) x^2 = 0.$$

Hence we see that

$$\lambda a^2 + \nu c^2 = 0,$$

and therefore, by (6), $\mu = 0$, which is contrary to the hypothesis.

(C). Suppose the generator (1) to intersect a consecutive generator. Then, at the point of intersection,

$$\frac{x^2}{\lambda^2} d\lambda = \frac{y^2}{\mu^2} d\mu = \frac{z^2}{\nu^2} d\nu.$$

But, by (5) and (6),

$$d\lambda + d\mu + d\nu = 0,$$

$$a^2 d\lambda + b^2 d\mu + c^2 d\nu = 0:$$

hence

$$\frac{\lambda^2}{x^2} + \frac{\mu^2}{y^2} + \frac{\nu^2}{z^2} = 0,$$

$$\frac{\lambda^2 a^2}{x^2} + \frac{\mu^2 b^2}{y^2} + \frac{\nu^2 c^2}{z^2} = 0,$$

and therefore $\lambda = 0$, $\mu = 0$, $\nu = 0$, values of λ, μ, ν, which are incompatible with the equation (5). Thus we see that consecutive generators do not intersect, or that the generation of the surface is effected by a skew movement.

(D). The generator (1), moving in accordance with the conditions (5) and (6), is evidently a particular instance of a generator, defined by the same equations, which moves in accordance with two general conditions

$$\left.\begin{array}{l}\phi(\lambda, \mu, \nu) = 0 \\ \chi(\lambda, \mu, \nu) = 0\end{array}\right\} \ldots\ldots\ldots\ldots\ldots (8).$$

[redacted]

Thus the locus of (1), the parameters of which are conn
by the conditions (8), constitutes a family of surfaces of
the wave-surface is a particular individual.

(E). The equations (1), if x, y, z, be now taken to
in place of x^2, y^2, z^2, respectively, become

$$\frac{y}{\mu} - \frac{z}{\nu} = b^2 - c^2,$$

$$\frac{z}{\nu} - \frac{x}{\lambda} = c^2 - a^2,$$

$$\frac{x}{\lambda} - \frac{y}{\mu} = a^2 - b^2.$$

Differentiating these equations, we have

$$\frac{dx}{\lambda} = \frac{dy}{\mu} = \frac{dz}{\nu} \text{ ....}$$

Since the general equation to the family of surface
involve only squares of the coordinates, as is evident
the equations to the generator, we may represent it by

$$f(x, y, z) = 0 \text{..............(}$$

Let us put

$$\frac{df}{dx} = U, \quad \frac{df}{dy} = V, \quad \frac{df}{dz} = W,$$

$$\frac{d^2f}{dx^2} = u, \quad \frac{d^2f}{dy^2} = v, \quad \frac{d^2f}{dz^2} = w,$$

$$\frac{d^2f}{dy\,dz} = u', \quad \frac{d^2f}{dz\,dx} = v', \quad \frac{d^2f}{dx\,dy} = w',$$

$$\frac{d^3f}{dx^3} = a, \quad \frac{d^3f}{dy^3} = b, \quad \frac{d^3f}{dz^3} = c,$$

$$\frac{d^3f}{dy^2\,dz} = a', \quad \frac{d^3f}{dz^2\,dx} = b', \quad \frac{d^3f}{dx^2\,dy} = c',$$

$$\frac{d^3f}{dy\,dz^2} = a_1, \quad \frac{d^3f}{dz\,dx^2} = b_1, \quad \frac{d^3f}{dx\,dy^2} = c_1,$$

$$\frac{d^3f}{dx\,dy\,dz} = h.$$

Differentiating (10) we have

$$U dx + V dy + W dz = 0,$$

erefore, since dx, dy, dz, may be supposed to be the tions of an elementary arc of the generator on the nate axes, we get, by (9),

$$\lambda U + \mu V + \nu W = 0 \ldots\ldots\ldots\ldots (11).$$

entiating (11) twice on the same hypothesis, we shall the two following equations:

$$\lambda^2 u + \mu^2 v + \nu^2 w$$
$$+ 2\mu\nu u' + 2\nu\lambda v' + 2\lambda\mu w' = 0 \ldots\ldots\ldots (12),$$
$$\lambda^3 a + \mu^3 b + \nu^3 c$$
$$+ 3\mu^2\nu a' + 3\nu^2\lambda b' + 3\lambda^2\mu c'$$
$$+ 3\mu\nu^2 a_1 + 3\nu\lambda^2 b_1 + 3\lambda\mu^2 c_1$$
$$+ 6\lambda\mu\nu h = 0 \ldots\ldots\ldots\ldots\ldots (13).$$

limination of λ, μ, ν, between (11), (12), (13), will he partial differential equation to the family of the surface.

minating ν between (11) and (12), we shall get

$$\lambda^2 A_\nu + \mu^2 B_\nu + 2\lambda\mu C_\nu = 0 \ldots\ldots\ldots (14),$$

$$A_\nu = W^2 u + U^2 w - 2WUv',$$
$$B_\nu = W^2 v + V^2 w - 2WVu',$$
$$C_\nu = UVw - WUu' - WVv' + W^2 w'.$$

ating ν between (11) and (13), we shall get

$$\lambda^3 D_\nu + 3\lambda^2\mu D'_\nu + 3\lambda\mu^2 E'_\nu + \mu^3 E_\nu = 0 \ldots\ldots (15),$$

$$W^3 a - U^3 c + 3WU^2 b' - 3W^2 U b_1,$$
$$- U^2 Vc + W U^2 a_1 + 2UVWb' - VW^2 b_1 + 3W^2 c' - 2W^2 Uh,$$
$$- V^2 Uc + W V^2 b' + 2UVWa_1 - UW^2 a' + 3W^2 c_1 - 2W^2 Vh,$$
$$W^3 b - V^3 c + 3WV^2 a_1 - 3W^2 Va'.$$

lying (14) by λ, μ, successively, we have

$$\lambda^3 A_\nu + \lambda\mu^2 B_\nu + 2\lambda^2\mu C_\nu = 0 \ldots\ldots\ldots (16),$$
$$\lambda^2\mu A_\nu + \mu^3 B_\nu + 2\lambda\mu^2 C_\nu = 0 \ldots\ldots\ldots (17).$$

lying (15) by $A_\nu B_\nu$, and then eliminating λ^3, μ^3, by of (16) and (17), we shall get, dividing the resulting

equation by $\lambda\mu$,

$$\lambda(3A_\nu B_\nu D'_\nu - 2B_\nu C_\nu D_\nu - A^2{}_\nu E_\nu)$$
$$= -\mu(3A_\nu B_\nu E'_\nu - 2A_\nu C_\nu E_\nu - B^2{}_\nu D_\nu) \dots (18)$$

By symmetry we have also

$$\mu(3A_\lambda B_\lambda D'_\lambda - 2B_\lambda C_\lambda D_\lambda - A^2{}_\lambda E_\lambda)$$
$$= -\nu(3A_\lambda B_\lambda E'_\lambda - 2A_\lambda C_\lambda E_\lambda - B^2{}_\lambda D_\lambda) \dots (19)$$

and

$$\nu(3A_\mu B_\mu D'_\mu - 2B_\mu C_\mu D_\mu - A^2{}_\mu E_\mu)$$
$$= -\lambda(3A_\mu B_\mu E'_\mu - 2A_\mu C_\mu E_\mu - B^2{}_\mu D_\mu) \dots (20)$$

Multiplying together the equations (18), (19), (20), we get

$$\begin{aligned} 0 = &(3A_\lambda B_\lambda D'_\lambda - 2B_\lambda C_\lambda D_\lambda - A^2{}_\lambda E_\lambda) \\ &(3A_\mu B_\mu D'_\mu - 2B_\mu C_\mu D_\mu - A^2{}_\mu E_\mu) \\ &(3A_\nu B_\nu D'_\nu - 2B_\nu C_\nu D_\nu - A^2{}_\nu E_\nu) \\ + &(3A_\lambda B_\lambda E'_\lambda - 2A_\lambda C_\lambda E_\lambda - B^2{}_\lambda D_\lambda) \\ &(3A_\mu B_\mu E'_\mu - 2A_\mu C_\mu E_\mu - B^2{}_\mu D_\mu) \\ &(3A_\nu B_\nu E'_\nu - 2A_\nu C_\nu E_\nu - B^2{}_\nu D_\nu), \end{aligned}$$

which is the symmetrical partial differential equation to the family of the wave-surface.

It may be observed that

$$B_\lambda = A_\nu, \quad B_\mu = A_\lambda, \quad B_\nu = A_\mu,$$
$$E_\lambda = -D_\nu, \quad E_\mu = -D_\lambda, \quad E_\nu = -D_\mu,$$
$$WA_\lambda = VC_\lambda + UC_\mu = WB_\mu,$$
$$UA_\mu = WC_\mu + VC_\nu = UB_\nu,$$
$$VA_\nu = UC_\nu + WC_\lambda = VB_\lambda.$$

From the nature of the preceding investigation it is evident that the differential equation to the family of the wave-surface is coincident in form with the differential equation to skew surfaces generated by the motion of a straight line passing through three given directors, the partial differential coefficients in the case of the latter surfaces involving simple powers of x, y, z, as those in the case of the family of the wave-surface involve squares.

Cambridge, Dec. 23, 1851.

ANTANEOUS PLANES AND LINES IN A BODY REVOLVING ABOUT A FIXED POINT.

By William Walton.

ONCEIVE a rigid body to be in motion under the action y forces, one point of the body being fixed. We know the actual motion of the body at any instant may be tructed by impressing upon it simultaneously three mo-, each of which, taken singly, would correspond to an ular velocity, common to all its molecules, about one of e fixed coordinate axes passing through the fixed point. e three angular velocities are ordinarily represented by symbols ω_x, ω_y, ω_z, and may be called the *constructive* ular velocities of the body or of its molecules. Let the *al* angular velocities of any molecule m, of which the dinates are x, y, z, be denoted by Ω_x, Ω_y, Ω_z.

1.) Let us determine the relations subsisting between the tructive and the actual angular velocities of any molecule. t is proved in the ordinary treatises that

$$\frac{dx}{dt} = z\omega_y - y\omega_z, \quad \frac{dy}{dt} = x\omega_z - z\omega_x, \quad \frac{dz}{dt} = y\omega_x - x\omega_y;$$

ce
$$(y^2 + z^2)\,\Omega_x = y\frac{dz}{dt} - z\frac{dy}{dt}$$

$$= (y^2 + z^2)\,\omega_x - x(y\omega_y + z\omega_z),$$

therefore

$$\Omega_x = \omega_x - \frac{x}{y^2 + z^2}(y\omega_y + z\omega_z) \;\ldots\ldots\ldots\; (1).$$

similar reasoning, we have also

$$\Omega_y = \omega_y - \frac{y}{z^2 + x^2}(z\omega_z + x\omega_x) \ldots\ldots\ldots(2),$$

$$\Omega_z = \omega_z - \frac{z}{x^2 + y^2}(x\omega_x + y\omega_y) \ldots\ldots\ldots(3).$$

the actual angular velocities of any proposed molecule xpressed in terms of the constructive angular velocities the coordinates of position. It is obvious from these ulæ that the constructive and actual angular velocities molecule of the body are generally different.

.) If we put $x = 0$, then, from (1), we see arly, if $y = 0$, $\Omega_y = \omega_y$, and, if $z = 0$, $\Omega_z =$

appears that the actual angular velocities of all molecules in the fixed coordinate planes, about the axes respectively normal to them, coincide with the constructive angular velocities.

We proceed now to prove the two following theorems:

(1) That, if instantaneous planes be drawn through the axes of x, y, z, perpendicular to the projections of the instantaneous axis upon the planes of yz, zx, xy, respectively, the actual and constructive angular velocities of all the molecules in these instantaneous planes about the axes of x, y, z, respectively, will be equal; and

(2) That there exist three instantaneous lines, equally inclined to the instantaneous axis, such that all molecules in them have equal constructive and actual angular velocities about the axes of y and z, z and x, x and y, respectively.

(III.) If we assume that

$$y\omega_y + z\omega_z = 0,$$

then, from (1), we see that $\Omega_x = \omega_x$.

But the equations to the instantaneous axis are

$$\frac{x}{\omega_x} = \frac{y}{\omega_y} = \frac{z}{\omega_z}.$$

Hence it appears that, if we draw a plane through the axis of x at right angles to the projection of the instantaneous axis on the plane of yz, the actual and constructive angular velocities of all molecules in this plane about the axis of x are coincident. Analogous remarks are applicable to the axes of y and z.

Let the three instantaneous planes thus determined be called the planes X, Y, Z.

(IV.) It is evident, from what has been said above, that all molecules lying in the line of intersection of the planes Y and Z will have equal actual and constructive angular velocities about the axes both of y and of z. The equations to this instantaneous line are evidently

$$\left.\begin{aligned} x\omega_x + y\omega_y &= 0 \\ x\omega_x + z\omega_z &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (4).$$

The equations to the other two analogous instantaneous lines are evidently

$$\left.\begin{aligned} y\omega_y + z\omega_z &= 0 \\ y\omega_y + x\omega_x &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (5),$$

and

$$\left.\begin{aligned} z\omega_z + x\omega_x &= 0 \\ z\omega_z + y\omega_y &= 0 \end{aligned}\right\} \quad\dots\dots\dots\dots\dots\quad (6).$$

Let these three instantaneous lines be represented by X', Y', Z'.

(V.) Let λ, μ, ν, be the inclinations of X', Y', Z', respectively, to the instantaneous axis. Then it is easily seen that, ω denoting the angular velocity of the body about the instantaneous axis,

$$\omega^2 \cos^2\lambda = \omega^2 \cos^2\mu = \omega^2 \cos^2\nu = \frac{1}{\dfrac{1}{\omega_x^2} + \dfrac{1}{\omega_y^2} + \dfrac{1}{\omega_z^2}},$$

and that accordingly the three instantaneous lines are equally inclined to the instantaneous axis.

(VI.) Let α, β, γ, be the angles between (Y', Z'), (Z', X'), (X', Y'), respectively. Then

$$\cos\alpha = \frac{\dfrac{1}{\omega_y^2} + \dfrac{1}{\omega_z^2} - \dfrac{1}{\omega_x^2}}{\dfrac{1}{\omega_x^2} + \dfrac{1}{\omega_y^2} + \dfrac{1}{\omega_z^2}},$$

$$\cos\beta = \frac{\dfrac{1}{\omega_z^2} + \dfrac{1}{\omega_x^2} - \dfrac{1}{\omega_y^2}}{\dfrac{1}{\omega_x^2} + \dfrac{1}{\omega_y^2} + \dfrac{1}{\omega_z^2}},$$

$$\cos\gamma = \frac{\dfrac{1}{\omega_x^2} + \dfrac{1}{\omega_y^2} - \dfrac{1}{\omega_z^2}}{\dfrac{1}{\omega_x^2} + \dfrac{1}{\omega_y^2} + \dfrac{1}{\omega_z^2}}.$$

These expressions shew that no two of the instantaneous lines can ever degenerate into one.

(VII.) It may readily be ascertained from the equations (1), (2), (3), that there is no molecule in the body of which the constructive and actual angular velocities coincide relatively to the three axes taken simultaneously.

Cambridge, Jan. 23, 1852.

ON THE METHOD OF VANISHING GROUPS.

By James Cockle.

[*Continued from* Vol. vi. p. 178.]

XVIII. Let $\Sigma_m . H^n + A\xi + B$ denote that state of a function which next precedes its final reduction. Then if A be of either of the forms

$$h_1^{n-1} + h_2^{n-1}, \text{ or } h_1^{n-1} + h_2^{n-1} + h_3^{n-1} + h_4^{n-1},$$

to say nothing of others, the requirements of the Method of Vanishing Groups are satisfied. The latter form is the one which we have hitherto supposed to hold, and in the case of $n = 3$ there is no essential difference between them; but for biquadratic and higher functions the former must be exclusively adopted. We shall thereby most materially reduce the number of indeterminate quantities.

XIX. Thus, in III., we may take the reduced value

$$(u_1) = 1 + 2(3.2^2 - 1) = 3.2^3 - 1 = 23,$$

and the other values of u will be almost inconceivably diminished.* The same reduction may be applied to w in IV., and, in short, to all the higher functions.

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* We shall have, for instance,

$$(u_2) = 3.2^{17} - 1,$$

in place of the value given in XIII. When considered in reference to its application to the Theory of Equations, the essentially *indeterminate* character of the method must be borne in mind. The more recent improvements in that theory have all partaken of this character; and, perhaps, where transformation only is in question it is useless to go beyond it. But, whenever we are considering the complete solution of a system of equations, it may be doubtful whether some general *determinate* method, some extension of that of Lagrange or Vandermonde for example, might not be successfully applied to illustrate and explain the principles of the solution. I am inclined to think that a general determinate method of my own (which I term the Method of Symmetric Products, and respecting which some information will be found at pp. 226—229 and 486, 487, of vol. LII. of the *Mechanics' Magazine*, and at the places there cited) possesses advantages in this respect. I have recently completed an investigation of its application to biquadratics. The results will be seen in the following detailed solution of a biquadratic by

The Method of Symmetric Products.

Let y_1, y_2, y_3, and y_4 be the roots of

$$y^4 + Ay^3 + By^2 + Cy + D = 0.$$

It is required to find three unequal values of the expression

$$Y_r = y_1 + \alpha_r y_2 + \beta_r y_3 + \gamma_r y_4,$$

whose product shall be a symmetric function of $y_1 \ldots y_4$.

XX. Many of our foregoing results may be arrived at *synthetically.* If we denote by n_s the number of indeter-

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Let Y_1, Y_2, and Y_3 be the values. Then the terms of the form y_1^3 give the following conditions of symmetry:

$$1 = a_1 a_2 a_3 = \beta_1\beta_2\beta_3 = \gamma_1\gamma_2\gamma_3;$$

those of the form $y_1^2 y_2$ give

$$\begin{aligned}\Sigma.a &= \Sigma.\beta = \Sigma.\gamma = \Sigma.a_1 a_2 = \Sigma.\beta_1\beta_2 = \Sigma.\gamma_1\gamma_2\\ &= \Sigma.a_1 a_2 \beta_3 = \Sigma.a_1 a_2 \gamma_3 = \Sigma.a_1\beta_2\beta_3\\ &= \Sigma.a_1\gamma_2\gamma_3 = \Sigma.\beta_1\beta_2\gamma_3 = \Sigma.\beta_1\gamma_2\gamma_3;\end{aligned}$$

and those of the form $y_1 y_2 y_3$ give

$$\begin{aligned}\Sigma.a_1\beta_2 + a_2\beta_1 &= \Sigma.a_1\gamma_2 + a_2\gamma_1\\ = \Sigma.\beta_1\gamma_2 + \beta_2\gamma_1 &= \Sigma.(a_1\beta_2 + a_2\beta_1).\gamma_3.\end{aligned}$$

Let E be the coefficient of $y_1^2 y_2$. Then the above relations shew that (a_1, a_2, and a_3), (β_1, β_2, and β_3) and (γ_1, γ_2, and γ_3) are systems of roots of the equation

$$z^3 - Ez^2 + Ez - 1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a).$$

From the form of the roots of (a) we are at liberty to assume

$$a_1 = m^{-1}, \;\; a_2 = m, \;\text{ and }\; a_3 = 1.$$

Substitute these values in the equations

$$\Sigma.\beta - \Sigma.a_1 a_2 \beta_3 = 0 = \Sigma.\gamma - \Sigma.a_1 a_2 \gamma_3,$$

and we obtain

$$\beta_2 = m\beta_1, \quad \gamma_2 = m\gamma_1,$$

and hence, by means of the first conditions of symmetry,

$$\beta_3 = m^{-1}\beta_1^{-2}, \quad \gamma_3 = m^{-1}\gamma_1^{-2}.$$

By substitution in the expressions

$$\Sigma.\beta_1\beta_2\gamma_3 \text{ and } \Sigma.\beta_1\gamma_2\gamma_3,$$

we now obtain

$$\left(\frac{\beta_1}{\gamma_1}\right)^2 + 2\frac{\gamma_1}{\beta_1} = \left(\frac{\gamma_1}{\beta_1}\right)^2 + 2\frac{\beta_1}{\gamma_1} = E,$$

and consequently

$$\beta_1\gamma_1^{-1} = \pm 1, \text{ and } E = 3 \text{ or } -1, \text{ and } \beta_2 = \gamma_2.$$

The first value of E is useless, since it would render the values of Y equal. Hence, adopting the last, we see that (a) is equivalent to

$$(z+1)^2(z-1) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (b);$$

and that

$$m = -1 = a_2 = a_1 = \beta_3 = \gamma_3;$$

and rejecting, as we must do, the positive sign in the expression for $\beta_1\gamma^{-1}$, we have, finally,

$$\beta_1 = -\gamma_1 = -\beta_2 = \gamma_2 = \pm 1 \text{ indifferently}.$$

It will perhaps be better to make $\beta_1 = 1$. All the conditions of symmetry are now satisfied, and the symmetric product is

$$Y_1 Y_2 Y_3 = -8C + 4AB - A^3 = P_3.$$

Let x be the root of any biquadratic; and, assuming

$$y = \lambda x + x^2,$$

by the known processes, the biquadratic in x into another in y, λ is determined (by a cubic, the use and meaning of the several

minates necessary for the reduction of a function to the form

$$h_1^n + h_2^n + \ldots + h_x^n,$$

we have (besides the anomalous result $1_x = y$, where y is indeterminate) the following:

$$2_{x+1} = 2_x + 1,\ 2_x = x;\quad 3_{x+1} = 2.3_x + 1,\ 3_x = 3.2^x - 1;$$

$$4_{x+1} = 3.2^{1+2+4_x} - 1;\quad 5_{x+1} = 3.2^{1+2+4_x} - 1;\ \&c.$$

The above two integrals are corrected. In forming the remaining equations we must employ the 'reduced' values of 4_1, 5_1, &c. By actual development we see that

$$3_x = 1 + 2\{1 + 2[1 + 2(1 + \ldots 2(1 + 2^2)\ldots)]\},$$

where there are x brackets; and, consequently,

$$3_x = 1 + 2 + 2^2 + \ldots + 2^{x-1} + 2^{x+1} = 3.2^x - 1,$$

which confirms our preceding result.

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roots of which I shall not now stop to discuss), so as to render $P_3 = 0$. Let a be any constant quantity, then, from the structure of P_3, we see that that product, and each of its factors, remains unchanged in value when $y + a$ or y' is written in place of y. Let $4a = -A$, and denote by an accent the changes corresponding to the change of y into y; then we have

$$P_3' = 0,\ A' = 0,\ \text{and } \textit{consequently } C' = 0.$$

Hence if, by means of $Y_1 = 0$ and $A' = 0$, we eliminate y_1 and y_4 from $C' = 0$, we have a result which admits of a (cubic) solution of the form $y_3 = ry_4$; and y_4 will be determined by substitution in

$$\Sigma.y_1'y_2' = B',$$

and by means of a quadratic. It is of course immaterial in what order the roots are eliminated; but y and y and z are completely determined—the latter by known processes.

The function P_3 is a critical function; it possesses this property, that if y and x be connected by the relation

$$y = a + f(x),$$

where f is integral and rational, then P_3 is free from a whatever be the value of a or the form of f. The corresponding symmetric product for cubics (viz. $P_2 = A^2 - 3B$) is also critical. These are facts which must not escape us when we come to consider the further application of the Method of Symmetric Products. Assuming the impossibility of solving equations of the fifth and higher degrees, a theory of *conjugate* equations extending to all degrees would seem to be fairly within the grasp of Algebra.

If, in the above investigations, we take the values of a as given above, we find that

$$\Sigma.a_1\beta_2 + a_2\beta_1 = \Sigma.a_1\gamma_2 + \gamma_1 a_2,\ \text{gives } \beta_2 = \gamma_2;$$

$$\Sigma.\beta_1\gamma_2 + \beta_2\gamma_1 = \Sigma.(a_1\beta_2 + a_2\beta_1)\gamma_3,\ \text{gives } \gamma_3 = -1;$$

and $$\Sigma.\beta_1\gamma_2\gamma_3 = -1,\ \text{gives } \beta_1\gamma_2 + \beta_2\gamma_1 = -2.$$

This confirms the above results. The last relation shews how the values of β and γ are connected.

[redacted]

XXI. In its application to the solution of a simultaneous uadratic and n^{ic} equation, the process of the method is dentical in principle with that indicated by Mr. Sylvester in a note at a previous portion of this *Journal* (vol. VI. pp. 14–15); and it gives that which is substantially equivalent to what Mr. Sylvester terms a 'linear solution' of the quadratic.

It must never be forgotten that we may always introduce as many indeterminates as we please into any equation or system of equations. When we have only one equation to deal with, the roots of the transformed equation are in general linear functions of the indeterminates, but non-linear with respect to the roots of the original equation. The theory of systems of *determinate* equations is yet in its infancy, but it is certain that there, as in the theory of single equations, the limitation of the number of solutions must extend implicitly to the transformed equations in their indeterminate form, and, by causing the occurrence of vanishing fractions, give rise to illusòry results. I shall not, however, at the present moment pursue the subject further than to suggest it as a topic for discussion.

XXII. Let U and U' be homogeneous quadratic functions. Then, in general,

$$U = \gamma(U) = h_1^2 + h_2^2 + u,$$
$$U' = \gamma(U') = h_1'^2 + h_2'^2 + u',$$

and, if the coefficient of ξ^2 in U and U' be unity, we have

$$h'_1 = h_1 + bh_2 + c, \text{ and } h'_2 = dh_2 + e;$$

consequently, if we form the expression

$$(\lambda - 1)\, U + U' = \lambda \mathrm{V},$$

we have a result that may be represented by

$$\lambda h_1^2 + 2bh_1h_2 + (\lambda - 1 + a)\, h_2^2 + 2ch_1 + 2fh_2 + l.$$

XXIII. Let λ_1 and λ_2 be the roots of the equation

$$\lambda(\lambda - 1 - a) = b^2,$$

and let $\quad b\lambda_1^{-1} = a_1 + a_2$, and $b\lambda_2^{-1} = a_1 - a_2$;

then we have two values of V included in an equation of the form

$$\mathrm{V} = \{h_1 + (a_1 \pm a_2)\, h_2\}^2 + 2\mathrm{C}h_1 + 2\mathrm{F}h_2 + p.$$

XXIV. Suppose that

$$h_1 + a_1h_2 = v, \quad a_2h_2 = w, \quad 2\,(\mathrm{F} - \mathrm{C}a_1) = 2\mathrm{F}'a_2,$$
$$\mathrm{C} + \mathrm{F}' = 2\lambda^{-1}m, \text{ and } \mathrm{C} - \mathrm{F}' = 2\lambda^{-1}n,$$

then we obtain

$$V = (v \pm w)^2 + 2\lambda^{-1}m(v+w) + 2\lambda^{-1}n(v-w) + p.$$

XXV. We now see that the solution of the system $U = 0 = U'$ is reduced to that of

$$(v + w + \lambda_1^{-1}m)^2 + 2\lambda_1^{-1}n(v-w) + p - \lambda_1^{-2}m^2 = 0,$$

$$(v - w + \lambda_2^{-1}n)^2 + 2\lambda_2^{-1}m(v+w) + q - \lambda_2^{-2}n^2 = 0.$$

In the particular case where U and U' involve only two unknowns this reduction enables us to effect the solution without having to solve an equation higher than a *cubic*. Its bearing upon the general case remains for consideration.

[*To be continued.*]

Erratum.—Vol. vi. p. 178, line 4, *for* k *read* h.

2, *Pump Court, Temple,*
Sept. 30, 1851.

ON SOME GEOMETRICAL CONSTRUCTIONS.

By H. J. S. Smith, Fellow of Balliol College, Oxford.

If a geometrical curve be completely traced on a sheet of paper, the principles of the Theory of Transversals enable us to assign its tangent line and radius of curvature at any point, without supposing its equation known, and without employing any operations excluded from the sphere of elementary geometry.

The construction given by M. Chasles for this purpose is the following. Let m be the point on the curve, M any point assumed in the plane, mp, mq, two transversals, and MP, MQ, two parallels to them. Let also p, q, P, Q, denote the continued products of the segments on mp, mq, MP, MQ, respectively, excepting the evanescent segments on mp, mq. Then if we take on mp, mq, two lines respectively proportional to $\frac{P}{p}, \frac{Q}{q}$, the line joining their extremities shall be parallel to the tangent at m. To find the osculating circle, let t denote the continued product of the segments on the tangent at m, excepting the two evanescent segments, T the continued product of the segments on MT drawn parallel to mt; then, if, on any transversal

ıp, we take mc equal to $\frac{T}{t} \cdot \frac{p}{P}$, the point c shall lie on the ısculating circle. For the diameter of this circle we have he expression $\frac{T}{t} \cdot \frac{n}{N}$; n, N denoting products of segments ɔn the normal and on a parallel to it.

If the point be a double point, the preceding constructions fail; but by slightly modifying them, we may determine the two tangents and two radii of curvature, if the point be nodal: or, if it be conjugate, we can assign the elements of an ellipse, whose imaginary asymptotes shall be the imaginary tangents in question; that is to say, an ellipse concentric, similar, and similarly placed, with the evanescent conic formed by the conjugate point. In this case the two radii of curvature are in general imaginary and therefore cannot be constructed: but, since they are conjugate imaginary magnitudes, any rational symmetrical functions of the two (for example, the rectangle under them, or their harmonic or arithmetic mean), may readily be determined. The process to be employed is as follows. Take three transversals mp, mq, mr, and three parallels to them MP, MQ, MR, and let MR cut mp, mq in A, B, and the two tangents in θ_1, θ_2. We shall have

$$A\theta_1 . A\theta_2 = (mA)^2 . \frac{R}{r} \cdot \frac{p}{P}, \quad B\theta_1 . B\theta_2 = (mB)^2 . \frac{R}{r} \cdot \frac{q}{Q}.$$

Now if the point m be conjugate, the products $A\theta_1$, $A\theta_2$, $B\theta_1$, $B\theta_2$, are essentially positive; and if it be nodal we can ensure their being so, by taking the three transversals mp, mq, mr, in one and the same pair of vertically opposite angles. Hence, if we put

$$(mA)^2 \frac{R}{r} \cdot \frac{p}{P} = a^2, \quad (mB)^2 \frac{R}{r} \cdot \frac{q}{Q} = b^2,$$

he lines a and b can always be constructed. Therefore to letermine θ_1, θ_2, describe two circles round A, and B, with adii a and b, respectively. The radical axis of these circles vill intersect MR at o the middle point of θ_1, θ_2; and any ircle of the system orthogonal to the two circles (A) and B), (*i.e.* any circle having its centre on the radical axis nd its radius equal to the tangential distance of its centre rom either of those two circles), will intersect MR in θ_1, θ_2. f (A) and (B) intersect in real points, their radical axis s instantly found; but in this case θ_1 and θ_2 are always

imaginary. Let s_1, s_2 be the points in which the ra[illegible] axis is cut by any one of the orthogonal circles; on m[illegible] mo' a mean proportional between os_1 and os_2 (*i.e.* equ[illegible] the tangential distance of o from any one of the orthog[illegible] circles): the ellipse having its centre at m, and mo, m[illegible] semi-conjugate diameters, will have the two imaginary tang[illegible] for its asymptotes. If (A) and (B) touch, the points [illegible] coincide in o, and the double point becomes a cusp, hav[illegible] mo for its tangent. Lastly, if (A) and (B) intersect [illegible] imaginary points, the radical axis, though not immedi[illegible] given, can always be determined by the ruler alone, [illegible] in this case, $\theta_1\theta_2$ being always real, the tangents $m\theta_1$, [illegible] can be directly constructed.

The direction of the tangents once ascertained, the ra[illegible] of curvature may be immediately found. In fact, if w[illegible] denote by c_1, c_2 the chords intercepted on mp by the tw[illegible] circles, and by $\theta_1\theta_2$ the two points in which the tangent[illegible] are cut by MP parallel to mp, we have

$$c_1 = \frac{\theta_1\theta_2}{m\theta_1} \cdot \frac{T_1}{t_1} \cdot \frac{p}{P}, \quad c_2 = \frac{\theta_1\theta_2}{m\theta_2} \cdot \frac{T_2}{t_2} \cdot \frac{p}{P};$$

and by making mp coincide successively with the two normals, we get the values of the two diameters of curvature. Or we may first determine one, and then obtain the second by the proportion, which is easily demonstrated,

$$R_1 : R_2 :: \frac{T_1}{t_1} : \frac{T_2}{t_2}.$$

If the point be triple the determination of the directions of its tangents, which in analysis depends on the solution of a cubic equation, is not in general possible by the intersections of right lines and circles. The problem in its simplest form is this: Given three points on a right line ABC, and given the products $A\theta_1 . A\theta_2 . A\theta_3$, $B\theta_1 . B\theta_2 . B\theta_3$, $C\theta_1 . C\theta_2 . C\theta_3$, find $\theta_1\theta_2\theta_3$. But whatever the order of the point, if the direction of its tangents be once known, the construction of its radii of curvature is very easy. If, for example, the order of the point be r, the chord determined on mp, by the circle tangent to $m\theta_1$, is readily seen to be given by the equation

$$c_1 = \frac{T_1}{t_1} \frac{p}{P} \frac{\theta_1\theta_2 . \theta_1\theta_3 \ldots \theta_1\theta_r}{(m\theta_1)^{r-1}};$$

which chord is therefore imaginary for an imaginary tangent, as it ought to be.

Returning to the case of double points, we see from the formula

$$c_1 = \frac{\theta_1\theta_2}{m\theta_1} \cdot \frac{T_1}{t_1} \frac{p}{P},$$

that if the two tangents coincide, the osculating circles become simultaneously evanescent, except a fourth segment of the tangent become evanescent also, that is, except the tangent cut the curve in four coincident points at m. In this case the point m is not cuspidal, but is a point of osculation, and possesses two radii of curvature, for which we proceed to give a graphical construction. If D_1D_2 be the two diameters, we find readily enough $D_1D_2 = \frac{T}{t} \cdot \frac{n}{N}$; but the theorem of Newton's, which has hitherto guided us, is perhaps insufficient immediately to furnish a second relation. Such a relation, however, may be obtained by the following considerations. It is well known that the polar conic of a point of inflexion breaks up into two lines: one of these is the tangent at the point of inflexion, the other will be found to be the locus of the harmonic centres of the $n-1$ points in which the curve is cut by a transversal through the point. Exactly in the same way the polar curve of the third order at a point m of the nature here considered, resolves itself into the tangent line and into a conic section; this conic touches the tangent at m, and is the locus of harmonic centres of the $n-2$ points in which the curve is cut by a transversal through m; consequently its curvature at m, multiplied by $n-2$, is precisely the sum of the curvatures sought. Now the diameter of curvature in a conic is to the chord intercepted on the normal, as the rectangle under the segments of a parallel to the tangent is to the rectangle under the segments of the normal chord. Hence, if mp_1 be any radius vector of the conic, p_1p_2 a chord parallel to the tangent at m, the radius of curvature is known as soon as the point p_2 has been constructed. To effect this, take any circle tangent to the conic at m; this circle and the conic being homological, their axis of homology may be first found, and then the line homologous to p_1p_2; this will give the point homologous to p_2, and therefore p_2 itself; in fact, the circle once described, p_2 may be found by the ruler alone. We now know the rectangle under the two radii of curvature, and the harmonic mean between them: the radii may therefore themselves be found by a simple and well-known construction.

It may be observed that the theory of polar curves leads to a construction for the tangent of a curve line, which is different from M. Chasles', and in fact linear. Through the given point P draw four transversals; each of these will cut the curve in $n-1$ points. Take the harmonic centre of each of these four groups with respect to P, and consider the four points thus obtained as determining a conic section passing through P. Pascal's theorem will then determine the tangent to this conic at P; that is, the tangent required. It is unnecessary to give the reciprocal construction, which enables us, when a curve of the n^{th} class is given tangentially, to determine with the ruler alone the point of contact on any one of its tangents, supposed not to be a double tangent. It should however be added that a method for the linear solution of these two problems has been long since given in a different and less explicit form by M. Poncelet, in his excellent memoirs on the Analysis of Transversals.

The radius of curvature of any point is, of course, by its nature, incapable of linear construction; but if we imagine ourselves to have constructed the normal at any point, and to have determined on it the centre of curvature of the given curve or of any one of its superior or inferior polar curves; and if, in addition, a line parallel to the normal be given, in order that the point at infinity on the normal may be known; we can find linearly the centre of curvature of every single curve of the polar system continued as far upwards as we please. This is a consequence of the following theorem: The distances of the centres of curvature of the successive polar curves from their common tangent form a harmonic progression, commencing at infinity and having the given point for its point of evanescence.

All the preceding methods admit of an easy application to the theory of surfaces. For example, to determine the tangent plane at m, we must draw three transversals mp, mq, mr, not in one plane, and then proceed as in the case of plane curves. If the surface be of the n^{th} order, its tangent plane will determine on it a curve of the same order; the point m being a double point in that curve nodal, cuspidal, or conjugate, according as the contact is hyperbolic, parabolic, or elliptic. Taking the last case, we must determine two conjugate semidiameters of an ellipse concentric, similar, and similarly placed with the evanescent conic in the tangent plane, and therefore with the indicatrix of the point m: the directions and magnitudes of the semi-

of this ellipse may now be deduced by a construction xtreme simplicity (vide. Note xxv. on M. Chasles' *His- of Geometry*), and therefore the ratio of the principal atures, and the traces of the principal normal sections he tangent plane are known. If now we construct the us of curvature in either of these normal sections, the re of one semi-axis of the indicatrix is found, and there- that curve may be regarded as completely determined. ence appears, that to find the tangent plane and the catrix of any point, it is requisite to draw sixteen trans- als; not that so many are absolutely essential, but the ble is rather increased than lessened by taking fewer.

rom their connexion with the present subject the follow- geometrical demonstrations of Meunier's and Euler's rems on curvature may find a place here. If we take oint P on a curve line, and if we consider an evanescent rd drawn parallel to the tangent at P as an infinitesimal he first order, this chord will be bisected by the normal P; that is to say, it will be divided into two segments ose difference will be infinitesimal of the second order. reover, if we take any sagitta perpendicular to the chord, intersecting it in a point distant only by an infinitesimal he second order from its centre, it is readily seen that square of either segment of the chord, divided by the itta, may be taken to represent the diameter of curva- e of the evanescent arc. Hence, if we take two plane tions of a surface intersecting in an evanescent chord, radii of curvature of the evanescent arcs are inversely any two sagittæ perpendicular to the chord, and bisect- it approximately. If, now, one of the sections be a rmal one, we may take for the sagitta in that section intercept on the normal to the surface. Consequently triangle formed by joining the extremities of the two ittæ will be right-angled; and therefore the radius of oblique section is equal to the orthogonal projection the normal radius upon the plane of the oblique section, ich is Meunier's theorem.

It follows also from what has been said, that if we draw lane parallel to a tangent plane, and distant from it by infinitesimal of the second order, the curve surface will, general, determine upon this plane an evanescent hyper- ic or elliptic oval; and that the chords of the oval, being mselves infinitesimals of the first order, will be bisected hin an infinitesimal of the second order at the point at ich the normal meets the plane, and which we will call C.

We may therefore consider the oval as a central c... having its centre at C: it only remains to shew that i... a conic section. This may be done as follows: Every t... versal passing through C and lying in the plane of the ... will cut the surface in two points belonging to the ... and in $n-2$ points whose distance from C is infinitely ... compared with that of the two first points. Now, if ... a moment we consider the diameters of the oval ... finite, the remaining $n-2$ points will lie at infinity, ... therefore an infinitely magnified representation of the se... we are considering would consist of a finite central c... replacing the oval, and of the line at infinity $n-2$ ti... repeated, replacing the $n-2$ branches which lie at a ... distance from C. Since, then, the radii of curvature ... the normal sections vary as the squares of the diam... of the evanescent oval, they vary as the squares of ... central radii vectores of a conic section.

If there be a double line upon the surface we can co... struct the two tangent planes at any point m by taking ... two plane sections passing through m and constructing the ... tangents of the double points. Each of these tangent plane... will cut the surface in a curve having a triple point at m; but as the direction of one of the three tangents is known *a priori*, being the intersection of the tangent planes, the directions of the remaining two may be found by the construction used for double points; consequently the directions of the tangents to the principal sections on each sheet of the surface are known, and the principal radii of curvature may be determined by the construction before given for finding either radius of curvature at a double point. The two indicatrices at the point m may therefore be considered as ascertained in magnitude and position.

If the osculating plane and radius of curvature of the double line itself be required, they may be obtained very simply by a method to be given below.

If the singular line be of the r^{th} order ($r > 2$), a little consideration will shew that though we cannot determine the tangent planes by any elementary construction, yet, if we assume these planes as known, the indicatrices upon each sheet may be found as easily as if the point were not singular. This is the more remarkable, since the expressions given by analysis for the principal radii of curvature at such points appear to be of great complexity.

Let us now take a point m on a curve surface where two sheets of the surface meet and have a common tan-

...t plane. This tangent plane will intersect the surface in ...urve having a quadruple point at m; but the directions ...the four tangents may always be ascertained by a quad-...c construction. For at such a point the polar surface ...the third order will resolve itself into the tangent plane ... into a surface of the second order. And it may be ...wn that the two generatrices (real or imaginary) of that ...face which lie in the tangent plane are in involution ...h the two pair of asymptotes of the two indicatrices; ...t is, with the four tangents before mentioned. Now ...se two generatrices may be determined by means of ... theory of homological figures, since that theory enables ... to assign a pair of semi-conjugate diameters of a section ... the surface of the second order parallel to the tangent ...ane at m, whence the directions of the asymptotes of that ...ection become known, and therefore the two generatrices ...equired. The problem now will be: Given four points in ... line $PQRS$, and the four products $P\theta_1.P\theta_2.P\theta_3.P\theta_4$, &c., ...etermine $\theta_1\theta_2\theta_3\theta_4$, a pair of points G_1G_2 being also given which form an involution with the two pairs $\theta_1\theta_2$ and $\theta_3\theta_4$. It is plain that, A being any point whatever, any symmetrical function of the distances $A\theta_1$, $A\theta_2$, $A\theta_3$, $A\theta_4$, may be constructed. Hence H_1H_2, the harmonic centres of the four points $\theta_1\theta_2\theta_3\theta_4$ with respect to G_1G_2, are known. But H_1H_2 form a pair of points in involution with the two pair sought: and therefore H_1H_2 together with G_1G_2 completely determine the involution. Therefore the centre and foci of the system are known, and consequently $\theta_1\theta_2$ and $\theta_3\theta_4$ may be now quadratically determined. Points of the nature here considered may exist isolated on a curve surface; but if there be a continuous series of them, we shall have a line along which two sheets of the surface envelope one another (not a cuspidal line, for any transversal plane will determine a section having not cusps, but points of osculation at its intersections with the singular line), and at any point on such a line the two generatrices before mentioned will be found to coincide: and consequently the surface of the second order will degenerate into a cone. The side of this cone, existing in the tangent plane at m, may be determined by proceeding as in the general case: and then, instead of a pair of points in involution with $\theta_1\theta_2\theta_3\theta_4$, we ...all have one focus of that involution given. The second ...may next be constructed (being the harmonic centre ...θ_4 with respect to the given focus), and the problem ...quadratic as before.

Lastly, let there be a point on a curve surface having a tangent cone of the second order. It will be possible to determine three conjugate diameters of that cone. If we take any two planes passing through the vertex, and having constructed the tangent lines of the double points in these planes, take the harmonic conjugates of the line of intersection of the two planes with respect to each pair of tangents. This will give the plane conjugate to the line of intersection; and by taking any two lines harmonically conjugate with respect to the two tangent lines existing in that plane, we shall obtain the directions of the three semidiameters required. Likewise, their ratios, or rather the ratios of their squares, may be found, since the two sides of the cone in each conjugate plane may be constructed. Hence, we may deduce the directions and the ratios of the squares of the principal semiaxes of the cone. But this determination, involving the solution of a cubic equation, requires the construction of a conic, and is consequently not within the limits of elementary geometry. (Vide the Note on M. Chasles' *History*, already quoted, and a paper by Mr. Townsend in this *Journal*.)

If a curve of double curvature be given in space the principles of the theory of transversals are not immediately applicable: but if we regard it as the intersection of two geometrical surfaces completely given, we may immediately find its tangent, osculating plane, and radius of curvature. The tangent at m is of course determined by the intersection of the two tangent planes; and if we take the two normal sections containing that tangent, and, having constructed their radii of curvature, let fall a perpendicular from m on the line joining the two centres, this perpendicular will represent in magnitude and direction the radius of curvature of the given curve. This (it will be seen) follows at once from Meunier's theorem, or from that known as Hachette's.

LAPLACE'S EQUATION AND ITS ANALOGUES.

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It has been long recognized by mathematicians that the arbitrary portions of the solutions of partial differential equations can, in general, be expressed under three distinct forms. *The first* gives the arbitrary portion of the general solution

as the sum of an infinite number of particular expressions, and, though open to objection, is recommended by the circumstances that it is unaffected by any signs of integration, and wholly free from arbitrary functions. *The second*, which is due to Laplace, expresses the arbitrary portion by means of Definite Integrals, under the signs of which arbitrary functions occur; and while it is, in the existing state of science, generally unattainable, yet when once arrived at, it has the advantage of enabling us to determine the arbitrary functions with considerable facility. *The third and most common* form is that which exhibits the same arbitrary portion in terms of arbitrary functions without any signs of integration; and, in contrast with the last, while in most cases it *can* be obtained by the aid of the Calculus of Operations, it is objectionable from the difficulty of determining the arbitrary functions. It may be observed that this last form is unique, whereas in each of the two preceding cases we have obviously as many general solutions as we can obtain particular solutions of distinct types. The order in which these forms have been stated seems to be that of their chronological employment, although the reverse of this order is that of their logical filiation.

It is proposed in the following paper to exhibit general solutions of the class of partial differential equations to which the Laplacian equation belongs, under these three several forms. It is obvious that, as this class of equations contains no term in its right-hand member involving only the independent variables, the general solutions will contain no determinate expression, but will reduce themselves to the arbitrary portions solely. It will be found that the particular solutions employed in the first and second forms are duplicate, each indicating a second in close correlation with itself. In the case of the équation of the simplest type, the duplicate solutions are omitted as being evident.

The instruments employed in arriving at these solutions are known by the names of Calculus of Duplets, Triplets, Quaternions, &c., which have of late occupied so considerable a share of the attention of the mathematical world. Frequent reference is made to a paper on Quaternions by Sir William Rowan Hamilton, published in the second volume of the *Proceedings of the Royal Irish Academy*, and the laws of the system of Triplets here employed are analogous to those which govern Quaternions. The writer is not aware that any use has as yet been made of these instruments in connexion with the subject of the integration

of partial differential equations, and believes the forms of the solutions themselves, as found by them, to be new. It will be seen that he has commenced with the equation of the simplest character, the solution of which is familiar to the reader, and by successively engaging the equations as they rise in order, has endeavoured to shew the identity of the method applied to all.

To the end of the paper are appended some additional examples of the applicability of imaginary symbols of operation to integration, and it will be evident from them that such symbols can be used to great advantage when the equations to be integrated are symmetrical.

1. It is known that a general solution of the equation

$$\frac{d^2 U}{dx^2} + \frac{d^2 U}{dy^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots \text{(I.)}$$

is given by $\qquad U = \Sigma A e^{m_1 x + m_2 y},$

where m_1, m_2 are connected by the relation

$$m_1^2 + m_2^2 = 0.$$

Now, by the ordinary Calculus of Imaginaries, or *Duplets*, this relation is equivalent to

$$(m_2 + i m_1)(m_2 - i m_1) = 0,$$

where i is such an operation that

$$i^2 = -1.$$

Thus m_2 has either of values $\pm i m_1$, and upon the introduction of these, the value of U becomes

$$\Sigma A e^{m_1 x + i m_1 y} + \Sigma B e^{m_1 x - i m_1 y},$$

or $$U = \left\{ \begin{array}{c} \Sigma A e^{m_1 x} (\cos m_1 y + i \,.\, \sin m_1 y) \\ + \\ \Sigma B e^{m_1 x} (\cos m_1 y - i \,.\, \sin m_1 y) \end{array} \right\} \ldots\ldots (a_1),$$

which is a general solution of (I.) in the *first* form. It is evident that this solution may be condensed, but we shall retain it in its present shape for the sake of symmetry.

Now since the arbitrary constants A, B, and m_1 are independent of each other, we may obviously substitute for A and B *arbitrary functions* of m_1, and the solution just found may then be put under the shape

$$U = \left\{ \begin{array}{c} \int \Phi(m_1)\, e^{m_1 x} . \cos m_1 y \,.\, dm_1 \\ + \\ \int \Psi(m_1)\, e^{m_1 x} . \sin m_1 y \,.\, dm_1 \end{array} \right\} \ldots\ldots (a_1)',$$

imits of the integrals being supposed independent of x y. At first sight this might be regarded as a solution e *second* form, that namely of a definite integral. It , however, more just to consider it as merely *another* of the first form, since the limits are indeterminate.

he general solution in the *third form* is given by

$$U = \left(\frac{d}{dy} - i\frac{d}{dx}\right)^{-1} \cdot \left(\frac{d}{dy} + i\frac{d}{dx}\right)^{-1} . \, 0,$$

$$U = \phi(x+iy) + \psi(x-iy). \,\ldots\ldots\ldots (c_1),$$

the coincidence of the previous with this is evident.

Similarly, a general solution of the higher equation, ered famous by its connexion with the name of Laplace,

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = 0 \,\ldots\ldots\ldots\ldots (\text{II.})$$

iven by

$$V = \Sigma A e^{m_1 x + m_2 y + m_3 z},$$

re m_1, m_2, m_3 are connected by the corresponding relation

$$m_1^2 + m_2^2 + m_3^2 = 0.$$

, by the Calculus of *Triplets*, this relation is equivalent to

$$(m_3 + im_1 + jm_2)(m_3 - im_1 - jm_2) = 0,$$

re i and j are such operations that

$$i^2 = -1, \quad j^2 = -1, \quad ij = -ji.$$

s m_3 has either of values $\pm(im_1 + jm_2)$, and the solution mes the form

$$V = \begin{cases} \Sigma A e^{m_1 x + m_2 y + (im_1 + jm_2) z} \\ + \\ \Sigma B e^{m_1 x + m_2 y - (im_1 + jm_2) z}. \end{cases}$$

obtain the form corresponding to (a_1), we assume

$$m_1 = r\cos a, \quad m_2 = r\sin a,$$

the value of V becomes

$$\Sigma A e^{m_1 x + m_2 y + i_p . rz} + \Sigma B e^{m_1 x + m_2 y - i_p . rz},$$

re

$$\left.\begin{aligned} i_p &= i\cos a + j\sin a \\ r^2 &= m_1^2 + m_2^2 \end{aligned}\right\};$$

a general solution of (II.) in the *first* form is

$$\left\{\begin{aligned} &\Sigma A e^{m_1 x + m_2 y}\{\cos\sqrt{(m_1^2+m_2^2)}z + i_p . \sin\sqrt{(m_1^2+m_2^2)}z\} \\ &+ \\ &\Sigma B e^{m_1 x + m_2 y}\{\cos\sqrt{(m_1^2+m_2^2)}z - i_p . \sin\sqrt{(m_1^2+m_2^2)}z\} \end{aligned}\right\} \ldots (a_2),$$

the duplicate of which, namely,

$$V = \left\{ \begin{array}{l} \Sigma A e^{\sqrt{(m_1^2+m_2^2)}z}\{\cos(m_1x+m_2y)+i_r\,.\sin(m_1x+m_2y)\} \\ \qquad + \\ \Sigma B e^{\sqrt{(m_1^2+m_2^2)}z}\{\cos(m_1x+m_2y)-i_r\,.\sin(m_1x+m_2y)\} \end{array} \right\} \ldots a_2$$

is had by an obvious modification.

It may be well to preserve the solutions in these shapes, as it is probable that the quantity i_r and those corresponding to it bear some relation to the character of the problem whose law is expressed by the equation of which this is the solution.

As before, substituting for A and B arbitrary functions of m_1 and m_2, we can throw these solutions into the following shapes:

$$V = \left\{ \begin{array}{l} \iint \Phi(m_1, m_2)\, e^{m_1x+m_2y}.\cos\sqrt{(m_1^2+m_2^2)}\,z\,.\,dm_1dm_2 \\ \qquad + \\ \iint \Psi(m_1, m_2)\, e^{m_1x+m_2y}.\sin\sqrt{(m_1^2+m_2^2)}\,z\,.\,dm_1dm_2 \end{array} \right\} \ldots (a_1)'$$

with its duplicate

$$V = \left\{ \begin{array}{l} \iint \Phi(m_1, m_2)\cos(m_1x+m_2y)\, e^{\sqrt{(m_1^2+m_2^2)}z}.\,dm_1dm_2 \\ \qquad + \\ \iint \Psi(m_1, m_2)\sin(m_1x+m_2y)\, e^{\sqrt{(m_1^2+m_2^2)}z}.\,dm_1dm_2 \end{array} \right\} \ldots (a_2)''$$

the limits of the integrals in both cases being supposed independent of the quantities x, y, and z, and their order being equal to the number of the quantities m_1, m_2. In this latter respect it will be found that these solutions are only particular cases of a general law.

The same remark applies to the solutions of (II.) just found as to the solution of (I.) represented by the formula $(a_1)'$. They are not to be regarded as solutions of (II.) in the *second form*, but only as *other shapes* of the solutions in the *first form*, since the limits of the integrals are indeterminate.

Poisson has furnished an integral of the Laplacian equation strictly in the *second* form,* namely

$$4\pi V = \left\{ \begin{array}{l} \int_0^{\pi}\int_0^{2\pi} x\Phi(y + x\sin u\cos v\sqrt{-1}, \\ \qquad\qquad z + x\sin u\sin v\sqrt{-1})\sin u\,du dv \\ \qquad + \\ \dfrac{d}{dx}\int_0^{\pi}\int_0^{2\pi} x\Psi(y + x\sin u\cos v\sqrt{-1}, \\ \qquad\qquad z + x\sin u\sin v\sqrt{-1})\sin u\,du dv \end{array} \right\} \ldots (b_2);$$

* *Mémoires de l'Institut*, 1818.

[redacted]

but he considers its value lessened by the circumstance of its containing imaginaries under the signs of the arbitrary functions, more especially since, in the application of these solutions, the arbitrary functions might to be discontinuous. The integral of equation I. might have been expressed in a form similar to this, but upon examination it will be seen that it is reducible to the third given form (c).

It now remains only to find the solution of the Laplacian equation in the *third* or simple functional form, and this is given by

$$V=\left(\frac{d}{dz}-i\frac{d}{dx}-j\frac{d}{dy}\right)^{-1}\left(\frac{d}{dz}-i\frac{d}{dx}-j\frac{d}{dy}\right)^{-1}0,$$

or is
$$V=\phi(x+iz,\ y+jz)+\psi(x-iz,\ y-jz)\ldots\ldots(c),$$

the coincidence of which with the first form is obvious. The necessity of *interpreting* all the results of this article, which are analytically complete, is sufficiently obvious, and the practical value of the results mainly depends on their susceptibility of such interpretation. The first form of solution is easily interpretable by the ordinary principles of triplets. As regards the two latter, the writer regrets his inability hitherto to satisfy the same demand, and would solicit the attention of those who may favour the present paper with a perusal, to this point.

3. Again, a general solution of the still higher equation of the same type

$$\frac{d^2W}{dx^2}+\frac{d^2W}{dy^2}+\frac{d^2W}{dz^2}+\frac{d^2W}{dw^2}=0\ \ldots\ldots(\text{III.})$$

is given by $\qquad W=\Sigma Ae^{m_1x+m_2y+m_3z+m_4w},$

where m_1, m_2, m_3, m_4 are connected by the relation

$$m_1^2+m_2^2+m_3^2+m_4^2=0.$$

Now, by the Calculus of *Quaternions*, this is equivalent to

$$(m_4+im_1+jm_2+km_3)(m_4-im_1-jm_2-km_3)=0,$$

where i, j, and k are such operations that

$$\left.\begin{array}{l} i^2=j^2=k^2=-1 \\ ij=-ji,\quad jk=-kj,\quad ki=-ik \end{array}\right\},$$

and the solution becomes

$$W=\left\{\begin{array}{l}\Sigma Ae^{m_1x+m_2y+m_3z+(im_1+jm_2+km_3)w} \\ \qquad + \\ \Sigma Be^{m_1x+m_2y+m_3z-(im_1+jm_2+km_3)w}\end{array}\right.$$

[redacted]

To obtain the form of this corresponding to (a_1) and (
we assume

$$m_1 = r\cos a, \quad m_2 = r\cos\beta, \quad m_3 = r\cos\gamma,$$

and we have

$$W = \begin{cases} \Sigma A e^{m_1x + m_2y + m_3z + i_R . rw}, \\ \quad + \\ \Sigma B e^{m_1x + m_2y + m_3z - i_R . rw}, \end{cases}$$

where
$$\left.\begin{aligned} i_R &= i\cos a + j\cos\beta + k\cos\gamma \\ r^2 &= m_1^2 + m_2^2 + m_3^2 \end{aligned}\right\},$$

and a general solution of (III.) in the *first* form is

$$W = \left\{\begin{aligned} &\Sigma A e^{m_1x + m_2y + m_3z}\left\{\cos\sqrt{(m_1^2 + m_2^2 + m_3^2)}\,w\right. \\ &\qquad\qquad \left. + i_R . \sin\sqrt{(m_1^2 + m_2^2 + m_3^2}\right\} w \\ &\quad + \\ &\Sigma B e^{m_1x + m_2y + m_3z}\left\{\cos\sqrt{(m_1^2 + m_2^2 + m_3^2)}\,w\right. \\ &\qquad\qquad \left. - i_R . \sin\sqrt{(m_1^2 + m_2^2 + m_3^2}\right\} w \end{aligned}\right\} \ldots (a_1$$

and its duplicate

$$W = \left\{\begin{aligned} &\Sigma A e^{\sqrt{(m_1^2 + m_2^2 + m_3^2)}\,w}\left\{\cos(m_1x + m_2y + m_3z)\right. \\ &\qquad\qquad \left. + i_R . \sin(m_1x + m_2y + m_3z)\right\} \\ &\quad + \\ &\Sigma B e^{\sqrt{(m_1^2 + m_2^2 + m_3^2)}\,w}\left\{\cos(m_1x + m_2y + m_3z)\right. \\ &\qquad\qquad \left. - i_R . \sin(m_1x + m_2y + m_3z)\right\} \end{aligned}\right\} \ldots (a$$

Again, as before, regarding A and B as arbitrary functio
of m_1, m_2, m_3, we can throw these solutions into the follo
ing shapes:

$$W = \left\{\begin{aligned} &\iiint \Phi(m_1, m_2, m_3)\, e^{m_1x + m_2y + m_3z} \\ &\qquad\qquad \cos\sqrt{(m_1^2 + m_2^2 + m_3^2)}\,w \,.\, dm_1 dm_2 dm_3 \\ &\quad + \\ &\iiint \Psi(m_1, m_2, m_3)\, e^{m_1x + m_2y + m_3z} \\ &\qquad\qquad \sin\sqrt{(m_1^2 + m_2^2 + m_3^2)}\,w \,.\, dm_1 dm_2 dm_3 \end{aligned}\right\} \ldots (a_1$$

with its duplicate

$$W = \left\{\begin{aligned} &\iiint \Phi(m_1, m_2, m_3)\cos(m_1x + m_2y + m_3z) \\ &\qquad\qquad e^{\sqrt{(m_1^2 + m_2^2 + m_3^2)}\,w} \,.\, dm_1 dm_2 dm_3 \\ &\quad + \\ &\iiint \Psi(m_1, m_2, m_3)\sin(m_1x + m_2y + m_3z) \\ &\qquad\qquad e^{\sqrt{(m_1^2 + m_2^2 + m_3^2)}\,w} \,.\, dm_1 dm_2 dm_3 \end{aligned}\right\} \ldots (a_2$$

the limits of the integrals in both cases being supposed independent of the quantities x, y, z, and w. The same remark which has been made upon the similar solutions of (I.) and (II.) will apply here with equal force.

By a modification of Poisson's solution of the equation of oscillatory motion in an unlimited gas, we should obtain a solution of equation (III.) strictly in the second form, viz.:

$$4\pi W^{*} = \left\{ \begin{array}{l} \displaystyle\int_0^{\pi}\!\!\int_0^{2\pi} w\Phi(x + w\cos u\sqrt{-1},\quad y + w\sin u\cos v\sqrt{-1}, \\ \qquad\qquad z + w\sin u\sin v\sqrt{-1})\sin u\,du\,dv \\ \qquad + \\ \displaystyle\frac{d}{dw}\int_0^{\pi}\!\!\int_0^{2\pi} w\Psi(x + w\cos u\sqrt{-1},\quad y + w\sin u\cos v\sqrt{-1}, \\ \qquad\qquad z + w\sin u\sin v\sqrt{-1})\sin u\,du\,dv \end{array} \right\} \quad \ldots\ldots\ldots (b_3).$$

To complete the discussion of equation (III.), it now only remains to find its solution in the *third* or simple functional

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* By a regular deduction of the integral from the equation

$$W = \left(\frac{d}{dw} - i\frac{d}{dx} - j\frac{d}{dy} - k\frac{d}{dz}\right)^{-1} \cdot \left(\frac{d}{dw} + i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}\right)^{-1} \cdot 0,$$

or, putting

$$i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz} = D,$$

from the equation

$$2W = (e^{wD} - e^{-wD})\,\phi(xyz) + (e^{wD} + e^{-wD})\,\psi(xyz),$$

a form can be obtained, which seems to the writer to possess somewhat greater generality, viz. $4\pi W =$

$$\left\{ \begin{array}{l} \displaystyle\int_0^{\pi}\!\!\int_0^{2\pi} w\Phi(x + iw\cos u,\ y + jw\sin u\cos v,\ z + kw\sin u.\sin v)\sin u\,du\,dv \\ \qquad + \\ \displaystyle\frac{d}{dw}\int_0^{\pi}\!\!\int_0^{2\pi} w\Psi(x + iw\cos u,\ y + jw\sin u\cos v,\ z + kw\sin u.\sin v)\sin u\,du\,dv, \end{array} \right.$$

and by a similar process applied to the equation of the last article, its solution would be

$$4\pi V = \left\{ \begin{array}{l} \displaystyle\int_0^{\pi}\!\!\int_0^{2\pi} z\Phi(x + iz\cos u,\ y + jz\sin u\cos v)\sin u\,du\,dv \\ \qquad + \\ \displaystyle\frac{d}{dz}\int_0^{\pi}\!\!\int_0^{2\pi} z\Psi(x + iz\cos u,\ y + jz\sin u\cos v)\sin u\,du\,dv. \end{array} \right.$$

From the distinct geometrical characters which we are able to assign to the several symbols i, j, k, it would seem that their occurrence, so far from being matter of objection, will yet be found to possess some important bearing upon the problems, in whose solutions such symbols are exhibited.

form, which is

$$W = \phi(x + iw,\ y + jw,\ z + kw) + \psi(x - iw,\ y - jw,\ z - kw).$$

and whose correspondence with the third forms of the so
tions of equations (I.) and (II.), as well as its coincide
with its own first form, are both obvious.

4. It is evident that, by the adoption of an extende system of operations, regulated by laws similar to tho already employed, the same methods of solution may applied to the general equation containing n independe variables. However, when the number of operations ind cated by the letters i, j, k, &c. exceeds *three*, we can longer attach to them distinct geometrical conceptions. Th the signification of i_n is purely analytical, and on that accou it seems unlikely that the examination of the general equ tion would lead to any practical result.

To adopt the language of Sir William Hamilton, t quantities $i, i_{\prime}, i_{\prime\prime}, i_{\prime\prime\prime}$, &c. are, severally, *imaginary unit* that is, their moduli are positive unity and their squa negative unity. If for each of these quantities we ha substituted the root of negative unity, the solutions in th form could have been obtained at once. Since however, has been before remarked, it is probable that these sever *units* bear some relation to the characters of the problem whose laws are represented, respectively, by the equatio (I.), (II.), (III.), it has been thought better to state th solutions in their present form. It will be noticed that th imaginary unit in the solution of (II.) has the same referenc to an unit circle, as that in the solution of (III.) has to unit sphere.

Again, it is to be observed that we are not at liberty write the second form of the solution of (II.) in the shape

$$V = \begin{cases} \Sigma A e^{m_1 x - m_2 y} (\cos m_1 z + i \,.\, \sin m_1 z)(\cos m_2 z + j \,.\, \sin m_2 z) \\ \qquad + \\ \Sigma B e^{m_1 x + m_2 y} (\cos m_1 z - i \,.\, \sin m_1 z)(\cos m_2 z - j \,.\, \sin m_2 z), \end{cases}$$

as at first sight it might be supposed; and a similar remar will apply to the second forms of the general solution of (III.) In fact, neither in the Calculus of Triplets, nor in that o Quaternions, does the ordinary property of exponentia functions, namely,

$$f(T) \,.\, f(T') = f(T + T'),$$
$$f(Q) \,.\, f(Q') = f(Q + Q'),$$

[illegible]

[illegible]

[illegible]

[illegible]

It is evidently equivalent to

[illegible]

the solution is given by the system

[illegible]

re γ_1 and γ_2 are arbitrary vectors.

I.) The general solution of the equation

$$\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2 = \frac{1}{m^2}$$

y an exactly similar process, given by the system

$$\left.\begin{aligned} z + ix + jy &= \frac{1}{m}\Phi(s) + \alpha_1 s + \beta_1 \\ z - ix - jy &= \frac{1}{m}\Psi(s) + \alpha_2 s + \beta_2 \\ \Phi''(s)\,\Psi''(s) &= 1 \end{aligned}\right\},$$

re, as before, α_1, β_1, α_2, β_2 are arbitrary vectors.

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Thus, by the combination of the system of the previo example with that just stated, we obtain the general repr sentation of the curve of double curvature, whose curvatu is constant.

(III.) To find the integral of the equation of vibrator motion of thin plates, namely

$$\frac{d^2z}{dt^2} + b^2\left(\frac{d^4z}{dx^4} + 2\,\frac{d^4z}{dx^2.dy^2} + \frac{d^4z}{dy^4}\right) = 0.$$

Let
$$i\,\frac{d}{dx} + j\,\frac{d}{dy} = D,$$

and the equation becomes

$$\frac{d^2z}{dt^2} + b^2D^4z = 0,$$

or
$$z = e^{i_v btD^2}.\phi(xy) + e^{-i_v btD^2}.\psi(xy).$$

Now
$$\int_{-\infty}^{\infty} dw\,.\,e^{-(w-x)^2} = \sqrt{\pi}\,;$$

and putting alternately

$$\alpha = \sqrt{(i_v\,.\,bt)}\left(i\,\frac{d}{dx} + j\,\frac{d}{dy}\right),$$

$$\alpha = \sqrt{(-i_v.bt)}\left(i\,\frac{d}{dx} + j\,\frac{d}{dy}\right),$$

we get for the required solution,

$$\sqrt{\pi}\,.\,z = \begin{cases} \int_{-\infty}^{\infty} dw.e^{-w^2}.\phi\{x + 2iw\sqrt{(i_v.bt)},\ y + 2jw\sqrt{(-i_v.bt)}\}, \\ \qquad + \\ \int_{-\infty}^{\infty} dw.e^{-w^2}.\psi\{x + 2iw\sqrt{(-i_v.bt},\ y + 2jw\sqrt{(-i_v.bt)}\}. \end{cases}$$

(IV.) By a similar process the integral of the equation

$$\frac{d^2v}{dt^2} + b^2\left(\frac{d^4v}{dx^4} + \frac{d^4v}{dy^4} + \frac{d^4v}{dz^4} + 2\,\frac{d^4v}{dy^2dz^2} + 2\,\frac{d^4v}{dz^2dx^2} + 2\,\frac{d^4v}{dx^2dy^2}\right) =$$

would be $\sqrt{\pi}v =$

$$\begin{cases} \int_{-\infty}^{\infty} dw.e^{-w^2}.\phi\{x + 2iw\sqrt{(i_n.bt},\ y + 2jw\sqrt{(i_n.bt)},\ z + 2kw\sqrt{(i_n.bt)} \\ \qquad + \\ \int_{-\infty}^{\infty} dw.e^{-w^2}.\psi\{x + 2iw\sqrt{(-i_n.bt)},\ y + 2jw\sqrt{(-i_n.bt)},\ z + 2kw\sqrt{(-i_n.bt)} \end{cases}$$

As regards the last two results, it may be again observed that they too demand interpretation, and that upon their susceptibility of such their practical value will depend.*

38, *Trinity College, Dublin.*

ON SOME PROPERTIES OF A PARABOLA TOUCHING THE THREE SIDES OF A TRIANGLE.

By WILLIAM B. COLTMAN, Trinity College, Cambridge.

1. Let P be any point in the plane of a given triangle ABC. Let the sides of this triangle be produced indefinitely, and let α, β, γ be the lengths of the perpendiculars from P on BC, CA, AB. Then it is evident that if α, β, γ are known, the position of the point P will be determined. BC, CA, AB are called lines of reference, and α, β, γ the trilinear coordinates of the point P. The quantities α, β, γ are not independent of each other, for they are connected by an identical equation. Thus, let

$$BC = a, \quad CA = b, \quad AB = c, \quad \Delta = \text{area of } ABC,$$

then, if P lies within the triangle ABC, we have

$$a\alpha + b\beta + c\gamma = 2\Delta PBC + 2\Delta PCA + 2\Delta PAB$$
$$= 2\Delta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

If P lies on the other side of one of the lines of reference, we must consider the corresponding perpendicular to be negative. Thus, if P lies on the other side of AB, we have

$$a\alpha + b\beta + c\gamma = 2\Delta PBC + 2\Delta PCA - 2\Delta PAB$$
$$= 2\Delta \text{ as before.}$$

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* After the present paper had been forwarded for publication, the writer became aware that Sir William Hamilton had called attention to the importance of the symbol

$$i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz},$$

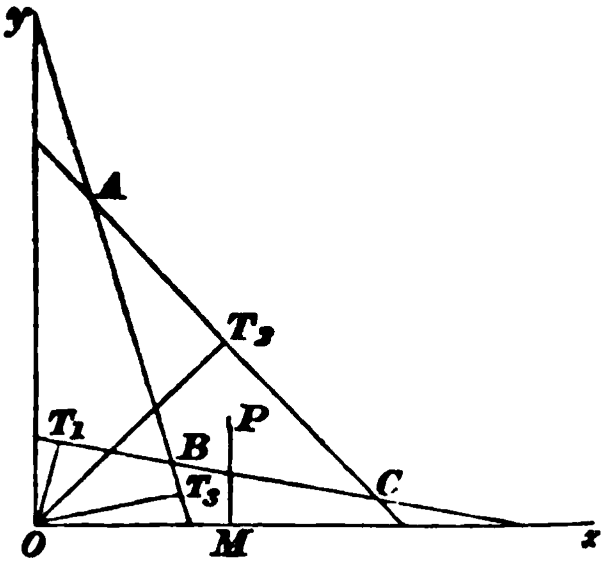
in the *Proceedings of the Royal Irish Academy* for July, 1846, as also to the law of combination of [illegible] the similar symbol

[illegible]

in the *Philosophical* [illegible]
like the former, [illegible]

2. We may also consider α, β, γ as functions of x and y. For let Ox, Oy be any lines at right angles to each other in the plane ABC. Let x, y be the coordinates of the point P referred to Ox, Oy as coordinate axes. Let OT_1, OT_2, OT_3 be perpendiculars on BC, CA, AB, respectively. Let

$$OT_1 = p_1, \quad OT_2 = p_2 \quad OT_3 = p_3;$$
$$T_1Ox = \omega_1, \quad T_2Ox = \omega_2, \quad T_3Ox = \omega_3.$$



Then, taking P within the triangle ABC, in order that α, β, γ may all be positive, it is easy to see that we shall have

$$\alpha = x \cos \omega_1 + y \sin \omega_1 - p_1,$$
$$\beta = p_2 - x \cos \omega_2 - y \sin \omega_2,$$
$$\gamma = x \cos \omega_3 + y \sin \omega_3 - p_3,$$
$$\omega_1 - \omega_2 = C; \quad \omega_2 - \omega_3 = A; \quad \omega_1 - \omega_3 = \pi - B.$$

Also the equations to BC, CA, AB, will be respectively

$$\alpha = 0, \quad \beta = 0, \quad \gamma = 0.$$

3. The general equation to any straight line referred to trilinear coordinates may be put under the form

$$A_1\alpha + B_1\beta + C_1\gamma = 0.$$

For it is a linear function of x and y, and contains two arbitrary constants. It is therefore the most general form of the equation to the straight line. If a curve of the second order be inscribed in the triangle ABC its equation will be

$$u = l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma - 2nl\gamma\alpha - 2lm\alpha\beta = 0 \ldots (2).$$

For the points at which AB meets (2) are got by combining (2) with the equation $\gamma = 0$, whence we get

$$0 = (l\alpha - m\beta)^2,$$

which shews that the two points of intersection coincide; that is, that the curve touches AB. Similarly we may shew that it touches AC and BC. Also it is of the second degree. For it is of the second degree in α, β, γ, which are linear functions of x and y. Mr. Salmon has discussed the properties of this equation when the curve which it represents is any conic section. The object of this paper is to establi certain equations which obtain when the curve is a p
It is however necessary first to prove sc
positions.

[redacted]

... in terms of x and y, we find that in the present case

$$a_1^2 + b_1^2 = (A_1 \cos\omega_1 - B_1 \cos\omega_2 + C_1 \cos\omega_3)^2 + (A_1 \sin\omega_1 - B_1 \sin\omega_2 + C_1 \sin\omega_3)^2$$
$$= A_1^2 + B_1^2 + C_1^2 - 2B_1C_1 \cos(\omega_2 - \omega_3) + 2C_1A_1 \cos(\omega_3 - \omega_1) - 2A_1B_1 \cos(\omega_1 - \omega_2)$$
$$= A_1^2 + B_1^2 + C_1^2 - 2B_1C_1 \cos A - 2C_1A_1 \cos B - 2A_1B_1 \cos C.$$

herefore the length of the perpendicular

$$= \frac{A_1\alpha' + B_1\beta' + C_1\gamma'}{\sqrt{(A_1^2 + B_1^2 + C_1^2 - 2B_1C_1 \cos A - 2C_1A_1 \cos B - 2A_1B_1 \cos C)}} \;\ldots\ldots\ldots\ldots (3).$$

Cor. The perpendiculars on two lines from any point of a line bisecting the angle between them are equal. Hence, if the equations to the lines are

$$A_1\alpha + B_1\beta + C_1\gamma = 0 \ldots\ldots\ldots\ldots (4),$$
$$A_2\alpha + B_2\beta + C_2\gamma = 0 \ldots\ldots\ldots\ldots (5),$$

e equations to their internal and external bisectors will be

$$\frac{A_1\alpha + B_1\beta + C_1\gamma}{\sqrt{(A_1^2+B_1^2+C_1^2-2B_1C_1\cos A-2C_1A_1\cos B-2A_1B_1\cos C)}} = \pm \frac{A_2\alpha + B_2\beta + C_2\gamma}{\sqrt{(A_2^2+B_2^2+C_2^2-2B_2C_2\cos A-2C_2A_2\cos B-2A_2B_2\cos C)}}.$$

Prop. 2. To find the condition that the two lines (4) and (5) may include an angle ϕ.

Proceeding as in the former proposition, we find that we must have

$$\tan\phi = \pm \frac{A_1(B_2c - C_2b) + B_1(C_2b - A_2c) + C_1(A_2b - B_2a)}{A_1(A_2 - B_2\cos C - C_2\cos B) + B_1(B_2 - C_2\cos A - A_2\cos C) + C_1(C_2 - A_2\cos B - B_2\cos A)}.$$

If (4) and (5) are parallel $\tan\phi = 0$, if they are perpendicular $\tan\phi = \infty$, in which cases we have respectively

$$0 = A_1(B_2c - C_2b) + B_1(C_2b - A_2c) + C_1(A_2b - Ba) \ldots\ldots\ldots\ldots (6),$$

and $0 = A_1(A_2 - B_2\cos C - C_2\cos B) + B_1(B_2 - C_2\cos A - A_2\cos C) + C_1(C_2 - A_2\cos B - B_2\cos A) \ldots (7).$

.. To find the equation to a tangent to any curve referred to trilinear coordinates.

... the line (4) cut the curve $v = 0$ in the points $(\alpha'\beta'\gamma')(\alpha''\beta''\gamma'')$, then we have

$$A_1\alpha' + B_1\beta' + C_1\gamma' = 0, \qquad A_1\alpha'' + B_1\beta'' + C_1\gamma'' = 0.$$

Therefore (eliminating A_1, B_1, and C_1, by cross-multiplication, from these equations and from (4)) we get

$$\alpha(\beta'\gamma'' - \gamma'\beta'') + \beta(\gamma'\alpha'' - \alpha'\gamma'') + \gamma(\alpha'\beta'' - \beta'\alpha'') = 0,$$

or $$\alpha\{\beta'(\gamma'' - \gamma') - \gamma'(\beta'' - \beta')\} + \beta\{\gamma'(\alpha'' - \alpha') - \alpha'(\gamma'' - \gamma')\} + \gamma\{\alpha'(\beta'' - \beta') - \beta'(\alpha'' - \alpha')\} = 0.$$

When the secant becomes a tangent, this equation becomes

$$\alpha(\beta' d\gamma' - \gamma' d\beta') + \beta(\gamma' d\alpha' - \alpha' d\gamma') + \gamma(\alpha' d\beta' - \beta' d\alpha') = 0, \quad \text{or } (\beta\gamma' - \gamma\beta')\, d\alpha' + (\gamma\alpha' - \alpha\gamma')\, d\beta + (\alpha\beta' - \beta\alpha')\, d\gamma = 0.$$

Also differentiating (1) and the equation to the curve, we get

$$a\,d\alpha' + b\,d\beta' + c\,d\gamma' = 0 \text{ and } \frac{dv}{d\alpha'}d\alpha' + \frac{dv}{d\beta'}d\beta' + \frac{dv}{d\gamma'}d\gamma' = 0.$$

Eliminating $d\alpha'$, $d\beta'$, $d\gamma'$, by cross-multiplication, we have

$$0 = \frac{dv}{d\alpha'}\{c(\gamma\alpha' - \alpha\gamma') + b(\beta\alpha' - \alpha\beta')\} + \frac{dv}{d\beta'}\{a(\alpha\beta' - \beta\alpha') + c(\gamma\beta' - \beta\gamma')\} + \frac{dv}{d\gamma'}\{b(\beta\gamma' - \gamma\beta') + a(\alpha\gamma' - \gamma\alpha')\},$$

$$= \frac{dv}{d\alpha'}\{\alpha'(b\beta + c\gamma) - \alpha(b\beta' + c\gamma')\} + \frac{dv}{d\beta'}\{\beta'(c\gamma + a\alpha) - \beta(c\gamma' + a\alpha')\} + \frac{dv}{d\gamma'}\{\gamma'(a\alpha + b\beta) - \gamma(a\alpha' + b\beta')\},$$

$$= \frac{dv}{d\alpha'}\{\alpha' - \alpha\}\, 2\Delta + \frac{dv}{d\beta'} 2\Delta(\beta' - \beta) + \frac{dv}{d\gamma'} 2\Delta(\gamma' - \gamma) \text{ by (1).}$$

...nce v is a homogeneous function of α, β, γ, $\alpha'\dfrac{dv}{d\alpha'} + \beta'\dfrac{dv}{d\beta'} + \gamma'\dfrac{dv}{d\gamma'} = 0$, and the equation to the tan-

—If there are n lines of reference $\alpha_1\alpha_2\alpha_3 \ldots \alpha_n$ the equation to the tangent will be

$$\alpha_1\frac{dv}{d\alpha_1'} + \alpha_2\frac{dv}{d\alpha_2'} + \ldots + \alpha_n\frac{dv}{d\alpha_n'} = 0.$$

gent becomes

$$\alpha \frac{dv}{d\alpha'} + \beta \frac{dv}{d\beta'} + \gamma \frac{dv}{d\gamma'} = 0 \dots\dots\dots (8).$$

COR. 1. The equation to the normal may be found at once from the general equation to the straight line by introducing the conditions, that it passes through the point $\alpha'\beta'\gamma'$, and that it is perpendicular to the tangent.

COR. 2. Let α, β, γ be a fixed point. Then it is easy to see from (8) that the points of contact of all the tangents which can be drawn through the point $\alpha\beta\gamma$ to touch the curve $v = 0$ lie on the curve

$$\alpha \frac{dv}{d\alpha'} + \beta \frac{dv}{d\beta'} + \gamma \frac{dv'}{d\gamma'} = 0 \dots\dots\dots (9),$$

where $\alpha'\beta'\gamma'$ are supposed to be variable. If $v = 0$ is of the second degree this equation becomes linear in $\alpha'\beta'\gamma'$, and is evidently the equation to the chord joining the points of contact of the two tangents through the point $\alpha\beta\gamma$. This point is called the pole of the chord of contact, and the chord of contact is called the polar of this point.

COR. 3. The equation (2) may be put under the form $(l\alpha)^{\frac{1}{2}} + (m\beta)^{\frac{1}{2}} + (n\gamma)^{\frac{1}{2}} = 0$. Hence the equation to a tangent to it is

$$\left(\frac{l}{\alpha'}\right)^{\frac{1}{2}}\alpha + \left(\frac{m}{\beta'}\right)^{\frac{1}{2}}\beta + \left(\frac{n}{\gamma'}\right)^{\frac{1}{2}} = 0.$$

Therefore if the line (4) touches (2), we must have

$$kl^{\frac{1}{2}} = A_1\alpha'^{\frac{1}{2}}, \quad km^{\frac{1}{2}} = B_1\beta'^{\frac{1}{2}}, \quad kn^{\frac{1}{2}} = C_1\gamma'^{\frac{1}{2}},$$

where k is indeterminate. Multiplying these equations by $A_1^{-1}l^{\frac{1}{2}}$, $B_1^{-1}m^{\frac{1}{2}}$, $C_1^{-1}n^{\frac{1}{2}}$, respectively, and adding, we have

$$\frac{l}{A} + \frac{m}{B} + \frac{n}{C} = 0 \dots\dots\dots\dots (10),$$

as the condition that (4) may touch (2).

4. We may now proceed to find the condition that (2) may be a parabola. From (1) and (2) we have, eliminating α,

$$0 = l^2(2\Delta - b\beta - c\gamma)^2 + a^2(m^2\beta^2 + n^2\gamma^2)$$
$$- 2mna^2\beta\gamma - 2la(m\beta + n\gamma)(2\Delta - b\beta - c\gamma).$$

Now if AB, AC be taken as coordinate axes, β and γ will ...rtional to the coordinates of a point in the curve. ...t the curve may be a parabola, we must have

...ient ...t of γ^2),

or $4(l^2bc + lmac + lnba - mna^2)^2 = 4(lb + ma)^2 - (lc + na)^2$
$$= 4(l^2bc + lmac + lnab + mna^2)^2,$$
or $$0 = lmn(lbc + mac + nba).$$
Now if either l, m, or n become zero, (2) becomes the equation to two straight lines. Hence the condition that (2) may be a parabola is
$$\frac{l}{a} + \frac{m}{b} + \frac{n}{c} = 0 \text{..............} (11).$$

Cor. 1. Comparing (10) and (11) we see that the geometrical property expressed by (11) is that the parabola (2) will touch the line $a\alpha + b\beta + c\gamma = 0$, which is a line at infinity.

Cor. 2. By similar reasoning we find, that if the general trilinear equation of the second order
$$A_1\alpha^2 + B_1\beta^2 + C_1\gamma^2 + A'\beta\gamma + B'\gamma\alpha + C'\alpha\beta = 0$$
represents a circle, we must have
$$B_1c^2 + C_1b^2 - A'bc = C_1a^2 + A_1c^2 - B'ac = A_1b^2 + B_1a^2 - C'ba;$$
and that the conditions that the curves
$$l\beta\gamma + m\alpha\gamma + n\alpha\beta = 0, \quad l^2\alpha^2 + m^2\beta^2 = n^2\gamma^2, \quad \alpha\gamma = k\beta^2$$
may be parabolæ, are respectively
$$(al)^{\frac{1}{2}} + (bm)^{\frac{1}{2}} + (cn)^{\frac{1}{2}} = 0, \quad \frac{a^2}{l} + \frac{b^2}{m} = \frac{c^2}{n}, \quad b^2 = 4kac.$$
The geometrical interpretation of the last three equations is similar to that of the equation (11).

5. To find the equations to the focus, director, and axis of the parabola (2).

Let the line (4) touch the curve (2) and be at right angles to AB. Then, from (7), we have
$$C_1 = A_1 \cos B + B_1 \cos A \text{..........} (12),$$
and, from (10) and (11), we have, by eliminating n,
$$A_1 \frac{a}{l}(cB_1 - C_1b) = B_1 \frac{b}{m}(aC_1 - cA_1) \text{......} (13).$$
Now from (12)
$$\begin{aligned} aC_1 - cA_1 &= A_1a \cos B + aB_1 \cos A - cA_1 \\ &= A_1(a \cos B - c) + aB_1 \cos A \\ &= (aB_1 - bA_1) \cos A. \end{aligned}$$
Similarly $$cB_1 - bC_1 = (aB_1 - bA_1) \cos B.$$
Therefore, from (13),
$$\frac{l}{a}\,\frac{\cos A}{A_1} = \frac{m}{b}\,\frac{\cos B}{B_1} \text{............} (14).$$

Let α_1, β_1, γ_1, be the coordinates of the focus. By a property of the parabola we know that the polar of any point in which two tangents at right angles to each other intersect, passes through the focus. Hence the polar of the point $\{\gamma = 0,\ A_1\alpha + B_1\beta = 0\}$ passes through the focus, therefore α_1, β_1, γ_1, satisfy its equation, which is by (9)

$$\frac{1}{A_1}\frac{du}{d\alpha} = \frac{1}{B_1}\frac{du}{d\beta}.$$

Therefore, from (14),

$$\frac{a}{l\cos A}\frac{du}{d\alpha_1} = \frac{b}{m\cos B}\frac{du}{d\beta_1},$$

or $$\frac{l\alpha_1 - m\beta_1 - n\gamma_1}{a^2 - b^2 - c^2} = \frac{m\beta_1 - n\gamma_1 - l\alpha_1}{b^2 - c^2 - a^2} = \frac{n\gamma_1 - l\alpha_1 - m\beta_1}{c^2 - a^2 - b^2} \text{ by symmetry,}$$

or $$\frac{l\alpha_1}{a^2} = \frac{m\beta_1}{b^2} = \frac{n\gamma_1}{c^2} = \lambda \ldots\ldots\ldots\ldots (15),$$

which are the equations to the focus. Also the director is the polar of the focus. Hence, by (9), its equation is

$$0 = \alpha_1\frac{du}{d\alpha} + \beta_1\frac{du}{d\beta} + \gamma_1\frac{du}{d\gamma}$$

and, by (15),

$$= a^2(l\alpha - m\beta - n\gamma) + b^2(m\beta - n\gamma - l\alpha) + c^2(n\gamma - l\alpha - m\beta)$$
$$= (a^2 - b^2 - c^2)\, l\alpha + m\beta\,(b^2 - c^2 - a^2) + n\gamma\,(c^2 - a^2 - b^2),$$

or $$0 = \frac{l\alpha}{a}\cos A + \frac{m\beta}{b}\cos B + \frac{n\gamma}{c}\cos C \ldots\ldots\ldots\ldots (16).$$

Let $\alpha_2\beta_2\gamma_2$ be the focus at infinity. Then, by a general property of conic sections,

$$\alpha_1\alpha_2 = \beta_1\beta_2 = \gamma_1\gamma_2.$$

Therefore, by (15),

$$\frac{a^2\alpha_2}{l} = \frac{b^2\beta_2}{m} = \frac{c^2\gamma_2}{n} \ldots\ldots\ldots\ldots (17).$$

Let the equation to the axis be

$$A_1\alpha + B_1\beta + C_1\gamma = 0.$$

Since $\alpha_1\beta_1\gamma_1$ is a point in the axis,

$$A_1\frac{a^2}{l} + B_1\frac{b^2}{m} + C_1\frac{c^2}{n} = 0,$$

and since $\alpha_2\beta_2\gamma_2$ is also a p

$$A_1\frac{l}{a^2} +$$

(This last condition may also be found from the condition that the axis is at right angles to the director.) Eliminating A_1, B_1, C_1, we have as the equation to the axis

$$\frac{\alpha a^3}{l}\left(\frac{b^4}{m^2}-\frac{c^4}{n^2}\right)+\frac{\beta b^3}{m}\left(\frac{c^4}{n^2}-\frac{a^4}{l^2}\right)+\frac{\gamma c^3}{n}\left(\frac{a^4}{l^2}-\frac{b^4}{m^2}\right)=0.$$

6. To find the magnitude of the latus-rectum L.

$\frac{1}{2}L$ = the length of the perpendicular from the focus on the director. Therefore, from (3) and (16),

$$\tfrac{1}{2}L=\frac{\alpha_1\frac{l}{a}\cos A+\beta_1\frac{m}{b}\cos B+\gamma_1\frac{n}{c}\cos C}{\sqrt{\left\{\frac{l^2}{a^2}\cos^2 A+\frac{m^2}{b^2}\cos^2 B+\frac{n^2}{c^2}\cos^2 C-2\cos A\cos B\cos C\left(\frac{lm}{ab}+\frac{mn}{bc}+\frac{nl}{ac}\right)\right\}}}$$

or, from (11) and (15),

$$=\frac{(a\cos A+b\cos B+c\cos C)\,\lambda}{\sqrt{\left\{\frac{l^2}{a^2}\cos A\,(\cos A+\cos B\cos C)+\frac{m^2}{b^2}\cos B\,(\cos B+\cos C\cos A)+\frac{n^2}{c^2}\cos C(\cos C+\cos A\cos B)\right\}}}.$$

But $a\cos A+b\cos B+c\cos C=2a(\cos A+\cos B\cos C)$; therefore

$$\frac{L}{2}=\frac{\lambda\sqrt{\{2\,(a\cos A+b\cos B+c\cos C)\,(abc)^2\}}}{\sqrt{(l^2b^2c^2\cos A+m^2c^2a^2\cos B+n^2a^2b^2\cos C)}};$$

from (1) and (15), $\lambda\left(\frac{a^2}{l}+\frac{b^2}{m}+\frac{c^2}{n}\right)=2\Delta$; therefore, if $\mu a=\sin A$,

$$L=(4\Delta)^{\frac{5}{2}}\{\mu(l^2b^2c^2\cos A+m^2c^2a^2\cos B+n^2a^2b^2\cos C)\}^{-\frac{1}{2}}\left(\frac{a^2}{l}+\frac{b^2}{m}+\frac{c^2}{n}\right)^{-1}.$$

[redacted]

7. By reasoning similar to the above, we may shew that the equations to the focus, director, and axis of the parabola whose equation is $4ac\alpha\gamma = b^2\beta^2$, are respectively

$$\frac{\alpha_1}{a} = \frac{b\beta_1}{2ac}\sec B = \frac{\gamma_1}{c}; \quad c\alpha - b\cos B\beta + a\gamma = 0,$$

$$2ac\left\{\alpha(a\cos B + c) - (a + c\cos B)\gamma\right\} = b\beta(a^2 - c^2),$$

and that its latus-rectum $= (4\Delta)^2 (b^2 + 2ac\cos B)^{-\frac{3}{2}}$.

8. The following problems serve to illustrate the use of the equations given above.

Prob. 1. The foci of all the parabolæ which can be escribed* to a triangle lie on the circle circumscribed about the triangle, and their directors will all pass through the point of intersection of perpendiculars from the angles on the opposite sides.

By (11) and (15) we have as the equation to the locus of the focus $\dfrac{a}{\alpha_1} + \dfrac{b}{\beta_1} + \dfrac{c}{\gamma_1} = 0$, the equation to the circle circumscribing ABC.

Also from (11) and (16) we see that the equation to the director may be put under the form

$$\frac{l}{a}(\rho - \alpha\cos A) + \frac{m}{b}(\rho - \beta\cos B) + \frac{n}{c}(\rho - \gamma\cos C) = 0,$$

and therefore the director passes through the point

$$\rho = \alpha\cos A = \beta\cos B = \gamma\cos C,$$

which is the point in which the perpendiculars from A, B, and C on BC, CA, AB intersect.

Prob. 2. Let S be the focus of the parabola (2). Join SA, SB, SC, and draw AA', AB', AC' at right angles to them respectively; AA', BB', CC' meet in a point.

The equation to SA is

$$\frac{m}{b^2}\beta = \frac{n}{c^2}\gamma.$$

Therefore, by (7), the equation to AA' is

$$\left(\frac{m}{b^2}\cos A + \frac{n}{c^2}\right)\beta + \left(\frac{n}{c^2}\cos A + \frac{m}{b^2}\right)\gamma = 0;$$

[* ...hing one side of a triangle and the two others produced.]

or, by (11),

$$\left(\frac{n}{c^2}\cos C - \frac{l}{a^2}\cos A\right)\frac{\beta}{b} = \left(\frac{l\cos A}{a^2} - \frac{m\cos B}{b^2}\right)\frac{\gamma}{c}.$$

By symmetry the equations to BB', CC' are

$$\left(\frac{l}{a^2}\cos A - \frac{m}{b^2}\cos B\right)\frac{\gamma}{c} = \left(\frac{m\cos B}{b^2} - \frac{n\cos C}{c^2}\right)\frac{\alpha}{a};$$

$$\left(\frac{m}{b^2}\cos B - \frac{n}{c^2}\cos C\right)\frac{\alpha}{a} = \left(\frac{n\cos C}{c^2} - l\frac{\cos A}{a^2}\right)\frac{\beta}{b}.$$

From the forms of their equations AA', BB', CC' evi
meet in a point.

PROB. 3. Let a parabola touch BC, CA, AB in D,
respectively. Join AD, BE, CF, and let them cut the
in L, M, N. Let LP, MQ, NR be tangents to the
and let them meet BC, CA, AB in P, Q, R. Then
$B'C$, $C'F$ meet in a point, the locus of which, for dif
parabolæ, is an ellipse circumscribing ABC. Also LP,
NR, and the polars of A, B, C, all pass through fixed p
and the points P, Q, R lie on a straight line passing th
the centre of gravity of ABC.

The equations to the point D are ($\alpha = 0$, $n\gamma = m\beta$),
fore the equation to AD is $n\gamma = m\beta$ Similarly, the equ
to BE and CF are $l\alpha = n\gamma$ and $m\beta = l\alpha$; therefore AD,
CF meet in a point $\rho = l\alpha = m\beta = n\gamma$. Therefore, by
the equation to its locus is $\dfrac{1}{a\alpha} + \dfrac{1}{b\beta} + \dfrac{1}{c\gamma} = 0$, the equ
to an ellipse circumscribing ABC. The equation to
polar of A is, by (9),

$$l\alpha - m\beta - n\gamma = 0,$$

which, by (10), may be put under the form

$$l\left(\alpha - \frac{\rho}{a}\right) - m\left(\beta + \frac{\rho}{b}\right) - n\left(\gamma + \frac{\rho}{c}\right) = 0.$$

Therefore the polar of A passes through the fixed
$\rho = a\alpha = -b\beta = -c\gamma$.

Similarly, the polars of B and C pass through fixed
Again, the equations to the point L are

$$\left\{(n\gamma)^{\frac{1}{2}} = (m\beta)^{\frac{1}{2}} = -\frac{\sqrt{l\alpha}}{2}\right\};$$

therefore, by (8), the equation to LP is $\frac{1}{2}l\alpha = m\beta + n\gamma$,

by (11) may be put in the form

$$\tfrac{1}{2}l\left(\alpha - \frac{2\rho}{a}\right) = m\left(\beta + \frac{\rho}{b}\right) + n\left(\gamma + \frac{\rho}{c}\right).$$

Therefore LP passes through the fixed point

$$\rho = \tfrac{1}{2}a\alpha = -\,b\beta = -\,c\gamma.$$

Similarly, MQ and NR pass through fixed points.

Also the equations to P, Q, R are

$$\{\alpha = 0,\ m\beta + n\gamma = 0\},\ \{\beta = 0,\ n\gamma + l\alpha = 0\},\ \{\gamma = 0,\ l\alpha + m\beta = 0\};$$

therefore P, Q, R lie on a line whose equation is

$$l\alpha + m\beta + n\gamma = 0.$$

This equation, by (11), may be put under the form

$$\frac{l}{a}(\rho - a\alpha) + \frac{m}{b}(\rho - b\beta) + \frac{n}{c}(\rho - c\gamma) = 0\,;$$

therefore PQR passes through the point $\rho = a\alpha = b\beta = c\gamma$, which is the centre of gravity of the triangle ABC.

December 23, 1851.

Postscript.—Since this paper was sent to the Editor, Mr. Salmon's work *On the Higher Plane Curves* has been published. In it Mr. Salmon has established the equation (6) by a different process.

ON THE KNIGHT'S MOVE AT CHESS.

By Ferdinand Minding (of Dorpat).

[From the *Bulletin de l'Académie des Sciences de St. Petersburg*, tom. vi. No. 14. (lu le 22 Janvier 1847.)]

This well-known problem was first treated scientifically by Euler in the *Memoirs of Berlin* of the year 1759, where he shews a simple method of finding a great number of solutions, particularly of recurring circuits, of the Knight's move ("von wiederkehrenden Umläufen des Springers"). This method consists essentially in first occupying a series of squares till the Knight can move no farther without recurring to a square it had already occupied; the squares thus preoccupied he then endeavours to arrange in groups, which in [illegible]ltered order may join each other as well as the squa[illegible]t touched. It is, however, not my object

to explain this method here, which is the less required, as it is fully done in Legendre's *Théorie des Nombres*, vol. II. p. 151, of the 2nd edition; (also in Klügel and Mollweide's *Mathematical Dictionary*, in the article, "Springer auf dem Schachbrette.")

In the *Memoirs of the Paris Academy* of 1771, Vandermonde (Remarques sur les problèmes de situation) justly draws attention to the importance of researches in geometry of position. He compares the course of the Knight with that of a thread, which, wrapped round a pin placed in the centre of each square, connects all squares according to the law assigned to the Knight's move; but he confines himself to the investigation of particular symmetrical arrangements. In the above-mentioned Dictionary there is also quoted a memoir by Colini (Solution du problème du cavalier au jeu des échecs, Mannheim 1773), which contains several solutions; but I have not seen the book.

Legendre (p. 165 of the above work) touches upon the question relating to the number of all possible solutions, and this indeed is the principal question which it is the province of analysis to answer. But as this distinguished analyst designates the problem as a difficult one, and has recourse to a preliminary solution, which clearly he himself did not consider satisfactory, the necessity arose of inquiring more accurately into the nature of this difficulty, and I beg permission to lay before the reader the results of my researches. I have, indeed, not arrived at an actual numerical result, or rather at a knowledge of the means, by which such a result, if required, could in all cases be obtained; but what follows will shew that the only obstacle in the way is the almost unlimited extent of a calculation that leads to very many and very large numbers. It would be a valuable acquisition to overcome this impediment by appropriate approximations, as has been done successfully in other, though, it would seem, less complicated cases.

I denote the squares of the chess-board by numbers which may be regarded as the coordinates of their centres, and I make a corner square the origin. For this there is $x = 0$, $y = 0$; for the one diagonally opposite, $x = 7$, $y = 7$. The square for which $x = a$, $y = b$ is denoted by (a, b): of course the order of these letters has to be attended to. The Knight may pass in one move from (a, b) to all those squares that are denoted by $(a \pm 2, b \pm 1)$ and $(a \pm 1, b \pm 2)$, provided their coordinates do not exceed the limits 0 and 7. I further affix to each square an index, *i.e.* one of the numbers from 1 to

in a perfectly arbitrary order, but each index corresponds
but one square.

Let a be the index of any square; β, γ, δ, &c. the indices
all those, which the Knight may reach in one move,
ginning from a; I shall call these the *neighbouring fields*
hich, consequently, do not mean the adjacent ones). If
w the Knight is on a particular square A, the question
ses, in how many different ways he may reach the square
by n moves, allowing him to touch the same square
peatedly. Let $w_n^{(a)}$ denote this number. If we separate
e squares into even and odd (black and white) ones, a must
of the same kind with A, or of the opposite kind, accord-
g as the number of moves n is even or odd. Whenever
is condition is not satisfied, $w_n^{(a)} = 0$.

Since the Knight can reach a only from β, γ, δ, &c., we
ve immediately

$$w_n^{(a)} = w^{(\beta)}_{n-1} + w^{(\gamma)}_{n-1} + w^{(\delta)}_{n-1} + \&c. \ldots\ldots (A),$$

hich reduces the n^{th} move to the $(n-1)^{th}$. Each square
rnishes us a similar equation by giving to a successively
e values 1, 2...64. Now we know the values of $w_n^{(a)}$ for
e first move, or for $n = 1$; this number being $= 1$ for all
e neighbouring squares of A, and $= 0$ for all others.
ence we can calculate $w_n^{(a)}$ by means of the 64 equations
r every square and every number of moves. Thus the
oblem is solved.

The same problem may be solved under the additional
ndition, that any particular square shall not be touched.
f β is such an excluded square, then $w_n^{(\beta)} = 0$ for all values
f n, and at the same time in the system (A) that equation
sappears which expresses the transition from the square A
β; hence we have in (A) one equation and one unknown
antity less. In general, as many equations and as many
nknown quantities will disappear from the system of equa-
ons (A) as the number of excluded squares amounts to;
d the number of equations (A) and of the unknown
antities they contain is in all cases equal to the number
f admissible (open) fields, which we shall call i. Let in-
ices $1, 2 \ldots i$ be assigned to them; a being one of them
signed to the square (a, b), let us form the product $w_n^{(a)} x^a y^b$
r it, and similarly for all other open squares, and let U_n
enote the sum of all these products, so that

$$U_n = \Sigma w_n^{(a)} x^a y^b \text{ from } a = 1 \text{ to } a = i$$

omprises all cases that are possible in n moves. To pass
om this to t[illegible] 1)th move, U_n has to be multiplied with

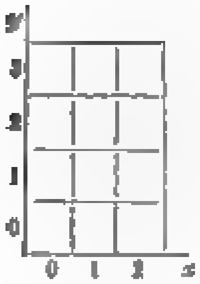
a factor U of a form related to the Knight's move, namely with

$$U = \left(x + \frac{1}{x}\right)\left(y^2 + \frac{1}{y^2}\right) + \left(x^2 + \frac{1}{x^2}\right)\left(y + \frac{1}{y}\right),$$

omitting in the product all terms in which negative powers of x or y and positive powers exceeding the 7^{th} occur; also those that refer to excluded squares. These omissions I denote by inclosing $U_n U$ in square brackets, and call $[U_n U]$ an abridged *product*. Thus we get

$$U_{n+1} = \Sigma w^a{}_{n+1} x^a y^b = [U_n U],$$

an equation which is nothing but a convenient form for embracing the equations (A). An example may serve to indicate some means of shortening the calculation.



In how many different ways is it possible, on a board containing three times four squares, to pass from a corner (0, 0) to the diagonally opposite one (2, 3) in 11 moves, if it is not allowed to touch again the starting-point (0, 0), nor to reach (2, 3) before the last (11^{th}) move?

In this case (0, 0) is excluded altogether and (2, 3) from the first 10 moves. The abridged products are

$$\begin{aligned}
U_1 &= \;[U] \;\; = x^2y + xy^2\\
U_2 &= [U_1U] = x^2 + y^2 + xy^3\\
U_3 &= [U_2U] = x + 2y + 2x^2y + xy^2\\
U_4 &= [U_3U] = 3x^2 + 3y^2 + 3x^2y^2 + 4xy^3\\
U_5 &= [U_4U] = 6x + 10y + 7x^2y + 3xy^2 + 3y^3\\
U_6 &= [U_5U] = 13x^2 + 3xy + 13y^2 + 19x^2y^2 + 17xy^3.
\end{aligned}$$

Instead of continuing in this manner, we may start from (2, 3) and develop a series of five moves, that is to meet the former. Denoting in this case the abridged products by V_1, V_2, &c., we get

$$V_1 = [x^2y^3U] = xy + y^2.$$

On account of the symmetry, it is unnecessary to calculate V_5; for V_n is found from U_n by changing x^a into x^{2-a} and y^b into y^{3-b}; hence we have at once

$$V_5 = 3x^2 + 3xy + 7y^2 + 10x^2y^2 + 6xy^3.$$

Hence since, starting from (0, 0), we can reach (2, 0) in 13 different ways by means of 6 moves, and by 5 moves in

different ways pass from (2, 3) to (2, 0) or from (2, 0) to 2, 3), there are $13 \times 3 = 39$ ways of passing in 11 moves from 0, 0) to (2, 3) under the restrictions of the problem, and occupying the square (2, 0) at the sixth move. If we apply this reasoning to all squares that may be reached in the same sixth move or all that occur in U_6, we get the total number C of ways in which it is possible to reach (2, 3) in the eleventh move, starting from (0, 0) and not touching this square again: we get

$$C = 13.3 + 3.3 + 13.7 + 19.10 + 17.6 = 431.$$

The closer investigation of the numbers $w_n^{(\alpha)}$ leads to certain recurring series, of which they are the terms.

We have, according to (A),

$$\begin{aligned}
w^{(\alpha)}_{n+1} &= w_n^{(\beta)} + w_n^{(\gamma)} + w_n^{(\delta)} + \&c. \\
w_n^{(\alpha)} &= w^{(\beta)}_{n-1} + w^{(\gamma)}_{n-1} + w^{(\delta)}_{n-1} + \&c. \\
&\vdots \\
w^{(\alpha)}_{n+1-i} &= w^{(\beta)}_{n-i} + w^{(\gamma)}_{n-i} + w^{(\delta)}_{n-i} + \&c.
\end{aligned}$$

Multiplying these by $1, \epsilon_1, \epsilon_2 \ldots \epsilon_i$ in order, adding and putting

$$w_n^{(\alpha)} + \epsilon_1 w^{(\alpha)}_{n-1} + \epsilon_2 w^{(\alpha)}_{n-2} + \ldots + \epsilon_i w^{(\alpha)}_{n-1} = W_n^{(\alpha)},$$

we get $\qquad W^{(\alpha)}_{n+1} = W_n^{(\beta)} + W_n^{(\gamma)} + W_n^{(\delta)} + \&c. \ldots . (B),$

an equation which represents i equations, arising from it by putting α successively $= 1, 2 \ldots i$, and then on the right-hand side in each case the indices of the neighbouring squares corresponding to each value of α. If now we determine the i coefficients $1, \epsilon_1 \ldots \epsilon_i$ by as many equations,

$$W^{(\alpha)}_{i+1} = 0 \text{ for } \alpha = 1, 2 \ldots i,$$

we get, by (B),

$$W^{(\alpha)}_{i+2} = 0, \text{ and generally } W_n^{(\alpha)} = 0,$$

for every $n > i$ and every α from 1 to i.

Thus the values of $w_n^{(\alpha)}$ for the several values of n form a recurring series, the scale of which is the same for all values of α. Hence they may also be considered as the coefficients of the development of i rational abgebraic fractions, all of which have the same denominator.

If t denotes an indeterminate quantity, this denominator is

$$\psi t = t^i + \epsilon_1 t^{i-1} + \epsilon_2 t^{i-2} + \&c. \ldots . + \epsilon_i,$$

and we have

$$\frac{\phi t}{\psi t} = \frac{w_1^{(\alpha)}}{t} + \frac{w_2^{(\alpha)}}{t^2} + \ldots = \Sigma \frac{w_n^{(\alpha)}}{t^n},$$

If we extend the meaning of the symbol W in some measure, by putting

$$W_i^{(x)} = w_i^{(x)} + \varepsilon_1 w^{(x)}_{i-1} + \varepsilon_2 w^{(x)}_{i-2} + \ldots + \varepsilon_{i-1} w_1^{(x)},$$

and generally

$$W^{(x)}_{i-k} = w^{(x)}_{i-k} + \varepsilon_1 w^{(x)}_{i-k-1} + \&c. \ldots\ldots + \varepsilon_{i-k-1} w_1^{(x)}$$

for $k = 0, 1, 2 \ldots i-1$, whence

$$W_1^{(x)} = w_1^{(x)},$$

the numerator ϕt will be

$$\phi t = W_i^{(x)} + W^{(x)}_{i-1} t + W^{(x)}_{i-2} t^2 + \ldots\ldots + W_1^{(x)} t^{i-1}.$$

By resolving $\dfrac{\phi t}{\psi t}$ into simple fractions, we obtain an independent expression for $w_n^{(x)}$. If the roots of ψt are all different and be denoted by $t_1, t_2 \ldots t_i$, we find

$$\frac{\phi t}{\psi t} = \frac{a_1}{t - t_1} + \frac{a_2}{t - t_2} + \ldots\ldots + \frac{a_i}{t - t_i},$$

hence $\qquad w_n^{(x)} = a_1 t_1^{n-1} + a_2 t_2^{n-1} + \ldots\ldots + a_i t_i^{n-1}.$

The determination of the quantities ε depended upon the equations

$$W^{(x)}_{i+1} = 0,$$

that is

$$\begin{array}{l} w^{(1)}_{i+1} + \varepsilon_1 w_i^{(1)} + \varepsilon_2 w^{(1)}_{i-1} + \ldots + \varepsilon_i w_1^{(1)} = 0, \\ w^{(2)}_{i+1} + \varepsilon_1 w_i^{(2)} + \ldots\ldots\ldots\ldots + \varepsilon_i w_1^{(2)} = 0, \\ \vdots \qquad \vdots \qquad\qquad\qquad \vdots \\ w^{(i)}_{i+1} + \varepsilon_1 w_i^{(i)} + \ldots\ldots\ldots\ldots + \varepsilon_i w_1^{(i)} = 0; \end{array}$$

hence there exists a definite system of values in all cases, unless the determinant $\Delta_i = \Sigma \pm w_1^{\,1} w_2^{\,2} w_3^{\,3} \ldots w_i^{\,i}$, that forms the denominator of each ε, is $= 0$. But in this case the following i equations may always be satisfied:

$$\left.\begin{array}{l} \Delta_{i-1} w_i^{(1)} + \varepsilon_1 w^{(1)}_{i-1} + \ldots\ldots + \varepsilon_{i-1} w_1^{(1)} = 0 \\ \Delta_{i-1} w_i^{(2)} + \varepsilon_1 w^{(2)}_{i-1} + \ldots\ldots + \varepsilon_{i-1} w_1^{(2)} = 0 \\ \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ \Delta_{i-1} w_i^{(i-1)} + \varepsilon_1 w^{(i-1)}_{i-1} + \ldots\ldots + \varepsilon_{i-1} w_1^{(i-1)} = 0 \\ (\Delta_{i-1} w_i^{(i)} + \varepsilon_1 w^{(i)}_{i-1} + \ldots\ldots + \varepsilon_{i-1} w_i^{(i)} = 0) \end{array}\right\} \ldots.(C),$$

therefore one at least must be a consequence of all the rest; let it be the last, which on this account has been put in brackets. The unknown quantities will now all get the

denominator

$$\Delta_{i-1} = \Sigma \pm w_1^{(1)} w_2^{(2)} w_3^{(3)} \ldots . w^{(i-1)}_{i-1},$$

which in the last system is put as a factor, so that the quantities $\varepsilon_1, \varepsilon_2 \ldots \varepsilon_{i-1}$ now denote only the corresponding numerators. Now if Δ_{i-1} is not $= 0$, the equations $W_n^{(a)} = 0$ commence to hold from $n = i$, and the common scale of the recurring series will contain one term less. But if $\Delta_{i-1} = 0$, another of the equations (C) (let it be the last but one) is a consequence of the remaining ones, and the scale is diminished by one more term. Continuing in this manner, we suppress successively all those equations of the system (C) that are consequences of the remaining ones. Now if all the Δ's from $p = 1$ to $p = i$ were $= 0$, all the equations (C) would be identical, *i.e.* every $w_i^{(a)}$ would $= 0$; that is the Knight would have no possible move at starting. This case may be excluded.

It follows that, if it should be impossible to satisfy the equations $w^{(a)}_{i+1} = 0$, because Δ_i is $= 0$, it will always be possible to satisfy a similar system of equations $w^{(a)}_{j+1} = 0$ for $a = 1, 2 \ldots i$, where j is $< i$; so that we again conclude that all the $w_n^{(a)}$ for different n's form recurring series, one for each value of a, and that all these series have the same scale of i terms at most.

In applications it is easy to distinguish those squares, to which, on account of the symmetry of their position, correspond equal values of w for the same values of n; and if they be denoted by the same index, we avoid the unnecessary accumulation of unknown quantities and the consequent necessity of suppressing identical equations. But when by the exclusion of some squares the symmetry is broken, these simplifications, of which I shall give an example, cease.

On a square board containing 25 squares, the Knight starts from the central field, which he is not allowed to occupy again; in how many ways can he reach each of the other squares, which he may occupy several times, in n moves?

4	3	2	3	4
3	2	1	2	3
2	1	0	1	2
3	2	1	2	3
4	3	2	3	4

The central square (0) being excluded, the remaining 24 may be arranged into four groups, marked by the figs. 1, 2, 3, 4. All squares of the same group have for any value of n the same w's: thus there are only 4 unknown quantities,

$$w_n^{(1)}, \quad w_n^{(2)}, \quad w_n^{(3)}, \quad w_n^{(4)}.$$

equation will be involved. This case is exemplified whenever we require to cause a surface of a given species or character to move in perpetual contact with another fixed surface, through the variation of the arbitrary contents in its equation. The previous cases relate to problems such as the determination of the conditions among the arbitrary contents, in order that a given surface may have a given envelope. All these cases are also related to the inverse problem of maxima and minima.

Of the theorems above stated the first appears to be the most important, and it really constitutes the basis upon which the others rest. It also involves as a particular consequence a theory of reciprocal polars, of which the received theory appears to be a special form.

Let $\phi(x, y, z, x', y', z') = 0$ be the equation of a surface in which x, y, z represent coordinates and x', y', z' parameters, the latter being subject to a condition $\psi(x', y', z') = 0$; and let the equation $\chi(x, y, z) = 0$ be thence deduced as the equation of the envelope of the surface given. Reciprocally, if in the primitive equation we regard x, y, z as parameters subject to the condition $\chi(x, y, z) = 0$, and x', y', z' as coordinates, then will $\psi(x', y', z') = 0$ express the envelope of the surface now represented by the primitive equation.

Suppose, then, that in that equation x, y, z and x', y', z' enter *symmetrically*, so that on changing x, y, z into x', y', z' respectively, and *vice versa*, the equation remains unchanged: also, any point x, y, z being given, let the above equation represent the *polar surface* of that point, x', y', z' being coordinates and x, y, z parameters; whence, by virtue of the symmetry, any point x', y', z' being given, the same equation, with x, y, z as coordinates, will represent the *polar surface* of that point. Now, by the theorem, if the surface $\chi(x, y, z) = 0$ is generated by the intersections of the polar surfaces of all the points contained on the surface $\psi(x', y', z') = 0$, then *reciprocally* the surface $\psi(x', y', z') = 0$ will be generated by the successive intersections of the polar surfaces of the points found on the surface $\chi(x, y, z) = 0$. In the received theory, the polar surface of a point, or pole as it is there termed, is a plane whose equation is of the form $xx' + yy' + zz' = 1$, or, in the most general case, of the form $axx' + byy' + czz' = 1$. The sole requisite condition of symmetry is therefore answered. But there is in the nature of things no necessity for the restriction of the equation of the polar of a point to the first degree. There is no reason why, in Geometry of Two Dimensions, the pole of a point

should not be an ellipse, or in Geometry of Three Dimensions an ellipsoid. I am not aware that the theory of reciprocal polars has been viewed in this universal manner before,—a manner which I think also to be more satisfactory than that which has previously been accepted, as well as more general.*

In the following demonstrations I have sometimes, when it appeared possible to avoid complexity without incurring the danger of misapprehension, substituted the three variables x, y, z for the more general system. Some of the conclusions stated in this paper may, I believe, be obtained by shorter methods than the one which is actually employed; certainly they may if they are regarded solely in their geometrical aspect and limited in expression accordingly. But if considered as purely quantitative relations, the special views of geometry are no longer applicable, and some new principle of investigation is needed. That which I have employed appears to me to be new and in some respects important. It is, that the equation or equations among the parameters are transformations of those which connect the coordinates, or speaking more generally, that the set of quantities among which relation is given, and the set among which relation is sought, are so connected, that the one relation or set of relations is but a transformation of the other. This principle being established, the known doctrines relative to the transformation of functions become applicable, and the subject is placed in that point of view which is apparently the most general.

Reciprocal Methods in the Differential Calculus.

THEOREM I. If two sets of quantities $x_1, x_2 \ldots x_n$ and $a_1, a_2 \ldots a_n$, equal in number, are connected by any relation or relations $u_1 = 0$, $u_2 = 0 \ldots u_r = 0$, and if the former set, further varying in subjection to a condition among themselves, $X = 0$, establish among the set $a_1, a_2 \ldots a_n$ a condition $A = 0$; then, conversely, the set $a_1, a_2 \ldots a_n$, varying in subjection to the condition $A = 0$, will establish among the quantities $x_1, x_2 \ldots x_n$, with which they are connected, the relation $X = 0$.

* Mr. Cayley informs me that the theory of reciprocal polars has actually been presented in this form by Druckenmuller (*Crelle*, tom. xxvi p. 1). The manuscript of the present memoir has been lying by me for about a year and a half, and I have during that period had but little opportunity of consulting foreign journals. [illegible] proper, however, to retain the above brief notice of the theory [illegible] possibly new to some of the readers of the *Journal*, and at [illegible] ding an interesting illustration of one of the general th[illegible] memoir.

The equations in which $x_1, x_2 \ldots x_n$ enter are

$$X = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$u_1 = 0, \quad u_2 = 0 \ldots u_r = 0 \ldots\ldots\ldots\ldots (2).$$

By affecting the latter set of equations with indeterminate multipliers, and adding to the former, we get

$$X + \lambda_1 u_1 + \lambda_2 u_2 \ldots + \lambda_r u_r = 0,$$

which we will represent by $U = 0$, and its differentiation will furnish the system of equations

$$\frac{dU}{dx_1} = 0, \quad \frac{dU}{dx_2} = 0, \quad \frac{dU}{dx_n} = 0 \ldots\ldots\ldots (3).$$

From the $r + n$ equations in (2) and (3), we can determine the values of $\lambda_1, \lambda_2 \ldots \lambda_r$ and $x_1, x_2 \ldots x_n$ in terms of $a_1, a_2 \ldots a_n$, and the substitution of the values thus found for $x_1, x_2 \ldots x_n$ in the equation (1) or $X = 0$ will give a relation among the constants $a_1, a_2 \ldots a_n$ which we may represent by $A = 0$. This is that relation which is so represented in the statement of the theorem, and which we are now to shew possessed of reciprocal properties with C.

Since by the above, A is a transformation of X, we may write

$$X = A,$$

or

$$X - A = 0 \ldots\ldots\ldots\ldots\ldots\ldots (4);$$

and, connecting this equation with the system (2) by indeterminate multipliers, we have

$$X - A + \lambda_1 u_1 + \lambda_2 u_2 \ldots + \lambda_r u_r = 0 \ldots\ldots (5),$$

or

$$U - A = 0.$$

Differentiate this equation with reference to $x_1, x_2 \ldots x_n$ and $a_1, a_2 \ldots a_n$, we get, on transposition,

$$\frac{dU}{dx_1} dx_1 + \frac{dU}{dx_2} x_2 \ldots + \frac{dU}{dx_n} dx_n$$

$$= \frac{d(A - U)}{da_1} da_1 + \frac{d(A - U)}{da_2} da_2 \ldots + \frac{d(A - U)}{da_n} da_n,$$

whence, as the number of the differentials $da_1, da_2 \ldots da_n$ is equal to the number of the differentials $dx_1, dx_2 \ldots dx_n$, the one set moreover being linear functions of the other, it appears that the two systems of equations

$$\frac{dU}{dx_1} = 0, \quad \frac{dU}{dx_2} = 0, \ldots \quad \frac{dU}{dx_n} = 0 \ldots\ldots\ldots (6),$$

$$\frac{d(A-U)}{da_1}=0, \quad \frac{d(A-U)}{da_2}=0, \ldots \quad \frac{d(A-U)}{da_n}=0 \ldots (7),$$

are equivalent; *i.e.* the satisfaction of the one set of relations involves that of the other.

But the set (6) is identical with (3), and its satisfaction furnishes the means of deducing A from X in accordance with the conditions of the problem stated in the outset. Again, if in (7) we write for U its value, viz.

$$X+\lambda_1 u_1+\lambda_2 u_2 \ldots +\lambda_r u_r,$$

and observe that X does not involve $a_1, a_2 \ldots a_n$, we get the system of equations,

$$\frac{d(A-\lambda_1 u_1-\lambda_2 u_2 \ \ldots -\lambda_r u_r)}{da_1}=0,$$

$$\frac{d(A-\lambda_1 u_1-\lambda_2 u_2 \ \ldots -\lambda_r u_r)}{da_2}=0,$$

&c. &c.

$$\frac{d(A-\lambda_1 u_1-\lambda_2 u_2 \ \ldots -\lambda_r u_r)}{da_n}=0.$$

But these are the very equations which we should form if we sought to determine the relation among $x_1, x_2 \ldots x_n$ consequent upon the variations of $a_1, a_2 \ldots a_n$ in the most general manner in subjection to the relation $A=0$, and to the connecting system of relations $u_1=0$, $u_2=0$, $u_n=0$.

Let $P=0$ represent the relation thus formed among $x_1, x_2 \ldots x_n$. Then it appears that the total system of relations necessary to transform X into A is equivalent to the total system necessary to retransform A into P. Wherefore P is equivalent to X, and the functions X and A possess reciprocal properties. Whence the theorem is proved.

Let $\phi(x, y, z, a, b, c)=0$ represent the equation of a proposed surface, x, y, z being the coordinates and a, b, c the constants. If the latter be made to vary in subjection to any condition $\psi(a, b, c)=0$, the form of the surface will vary, and the locus of its successive intersections will in general constitute a new surface $\chi(x, y, z)=0$, which is said to be the envelope of the system of surfaces defined by the equation $\phi(x, y, z, a, b, c)=0$ under the proposed condition. The constants a, b, c, whether to vary in the manner above described, or to be striction, are termed the *parameters*.

To apply to this case the general theorem, it is only necessary to observe that the system of equations (2) here is reducible to the single equation $\phi(x, y, z, a, b, c) = 0$, the equation X will be represented by $\chi(x, y, z) = 0$, and the equation $A = 0$ by $\psi(a, b, c) = 0$. We are thus led to the following theorem.

THEOREM. If $\chi(x, y, z) = 0$ is the equation of the envelope of surface whose equation is $\phi(x, y, z, a, b, c) = 0$, wherein a, b, c are parameters varying in subjection to the sole condition $\psi(a, b, c) = 0$, then is $\psi(a, b, c) = 0$ the equation of the envelope of the surface $\phi(x, y, z, a, b, c) = 0$, when x, y, z are regarded as parameters, subject to the condition $\chi(x, y, z) = 0$, and a, b, c are coordinates.

In other words, if the equation of a curve or surface involve an equal number of coordinates and parameters, the latter being subject to an equation of condition, then, on changing in the primitive equation the parameters into coordinates and the coordinates into parameters, the equation of condition and the equation of the envelope will mutually, but merely, change their places.

Hence to determine the condition under which a given curve or surface in which the number of parameters is equal to that of coordinates shall have a given envelope, it is only necessary to seek the envelope of that surface, regarding the coordinates as parameters and the equation of the envelope as the equation of condition among those parameters.

We might also arrive at the result required, by considering that a proposed curve or surface, which by the variation of its parameters is made to produce a given envelope, is always in contact with that envelope, and therefore has always along the line of contact the same tangent planes as the envelope. I am not sure that we should not by this method arrive somewhat more readily at the rule for determining the required relations among the parameters, but we should, in following this track, in a great measure lose sight of that principle of *reciprocity* which the more purely analytical treatment of the subject sets in so clear a light before us.

Examples.

Ex. 1. Let it be required to determine the condition under which the straight line whose equation is

$$\frac{x}{m} + \frac{y}{n} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

[redacted]

will have for its envelope an ellipse whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ldots\ldots\ldots\ldots\ldots (9).$$

Here the equation of the straight line involves two coordinates x, y, and two parameters m, n, and it is the relation connecting the two latter which is sought.

Regarding then x and y as *parameters*, and considering the equation of the envelope (12) as an equation between those parameters, we get in the ordinary way

$$\frac{1}{m}\,dx + \frac{1}{n}\,dy = 0,$$

$$\frac{x}{a^2}\,dx + \frac{y}{b^2}\,dy = 0,$$

whence $$\frac{y}{mb^2} - \frac{x}{na^2} = 0 \ldots\ldots\ldots\ldots\ldots (10).$$

Eliminating x and y between (8), (9), and (10), we have

$$\frac{a^2}{m^2} + \frac{b^2}{n^2} = 1 \ldots\ldots\ldots\ldots\ldots (11).$$

To verify this it is only necessary to seek the envelope of (8), regarding m and n as parameters subject to the condition (11). The result is (9).

Ex. 2. Under what conditions will the plane whose equation is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \ldots\ldots\ldots\ldots\ldots (12),$$

have for its envelope the surface whose equation is

$$xyz = m^3 \ldots\ldots\ldots\ldots\ldots (13).$$

We have, in obedience to the rule,

$$\frac{dx}{a} + \frac{dy}{b} + \frac{dz}{c} = 0,$$

$$yz\,dx + zx\,dy + xy\,dz = 0;$$

whence, on multiplying the first equation by λ, adding to it the second, and equating to 0 the coefficients of the differentials, we get

$$\frac{\lambda}{a} + y \qquad\qquad \frac{\lambda}{c} + xy = 0,$$

whence $$ayz = bxz = cxy.$$

Eliminating x, y, and z between these equations and the two primitives, we find

$$abc = \frac{m^3}{27} \dots\dots\dots\dots\dots\dots (14)$$

for the condition sought, a result easily verified.

Ex. 3. Required the conditions under which the circle whose equation is $(x-a)^2+(y-b)^2=r^2$ shall have for its envelope the circle whose equation is $x^2+y^2=1$, a and b being the variable parameters.

We have $$(x-a)^2+(y-b)^2=r^2 \dots\dots\dots (15),$$

$$x^2+y^2=1 \dots\dots\dots\dots (16),$$

whence, regarding x and y as parameters,

$$\left.\begin{aligned}(x-a)\,dx+(y-b)\,dy&=0\\ x\,dx+y\,dy&=0\end{aligned}\right\};$$

and, eliminating dx and dy,

$$(x-a)\,y-(y-b)\,x=0,$$

or $$bx=ay \dots\dots\dots\dots\dots\dots (17).$$

From (16) and (17), we have

$$x=\frac{a}{\sqrt{(a^2+b^2)}},\quad y=\frac{b}{\sqrt{(a^2+b^2)}};$$

and substituting these values in (15), and reducing, we find

$$a^2+b^2=(1\pm r)^2 \dots\dots\dots\dots (18)$$

as the condition sought. This implies that the centre of the primitive circle whose radius is r must move along the circumference of another circle whose radius is $1+r$ or $1\sim r$, in order to have for its envelope the proposed circle whose radius is unity. This is obviously true. It will be remarked that the proposed envelope, the circle whose radius is unity, is but part of the entire envelope which would be produced by the above motion, and which would in fact consist of three concentric circles whose respective radii are $1-2r$, 1, and $1+2r$.

As the sole condition recognised in the theorem which the above examples serve to illustrate is that the number of the parameters shall be equal to the number of the coordinates, and as in the last example there occur three

constants a, b, r, and two coordinates x, y; it is apparent that instead of employing a and b as parameters, we might have employed either of the other pairs a, r and b, r, and that the same solution will apply to all the three cases. Thus it appears that the circle represented by (18) will produce the envelope (19) when any two of the constants a, b, r are made to vary in subjection to the condition (18). We have considered the case in which a and b vary. Let us next suppose that a and r are the varying parameters, and merely for convenience let us suppose that r is greater than unity. Then, writing (17) and (18) in the forms

$$(x-a)^2+(y-b)^2-r^2=0 \quad \text{........} \quad (19),$$

$$a^2+b^2-(r\pm 1)^2=0 \quad \text{..........} \quad (20);$$

and, differentiating with reference to a and r, we get

$$-(x-a)\,da-rdr=0,$$

$$ada-(r\pm 1)\,dr=0;$$

and, eliminating the differentials,

$$(x-a)(r\pm 1)+ar=0.$$

If now between this equation and the two primitives (19) and (20) we eliminate a and b, we obtain a result equivalent to the two following equations, viz.

$$x^2+y^2=1,$$

$$x^2+(y-2b)^2=1.$$

The former of these equations agrees with (16), the latter of them represents another circle having the same radius and having its centre situated in the axis of y at a distance $2b$ from the origin. Now it is obvious that the variable circle (19) cannot, under the circumstances supposed, viz. b *not* varying, generate the first of the above circles, which was the one required to be generated, without also generating the second. The following is the true theory of the case.

Whenever we apply our theorem to determine the condition under which a given curve or surface shall by its successive intersections generate a given envelope, we ascertain that condition in the form of an equation among the constants of the given curve or surface. Provided that this equation is satisfied, it is indif[...]h two (in the curve) or which three (in the surf[...]onstants we suppose to vary. In *every* ca[...] envelope, but in *any* case we [...]

branches, and these will differ as different sets of constants are supposed to vary.

A part of these conclusions is indeed sufficiently obvious. We cannot cause a sphere to move so as to produce a given sphere for its envelope, without its also generating some other surface as another envelope. Our analysis gives the common condition for determining both these.

[*To be continued.*]

ON THE SINGULARITIES OF SURFACES.

By Arthur Cayley.

In the following paper, for symmetry of nomenclature and in order to avoid ambiguities, I shall, with reference to plane curves and in various phrases and compound words, use the term "node" as synonymous with double point, and the term "spinode" as synonymous with cusp. I shall, besides, have occasion to consider the several singularities which I call the "flecnode," the "oscnode," the "fleflecnode," and the "tacnode:" the flecnode is a double point which is a point of inflexion on one of the branches through it; the oscnode is a double point which is a point of osculation on one of the branches through it; the fleflecnode is a double point which is a point of inflexion on each of the branches through it; and the tacnode is a double point where two branches touch. And it may be proper to remark here, that a tacnode may be considered as a point resulting from the coincidence and amalgamation of two double points (and therefore equivalent to twelve points of inflexion); or, in a different point of view, as a point uniting the characters of a spinode and a flecnode. I wish to call to mind here the definition of conjugate tangent lines of a surface, viz. that a tangent to the curve of contact of the surface with any circumscribed developable and the corresponding generating line of the developable, are conjugate tangents of the surface.

Suppose, now, that an absolutely arbitrary surface of any order be intersected by a plane: the curve of intersection has not in general any singularities other than such as occur in a perfectly arbitrary curve of the same order; but as a plane can be made to satisfy one, two, or three conditions, the curve may be made to acquire singularities which do not occur in such absolutely arbitrary curve.

Let a single condition only be imposed on the plane. We may suppose that the curve of intersection has a node; the plane is then a tangent plane and the node is the point of contact—of course any point on the surface may be taken for the node. We may if we please use the term "nodes of a surface," "node-planes of a surface," as synonymous with the points and tangent planes of a surface. And it will be convenient also to use the word node-tangents to denote the tangents to the curve of intersection at the node; it may be noticed here that the node-tangents are conjugate tangents of the surface.

Next let two conditions be imposed upon the plane: there are three distinct cases to be considered.

First, the curve of intersection may have a flecnode. The plane (which is of course still a tangent plane at the flecnode) may be termed a flecnode-plane; the flecnodes are singular points on the surface lying on a curve which may be termed the "flecnode-curve,"* and the flecnode-planes give rise to a developable which may be termed the flecnode-develope. The "flecnode-tangents" are the tangents to the curve of intersection at the flecnode; the tangent to the inflected branch may be termed the "singular flecnode-tangent," and the tangent to the other branch the "ordinary flecnode-tangent."

Secondly, the curve of intersection may have a spinode. The plane (which is of course still a tangent plane at the spinode) may be termed a spinode-plane; the spinodes are singular points on the surface lying on a curve which may be termed the "spinode-curve."† And the spinode-planes give rise to a developable which may be termed the "spinode-develope." Also the "spinode-tangent" is the tangent to the curve of intersection at the spinode.

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* The flecnode-curve, defined as the locus of the points through which can be drawn a line meeting the surface in four consecutive points, was, so far as I am aware, first noticed in Mr Salmon's paper On the Triple Tangent Planes of a Surface of the Third Order (*Journal*, tom. IV. p. 252), where Mr. Salmon, among other things, shews that the order of the surface being n, the curve in question is the intersection of the surface with a surface of the order $11n-24$.

† The notion of a spinode, considered as the point where the indicatrix is a parabola (on which account the spinode has been termed a parabolic point) may be found in Dupin's *Developpements de Geometrie*: the most important step in the theory of these points is contained in Hesse's memoir "Ueber die Wendepuncte der Curven dritter Ordnung," (*Crelle*, tom. XXVIII. p. 97), where it is shewn that the spinode-curve is the curve of intersection of the surface supposed as bef[illegible] order n, with a certain surface of the order $4(n-2)$. See also [illegible] memoir "On the Condition that a Plane should touch a Surfa[illegible]" (*Journal*, tom. III. p. 44.)

Thirdly, the curve of intersection may have two nodes, or what may be termed a "node-couple." The plane (which is a tangent plane at each of the nodes and therefore a double tangent plane) may be also termed a "node-couple-plane." The node-couples are pairs of singular points on the surface lying in a curve which may be termed the "node-couple-curve," and the node-couple-planes give rise to a developable which may be termed the "node-couple-develope." The tangents to the curve of intersection at the two nodes of a node-couple might, if the term were required, be termed the "node-couple-tangents." Also one of the nodes of a node-couple may be termed a "node-with-node," and the tangents to the curve of intersection at such point will be the "node-with-node-tangents."

It is hardly necessary to remark that the flecnode-curve is *not* the edge of regression of the flecnode-develope, and the like remark applies *m.m.* to the spinode-curve and the node-couple curve.

Finally, let three conditions be imposed upon the plane: there are six distinct cases to be considered, in each of which we have no longer curves and developes, but only singular points and singular tangent planes determinate in number.

First, the curve of intersection may have an oscnode. The plane (which continues a tangent plane at the oscnode) is an "oscnode-plane." The "oscnode-tangents" are the tangents to the curve of intersection at the oscnode; the tangent to the osculating branch is the "singular oscnode-tangent;" and the tangent to the other branch the "ordinary oscnode-tangent."

Secondly, the curve of intersection may have a fleflecnode. The plane (which continues a tangent plane at the fleflecnode) is a "fleflecnode-plane." The "fleflecnode-tangents" are the tangents to the curve of intersection at the fleflecnode.

Thirdly, the curve of intersection may have a tacnode. The plane (which continues a tangent plane at the tacnode) is a "tacnode-plane." The "tacnode-tangent" is the tangent to the curve of intersection at the tacnode.

Fourthly, the curve of intersection may have a node and a flecnode, or what may be termed a node-and-flecnode. The plane (which is a tangent plane at the node and also at the flecnode, where it is obviously a flecnode-plane) is a "node-and-flecnode-plane." The "node-and-flecnode-tangents," if the term were required, would be the tangents

the curve of intersection at the node and at the flecnode the node-and-flecnode. The node of the node-and-flecnode ay be distinguished as the node-with-flecnode, and the ecnode as the flecnode-with-node, and we have thus the rms "node-with-flecnode-tangents," "flecnode-with-node-ngents," "singular flecnode-with-node-tangent," and "ordi-ary flecnode-with-node-tangent."

Fifthly, the curve of intersection may have a node and o a spinode, or what may be termed a "node-and-spinode." he plane (which is a tangent plane at the node and also tangent plane at the spinode, where it is obviously a inode-plane) is a "node-and-spinode-plane." The node-d-spinode-tangents, if the term were required, would be e tangents at the node and the tangent at the spinode the node-and-spinode to the curve of intersection. The de of the node-and-spinode may be distinguished as the node-with-spinode," and the spinode as the "spinode-with-de," and we have thus the terms "node-with-spinode-ngent," "spinode-with-node-tangent."

Sixthly, the curve of intersection may have three nodes, what may be termed a "node-triplet." The plane (which a triple tangent plane touching the surface at each of e nodes) is a "node-triplet-plane." The "node-triplet-ngents," if the term were required, would be the tangents the curve of intersection at the nodes of the node-triplet. ach node of the node-triplet may be distinguished as a node-with-node-couple," and the tangents to the curve of tersection at such nodes are "node-with-node-couple-tan-nts." The terms "node-couple-with-node," "node-couple-ith-node tangent", might be made use of if necessary.

It should be remarked that the oscnodes lie on the flecnode-rve, as do also the fleflecnodes; these latter points are al double points of the flecnode-curve. The tacnodes are oints of intersection and (what will appear in the sequel) oints of contact of the flecnode-curve, the spinode-curve, nd the node-couple-curve. The spinode-with-nodes are oints of intersection of the spinode-curve and node-couple-urve, and the flecnode-with-nodes are points of intersection the flecnode-curve and node-couple curve; the node-with-ode-couples are real double points (entering in triplets) the node-couple-curve.

Consider for a moment an arbitrary curve on the surface, e locus of the node-tangents at the different points of this rve is in general a skew surface, which may however, in es to be presently considered, degenerate in different ways.

Reverting now to the flecnode-curve, it may be shew that the singular flecnode-tangent coincides with the ta gent of the flecnode-curve. For consider on a surface tw consecutive points such that the line joining them mee the surface in two points consecutive to the first-mention two points. The line meets the surface in four consecuti points, it is therefore a singular flecnode-tangent; *each o* the first-mentioned two points must be on the flecnode-curv or the singular flecnode-tangent touches the flecnode-curv The two flecnode-tangents are by a preceding observati conjugate tangents. It follows that the skew surface, locu of the flecnode-tangents, breaks up into two surfaces, ea of which is a developable, viz. the locus of the singul flecnode-tangents is the developable having the flecnod curve for its edge of regression, and the locus of th ordinary flecnode-tangents is the flecnode-develope. O course at the tacnode, the tacnode-tangent touches th flecnode-curve.

Passing next to the spinode-curve, the spinode-plane an the tangent-plane at a consecutive point along the spinod tangent are identical,* or their line of intersection is in determinate. The spinode tangent is therefore the co jugate tangent to *any* other tangent line at the spinod and therefore to the tangent to the spinode-curve. I follows that the surface locus of the spinode-tangen degenerates into a developable surface twice repeated, vi the spinode-develope. Consider the tacnode as two co incident nodes; each of these nodes, by virtue of its co stituting, in conjunction with the other, a tacnode, is o the spinode-curve; or, in other words, the tacnode-tange touches the spinode-curve, and the same reasoning prov that it touches the node-couple-curve. It has already be seen that the tacnode-tangent touches the flecnode-curv consequently the tacnode is a point not of simple inte section only, but of contact of the flecnode-curve, the spinod curve, and the node-couple-curve.

In virtue of the principle of the spinode-plane bei identical with the tangent plane at a consecutive poi along the spinode tangent, it appears that the tacnode-pla is a stationary plane, as well of the flecnode-develope a of the spinode-develope, and it would at first sight appe that it must be also a stationary tangent plane of the node

* It must not be inferred that the tangent plane at such consecu point is a spinode plane; this is obviously not the case.

develope. But this is not so; the node-with-node-envelope, not the node-couple-develope, but the uple-develope twice repeated: the tacnode-plane is e a stationary plane on such duplicate developable, in any manner on the single developable. The plane is an ordinary tangent plane of the node-develope.

der now a spinode-with-node, which we have seen nt of intersection of the spinode-curve and node-curve. The tangent plane at a consecutive point he spinode-with-node-tangent, is *identical* with the with-node-plane; the curve of intersection of the plane at such consecutive point has therefore a the node-with-spinode, or the tangent plane in is a node-couple-plane, and the point of contact int on the node-couple-curve. Consequently the with-node-tangent touches the node-couple-curve, ce also the spinode-with-node-plane is a stationary plane of the node-couple-develope.

uld be remarked that no circumscribed developable a stationary tangent plane except the tangent planes ints where the curve of contact meets the spinode-nd any one of these planes is only a stationary hen the curve of contact touches the spinode-tangent; the node-couple-curve and the flecnode-curve do sect the spinode-curve except in the points which en discussed.

itulating, the node-couple-curve and the spinode-uch at the tacnodes, and intersect at the spinode-des: moreover, the tacnode-planes are stationary f the spinode-develope, and the spinode-with-node-re stationary planes of the node-couple-develope. this, the two curves are touched at the tacnodes lecnode-curve, and the tacnode-planes are stationary f the flecnode-develope.

ON THE THEORY OF SKEW SURFACES.

By Arthur Cayley.

RFACE of the n^{th} order is a surface which is met by terminate line in n points. It follows immediately urface of the n^{th} order is met by an indeterminate a curve of the [illegible] order.

Consider a skew surface or the surface generated by a singly infinite series of lines, and let the surface be of the n^{th} order. Any plane through a generating line meets the surface in the line itself and in a curve of the $(n-1)^{th}$ order. The generating line meets this curve in $(n-1)$ points. Of these points one, viz. that adjacent to the intersection of the plane with the consecutive generating line, is a unique point; the other $(n-2)$ points form a system. Each of the $(n-1)$ points are *sub modo* points of contact of the plane with the surface, but the proper point of contact is the unique point adjacent to the intersection of the plane with the consecutive generating line. Thus every plane through a generating line is an ordinary tangent plane, the point of contact being a point on the generating line. It is not necessary for the present purpose, but I may stop for a moment to refer to the known theorems that the anharmonic ratio of any four tangent planes through the same generating line is equal to the anharmonic ratio of their points of contact, and that the locus of the normals to the surface along a generating line is a hyperbolic paraboloid. Returning to the $(n-2)$ points in which, together with the point of contact, a generating line meets the curve of intersection of the surface and a plane through the generating line, these are fixed points independent of the particular plane, and are the points in which the generating line is intersected by other generating lines. There is therefore on the surface a double curve intersected in $(n-2)$ points by each generating line of the surface—a property which, though insufficient to determine the order of this double curve, shews that the order cannot be less than $(n-2)$. (Thus for $n=4$, the above reasoning shews that the double-curve must be at least of the second order: assuming for a moment that it is in any case precisely of this order, it obviously cannot be a plane curve, and must therefore be two nonintersecting lines. This suggests at any rate the existence of a class of skew surfaces of the fourth order generated by a line which always passes through two fixed lines and by some other condition not yet ascertained; and it would appear that surfaces of the second order constitute a degenerate species belonging to the class in question).

In particular cases a generating line will be intersected by the consecutive generating line. Such a generating line touches the double curve.

Consider now a point not on the surface, the planes determined by this point and the generating lines of the

surface are the tangent planes through the point, the intersections of consecutive tangent planes are the tangent lines through the point, and the cone generated by these tangent lines or enveloped by the tangent planes is the tangent cone corresponding to the point. This cone is of the n^{th} class. For considering a line through the point, this line meets the surface in n points, *i.e.* meets n generating lines of the surface and the planes through the line and these n generating lines, are of course tangent planes to the cone; that is, n tangent planes can be drawn to the cone through a given line passing through the vertex. The cone has not in general any lines of inflexion, or, what is the same thing, stationary tangent planes. For a stationary tangent plane would imply the intersection of two consecutive generating lines of the surface. And since the number of generating lines intersected by a consecutive generating line, and therefore the number of planes through two consecutive generating lines is finite, no such plane passes through an indeterminate point. The tangent cone will have in general a certain number of double tangent planes; let this number be x. We have therefore a cone of the class n, number of double tangent planes x, number of stationary tangent planes 0. Hence, if m be the order of the cone, a the number of its double lines, and β the number of its cuspidal or stationary lines,

$$m = n(n-1) - 2x,$$

$$\beta = 3n(n-2) - 6x,$$

$$a = \tfrac{1}{2}n(n-2)(n^2-9) - 2x(n^2-n-6) + 2x(x-1).$$

This is the proper tangent cone, but the cone through the double curve is *sub modo* a tangent cone, and enters as a square factor into the equation of the general tangent cone of the order $n(n-1)$. Hence, if X be the order of the double curve, and therefore of the cone through this curve,

$$m + 2X = n(n-1), \quad \text{and therefore } X = x;$$

that is, the number of double tangent planes to the tangent cone is equal to the order of the double curve. It does not appear that there is anything to determine x. And if this is so, skew surfaces of the n^{th} order may be considered as forming different families according to the order of the double curve upon them.

To complete the theory, it should be added that a plane intersects the surface in a curve of the n^{th} order having x double points but no cusps.

ON CERTAIN MULTIPLE INTEGRALS CONNECTED WITH THE THEORY OF ATTRACTIONS.

By Arthur Cayley.

It is easy to deduce from Mr. Boole's formula, given in my paper "On a Multiple Integral connected with the theory of Attractions," *Journal*, tom. II. p. 219, the equation

$$\int \frac{d\xi\, d\eta \dots}{[(\xi - a)^2 + (\eta - \beta)^2 + \dots v^2]^{\frac{1}{2}n-q}}$$

$$= \frac{fg\dots\pi^{\frac{1}{2}n}}{\theta_1^{\,n}\,\Gamma(\frac{1}{2}n - q)\,\Gamma(q+1)} \int_\epsilon^\infty \frac{s^{q-1}(\theta_1^{\,2} - \sigma)^q\, ds}{\sqrt{\left\{\left(s + \dfrac{f^2}{\theta_1^{\,2}}\right)\left(s + \dfrac{g^2}{\theta_1^{\,2}}\right)\right\}\dots}}$$

where n is the number of variables of the multiple integral, and the condition of the integration is

$$\frac{(\xi - a_1)^2}{f^2} + \frac{(\eta - \beta_1)^2}{g^2} + \dots \overline{\gtrless}\ 1.$$

Also where

$$\sigma = \frac{(a - a_1)^2}{s + \dfrac{f^2}{\theta_1^{\,2}}} + \frac{(\beta - \beta_1)^2}{s + \dfrac{g^2}{\theta_1^{\,2}}} \dots + \frac{v^2}{s},$$

and ϵ is the positive root of

$$\theta_1^{\,2} = \frac{(a - a_1)^2}{\epsilon + \dfrac{f^2}{\theta_1^{\,2}}} + \frac{(\beta - \beta_1)^2}{\epsilon + \dfrac{g^2}{\theta_1^{\,2}}} \dots + \frac{v^2}{\epsilon}.$$

Suppose $f = g \dots = \theta_1$, and write $(a - a_1)^2 + \dots = k^2$, we obtain

$$\int \frac{d\xi \dots}{[(\xi - a)^2 + \dots v^2]^{\frac{1}{2}n-q}} = \frac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n - q)\,\Gamma(q+1)} \int_\epsilon^\infty \frac{s^{q-1}(\theta_1^{\,2} - \sigma)^q\, ds}{(1+s)^{\frac{1}{2}n}},$$

the limiting condition for the multiple integral being

$$(\xi - a_1)^2 + \dots \overline{\gtrless}\ \theta_1^{\,2},$$

and the function σ and limit ϵ being given by

$$\sigma = \frac{k^2}{1+s} + \frac{v^2}{s}, \quad \theta_1^{\,2} = \frac{k^2}{1+\epsilon} + \frac{v^2}{\epsilon},$$

ϵ denoting, as before, the positive root. Observing that the quantity under the integral sign on the second side vanishes for $s = \epsilon$, there is no difficulty in deducing, by a differentiation with respect to θ_1, the formula

[redacted]

$$\int \frac{d\Sigma}{[(\xi - a)^2 \ldots + v^2]^{\frac{1}{2}n-q}} = \frac{2\theta_1 \pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n - q)\,\Gamma(q)} \int_\varepsilon^\infty \frac{s^{q-1}(\theta_1^2 - \sigma)^{q-1}\, ds}{(1+s)^{\frac{1}{2}n}},$$

where $d\Sigma$ is the element of the surface $(\xi - a_1)^2 + \ldots = \theta_1^2$, and the integration is extended over the entire surface.

A slight change of form is convenient. We have

$$\theta_1^2 - \sigma = \theta_1^2 - \frac{k^2}{1+s} - \frac{v^2}{s} = \frac{1}{s(1+s)}(\theta_1^2 s^2 + \chi s - v^2),$$

if we suppose $\qquad \chi = \theta_1^2 - k^2 - v^2.$

The formulæ become

$$\int \frac{d\xi \ldots}{[(\xi - a)^2 \ldots + v^2]^{\frac{1}{2}n-q}} = \frac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n - q)\,\Gamma(q+1)} \int_\varepsilon^\infty \frac{(\theta_1^2 s^2 + \chi s - v^2)^q\, ds}{s(1+s)^{\frac{1}{2}n+q}},$$

$$\int \frac{d\Sigma}{[(\xi - a)^2 \ldots + v^2]^{\frac{1}{2}n-q}} = \frac{2\pi^{\frac{1}{2}n}\theta_1}{\Gamma(\frac{1}{2}n - q)\,\Gamma q} \int_\varepsilon^\infty \frac{(\theta_1^2 s^2 + \chi s - v^2)^{q-1}\, ds}{(1+s)^{\frac{1}{2}n+q-1}},$$

in which ε is the positive root of the equation

$$\theta_1^2 \varepsilon^2 + \chi\varepsilon - v^2 = 0.$$

I propose to transform these formulæ by means of the theory of images; it will be convenient to investigate some preliminary formulæ. Suppose $\lambda^2 = \alpha^2 + \beta^2 \ldots$, $\lambda_1^2 = \alpha_1^2 + \beta_1^2 \ldots$. Also consider the new constants $a, b, \ldots, a_1, b_1 \ldots u, f_1$, determined by the equations

$$\frac{\delta^2 \alpha}{\lambda^2 + v^2} = a, \qquad \frac{\delta^2 \alpha_1}{\lambda_1^2 - \theta_1^2} = a_1,$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{\delta^2 v}{\lambda^2 + v^2} = u, \qquad \frac{\delta^2 \theta_1}{\lambda_1^2 - \theta_1^2} = f_1,$$

where δ is arbitrary. Then, putting

$$l^2 = a^2 + b^2 \ldots, \quad l_1^2 = a_1^2 + b_1^2 \ldots,$$

it is easy to see that

$$(\lambda^2 + v^2)(l^2 + u^2) = \delta^4, \quad (\lambda_1^2 - \theta_1^2)(l_1^2 - f_1^2) = \delta^4$$

and

$$\frac{\delta^2 a}{l^2 + u^2} = \alpha, \qquad \frac{\delta^2 a_1}{l_1^2 - f_1^2} = \alpha_1,$$

$$\vdots$$

$$\frac{\delta^2 f_1}{f_1^2} = \theta_1.$$

Proceeding to express the single integrals in terms of the new constants, we have in the first place $k^2 = \delta^4 k^2$, where

$$k^2 = \left(\frac{a}{l^2 + u^2} - \frac{a_1}{l_1^2 - f_1^2}\right)^2 + \dots;$$

or if we write

$$aa_1 + bb_1 \dots = ll_1 \cos\omega,$$

we have

$$k^2 = \frac{l^2}{(l^2 + u^2)^2} + \frac{l_1^2}{(l_1^2 - f_1^2)^2} - \frac{2ll_1 \cos\omega}{(l^2 + u^2)(l_1^2 - f_1^2)}.$$

Hence also $\chi = \delta^4 j$, where

$$j = \frac{f_1^2}{(l_1^2 - f_1^2)^2} - k^2 - \frac{u^2}{(l^2 + u^2)^2},$$

whence

$$-j = \frac{1}{l^2 + u^2} + \frac{1}{l_1^2 - f_1^2} + \frac{2ll_1 \cos\omega}{(l^2 + u^2)(l_1^2 - f_1^2)},$$

$$= \frac{1}{(l^2 + u^2)(l_1^2 - f_1^2)} [p^2 + u^2 - f_1^2],$$

where $p^2 = l^2 + l_1^2 - 2ll_1 \cos\omega$, that is

$$p^2 = (a - a_1)^2 + (b - b_1)^2 + \dots$$

Consequently $\theta_1^2 s^2 + \chi s - v^2 = \delta^4 \Pi$, where Π is given by

$$\Pi = \frac{f_1^2}{(l_1^2 - f_1^2)^2} s^2 - \frac{(p^2 + u^2 - f_1^2)}{(l^2 + u^2)(l_1^2 - f_1^2)} s - \frac{u^2}{(l^2 + u^2)^2}.$$

And it is clear that ε will be the positive root of

$$0 = \frac{f_1^2}{(l_1^2 - f_1^2)^2} \varepsilon^2 - \frac{(p^2 + u^2 - f_1^2)}{(l^2 + u^2)(l_1^2 - f_1^2)} \varepsilon - \frac{u^2}{(l^2 + u^2)^2}.$$

And it may be noticed that, in the particular case of $u = 0$, the roots of this equation are 0, and $\dfrac{(p^2 - f_1^2)(l_1^2 - f_1^2)}{l^2 f_1^2}$. Consequently if $p^2 - f_1^2$ and $l_1^2 - f_1^2$ are of opposite signs, we have $\varepsilon = 0$; but if $p^2 - f_1^2$ and $l_1^2 - f_1^2$ are of the same sign, $\varepsilon = \dfrac{(p^2 - f_1^2)(l_1^2 - f_1^2)}{l^2 f_1^2}$.

In order to transform the double integrals, considering the new variables $x, y \dots$ I write $x^2 + y^2 \dots = r^2$ and

$$\xi = \frac{\delta^2 x}{r^2}, \dots$$

whence also, if $\xi^2 + \eta^2 + \ldots = \rho^2$ (which gives $r\rho = \delta^2$), we have

$$x = \frac{\delta^2 \xi}{\rho^2} \ldots$$

Also it is immediately seen that

$$(\xi - \alpha)^2 + \ldots v^2 = \frac{\delta^4}{(l^2 + u^2)\, r^2} \{(x - a)^2 + \ldots u^2\},$$

$$(\xi - \alpha_1)^2 \ldots - \theta_1^{\,2} = \frac{\delta^4}{(l_1^{\,2} - f_1^{\,2})\, r^2} \{(x - a_1)^2 + \ldots - f_1^{\,2}\}.$$

And from the latter equation it follows that the limiting condition for the first integral is $(x - a_1)^2 + \ldots \gtreqless f_1^{\,2}$ (there is no difficulty in seeing that the sign $<$ in the former limiting condition gives rise here to the sign $>$), and that the second integral has to be extended over the surface $(x - a_1)^2 + \ldots = f_1^{\,2}$. Also if dS represent the element of this surface, we may obtain

$$d\xi\, d\eta \ldots = \frac{\delta^{2n}}{r^{2n}}\, dx\, dy \ldots, \quad d\Sigma = \frac{\delta^{2n-2}}{r^{2n-2}}\, dS;$$

and, combining the above formulæ,

$$\int \frac{dx\, dy \ldots}{(x^2 + y^2 \ldots)^{\frac{1}{2}n+q} \{(x - a)^2 + (y - b)^2 \ldots + u^2\}^{\frac{1}{2}n-q}}$$

$$= \frac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n - q)\, \Gamma(q + 1)\, (l^2 + u^2)^{\frac{1}{2}n-q}} \int_\sigma^\infty \frac{\Pi^q\, ds}{s\,(1 + s)^{\frac{1}{2}n+q}},$$

the limiting condition of the multiple integral being

$$(x - a_1)^2 + (y - b_1)^2 \ldots \gtreqless f_1^{\,2},$$

and

$$\int \frac{dS}{(x^2 + y^2 \ldots)^{\frac{1}{2}n+q-1} \{(x - a)^2 + (y - b)^2 + u^2\}^{\frac{1}{2}n-q}}$$

$$= \frac{2\pi^{\frac{1}{2}n} f_1}{\Gamma(\frac{1}{2}n - q)\, \Gamma q\, (l^2 + u^2)^{\frac{1}{2}n-q} (l_1^{\,2} - f_1^{\,2})} \int_\sigma^\infty \frac{\Pi^{q-1}\, ds}{(1 + s)^{\frac{1}{2}n+q-1}},$$

where dS is the element of the surface $(x - a_1)^2 + (y - b_1)^2 \ldots = f_1^{\,2}$ and the integration extends over the entire surface.

And, recapitulating,

$$l^2 = a^2 + b^2 + \ldots, \quad l_1^{\,2} = a_1^{\,2} + b_1^{\,2} + \ldots, \quad p^2 = (a - a_1)^2 + (b - b_1)^2 + \ldots,$$

$$\Pi = \frac{f_1^{\,2}}{(l^2 - f_1^{\,2})^2}\, s^2 - \frac{(p^2 + u^2 - f_1^{\,2})}{(l_1^{\,2} - f_1^{\,2})(l^2 + u^2)}\, s - \frac{u^2}{(l^2 + u^2)^2};$$

and ε is the positive root of the equation $\Pi = 0$. The only obviously integrable case is that for which in the second formula $q = 1$; this gives

$$\int \frac{dS}{(x^2+y^2\ldots)^{\frac{1}{2}n}\,\{(x-a)^2+(y-b)^2+u^2\}^{\frac{1}{2}n-1}}$$

$$= \frac{2\pi^{\frac{1}{2}n} f_1}{\Gamma(\frac{1}{2}n)\,(l^2+u^2)^{\frac{1}{2}n-1}\,(l_1^2 \sim f_1^2)\,(1+\varepsilon)^{\frac{1}{2}n-1}}.$$

In the case of $u = 0$, we have, as before, when $p^2 - f_1^2$ and $l_1^2 - f_1^2$ are of opposite signs, $\varepsilon = 0$, and therefore $1 + \varepsilon = 1$; but when $p^2 - f_1^2$ and $l_1^2 - f_1^2$ are of the same sign, the value before found for ε gives

$$1 + \varepsilon = \frac{1}{l^2 f_1^2}\{l^2 f_1^2 + (p^2 - f_1^2)(l_1^2 - f_1^2)\}.$$

Consider the image of the origin with respect to the sphere $(x-a_1)^2 + (y-b_1)^2 \ldots = f_1^2$, the coordinates of this image are

$$\frac{a_1}{l_1^2}(l_1^2 - f_1^2), \quad \frac{b_1}{l_1^2}(l_1^2 - f_1), \ldots$$

Consequently, if μ be the distance of this image from the point $(a, b \ldots)$, we have

$$\mu^2 = \left\{a - \frac{a}{l_1^2}(l_1^2 - f_1^2)\right\}^2 + \ldots$$

$$= \frac{1}{l_1^2}\{l^2 f_1^2 + (p^2 - f_1^2)(l_1^2 - f_1^2)\};$$

whence, by a simple reduction,

$$1 + \varepsilon = \frac{l_1^2 \mu^2}{l^2 f_1^2},$$

or the values of the integral are

$$p^2 - f_1^2 \text{ and } l_1^2 - f_1^2 \text{ opposite signs, } I = \frac{2\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}\,\frac{f_1}{l^{n-2}(l_1^2 \sim f_1^2)},$$

$$p^2 - f_1^2 \text{ and } l_1^2 - f_1^2 \text{ the same sign, } I = \frac{2\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}\,\frac{f_1^{n-1}}{l_1^{n-2}\mu^{n-2}(l_1^2 \sim f_1^2)},$$

where μ is the distance from the point $(a, b\ldots)$ of the image of the origin with respect to the sphere $(x-a_1)^2 + \ldots - f_1^2 = 0$.

No. 2, *Stone Buildings,*
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ON THE PRINCIPLES OF THE CALCULUS OF FORMS.

By J. J. Sylvester, Barrister-at-Law.

RT I. SECT. IV.—*Reciprocity, also Properties and Analogies of certain Invariants, &c.*

IT will hereafter be found extremely convenient to repre-nt all systems of variables cogredient with the original stem in the primitive form by letters of the Roman, and l contragredient systems by letters of the Greek alphabet; e rules for concomitance may then be applied without ying any regard to the distinction between the direction the march of the substitutions, the variables at the close each operation as it were telling their own tale in respect being cogredients or contragredients. This distinction s not (as it should have been) been uniformly observed the preceding sections; as, for instance, in the notation emanants which have been derived by the application of e symbol $\left(\xi\frac{d}{dx}+\eta\frac{d}{dy}+\text{\&c.}\right)^i$, instead of the more appro-iate one $\left(x'\frac{d}{dx}+y'\frac{d}{dy}+\text{\&c.}\right)^i$.

The observations in this section will refer exclusively to ints of doctrine which have been started in the preceding ctions in such order as they more readily happen to present emselves. And, first, as to some important applications the reciprocity method referred to in notes (6) and (8) of e Appendix.

The practical application of this method will be found eatly facilitated by the rule that x, y, z, &c. may always any combination of concomitants be replaced respectively $\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}$, &c., and *vice versâ*. I shall apply this olific principle of reciprocity to elucidate some of the pro-rties and relations of Aronhold's S and T, and certain her kindred forms. This S and T are the quartinvariant d sextinvariant respectively of a cubic of three variables. give the names of s and t to the quadrinvariant and cubin-riant of the quartic function of two variables. Further-ore, whoever will consider attentively the remarks made Section II. of the foregoing relative to reciprocal polars, ll apprehend without any difficulty that to every invariant a function of any degree of any number of variables will a contravariant of a function of the same degree one more in number, and that between such

contravariants, whatever relations exist expressed indep- dently of all other quantities, precisely the same relatio must exist between the corresponding contravariants. Th then, to s and t the two invariants of $(x, y)^4$ will correspo two contravariants σ and θ two contravariants to $(x, y,$ and to S and T the two invariants of $(x, y, z)^3$ will correspo Σ and ϑ two contravariants of $(x, y, z, t)^3$. Calling r th resultant of $(x, y)^4$, R the resultant of $(x, y, z)^3$, ρ the pol reciprocal, or, more briefly, the reciprocant of $(x, y, z)^3$, an (R) the reciprocant of $(x, y, z, t)^3$, we have the followi equations (presuming that all the quantities are previous affected with the proper numerical multipliers), viz.

$$r = s^3 + t^2, \qquad \rho = \sigma^3 + \tau^2,$$
$$R = S^3 + T^2, \quad (R) = \Sigma^3 + \vartheta^2,$$

I propose in this First Annotation to point out the remar able analogies which exist between the modes of generati the four pairs of quantities s, t, &c., the functions severa corresponding to which I shall call u, ω, U, Ω. The Hessi corresponding to any of these functions will be denoted b an H prefixed, and when we have to consider, not the pu Hessian, but the matrix formed from it by adding a vertic and horizontal border of variables, the same in number b contragredient to the variable of the function (as, for instanc the Hessian of u bordered with ξ, η horizontally and vert cally, or of U with ξ, η, ζ), then I shall denote the res by the prefix symbol $\overline{H}$, and if there be occasion to add tw borders, as ξ, η, ζ; ξ', η', ζ', both repeated in the horizont and vertical direction, the result will be typified by th prefix $\overline{\overline{H}}$.

Now, in the first place, as observed by me in note (8) o the Appendix in the last number; if we call the coefficient of U (10 in number) a, b, c, d, &c., we have

$$S = \overline{H}\left\{\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}; \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right\} \overline{H}\left\{x, y, z; \xi, \eta, \zeta\right\}$$

also $$T = \frac{dS}{da} \cdot \frac{d^3H}{dx^3} + \frac{dS}{db} \cdot \frac{d^3H}{d^2x.dy} + \frac{dS}{dc} \cdot \frac{d^3H}{d^2xdz} + \&c.$$

I will now add the further important relation

$$S^2 = \frac{dT}{da} \cdot \frac{d^3H}{dx^3} + \frac{dT}{db} \cdot \frac{d^3H}{d^2xdy} + \frac{dT}{dc} \cdot \frac{d^3H}{d^2xdz} + \&c.*$$

* It will be found hereafter convenient to designate contravaria formed in this manner from invariants as Everts of such invariants

[redacted]

so that it will be observed if all the derivatives of S are zero, T is zero, and *vice versâ.*

Precisely in the same way, using h and $\bar{\text{h}}$ to denote respectively the Hessian of u and the same bordered with ξ, η, we have

$$s = \bar{\text{h}}\left(\frac{d}{d\xi}, \frac{d}{d\eta}; \frac{d}{dx}, \frac{d}{dy}\right)\bar{\text{h}}(x, y; \xi, \eta),$$

$$t = \frac{ds}{da} \cdot \frac{d^4h}{dx^4} + \frac{ds}{db} \cdot \frac{d^4h}{dx^3.dy} + \frac{ds}{dc} \cdot \frac{d^4h}{dx^2.dy^2} + \&c.$$

$$s^2 = \frac{dt}{da} \cdot \frac{d^4h}{dx^4} + \frac{dt}{db} \cdot \frac{d^4h}{dx^3.dy} + \frac{dt}{dc} \cdot \frac{d^4h}{dx^2.dy^2} + \&c.$$

Again, taking $(\bar{\bar{H}})$ the second bordered Hessian of Ω; that is, Ω bordered as well horizontally as vertically with the double lines and columns ξ, η, ζ, θ; $\xi', \eta', \zeta', \theta'$,

$$\Sigma = (\bar{\bar{H}})\left(\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}, \frac{d}{d\theta}; \frac{d}{dx}, \frac{d}{dy}, \frac{d}{d\theta}, \frac{d}{dz}; \xi', \eta', \zeta', \theta'\right)$$
$$.(\bar{\bar{H}})(x, y, z, t; \xi, \eta, \zeta, \theta; \xi', \eta', \zeta', \theta'),$$

$$\vartheta = \frac{d\Sigma}{da} \cdot \frac{d^3\bar{H}}{dx^3} + \frac{d\Sigma}{db} \cdot \frac{d^3\bar{H}}{dx^2.dy} + \frac{d\Sigma}{dc} \cdot \frac{d^3\bar{H}}{dx^2.dz} + \frac{d\Sigma}{dd} \cdot \frac{d^3\bar{H}}{dx^2.dt} + \&c.,$$

$$\Sigma^2 = \frac{d\vartheta}{da} \cdot \frac{d^3\bar{H}}{dx^3} + \frac{d\vartheta}{db} \cdot \frac{d^3\bar{H}}{dx^2.dy} + \&c.$$

In like manner again

$$\sigma = (\bar{\bar{h}})\left\{\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}; \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}; \xi', \eta', \zeta'\right\}$$
$$.\bar{\bar{h}}\{x, y, z; \xi, \eta, \zeta; \xi', \eta', \zeta'\},$$

$$\theta = \frac{d\sigma}{da} \cdot \frac{d^4.(\bar{h})}{dx^4} + \&c.,$$

$$\sigma^2 = \frac{d\theta}{da} \cdot \frac{d^4.\bar{h}}{dx^4} + \&c.,$$

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contravariants, and according to the number of times that such process of derivation is applied, 1st, 2nd, 3rd, &c. evects. Such evects form a peculiar class, and when considered generally, without reference to the base to which they refer, they may be termed evectants. Evectants will be again distinguishable according [illegible] invariant simply or a contravariant. Perhaps the te[illegible] evectants may serve to mark this distinction. F[illegible] see Note (1).

σ and ϑ are the same quantities as are calculated by Mr. Salmon, in his inestimable work *On Higher Plane Curves*, but are there expressed under the names of S and T, with the sole difference that in place of x, y, z, used by Mr. Salmon, the contragredient variables ξ, η, ζ are used in the expressions above. Mr. Salmon has also pointed out to me that σ may be obtained by operating with

$$\left(\xi^4 \frac{d}{da} + \xi^3\eta \frac{d}{db} + \xi^3\zeta \frac{d}{dc} + \&c.\right)$$

directly upon I a cubic invariant of the function u, or $(x, y, z)^4$. This I is no other than the simple commutant obtained by operating upon u with the commutantive symbol formed by taking four times over the line $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$; agreeable to the remark made in the third section that every function of an even degree of n variables possesses an invariant of the n^{th} order in extension of Mr. Cayley's observation that every such function of two variables possesses a quadrinvariant, *i.e.* an invariant of the second order.

I need hardly remark that σ is of 2 dimensions in the coefficients and of 4 in the contragredient variables, τ of 3 in the coefficients and of 5 in the contragredients, Σ of 4 in the constants and 4 in the contragredients, ϑ of 6 in the constants and 6 in the contragredients, or that the single-bordered Hessians of u and U and the double-bordered Hessians of ω and Ω are each of them quadratic in respect of the x &c. as well as of the ξ &c. systems.

If the right numerical factors be attributed to S, T, Aronhold has shewn that

$$H.H(U) + T.H(U) + S^2.U = 0,$$

and in my paper in the last May No., I gave the equation

$$h.h(u) + s.h(u) + t(u) = 0.$$

I think it highly probable that it will be found that the analogous equations obtain, viz.

$$(H)(\bar{H})\,\Omega + \vartheta.\bar{H}\Omega + \Sigma^2\Omega = 0,$$

$$(h)\,(\bar{h})\,\omega + \sigma.(\bar{h}).\omega = \vartheta\omega = 0.$$

These remarkable equations, if verified, (of which I can scarcely doubt) will be most powerful aids to the dissection of the forms ω, Ω, and thereby to the detection of the

fundamental properties of curves of the fourth and surfaces of the third degree, of which at present so little is known. It will have been observed that in the preceding developments the contravariants of ω and Ω were derived in precisely the same way from ω and Ω as the corresponding invariants of u and U from u and U, with the sole difference that the Hessian used in the two latter cases is replaced by a single-bordered Hessian in the two former cases, and a single-bordered Hessian in the two latter by a double-bordered Hessian in the two former. The analogies are not even yet stated exhaustively; for it will be remembered (as shewn in the third section), that T and S can be derived directly and concurrently by means of operating with the commutantive symbol

$$\left.\begin{matrix} \dfrac{d}{dx}, & \dfrac{d}{dy}, & \dfrac{d}{dz} \\ \dfrac{d}{dx}, & \dfrac{d}{dy}, & \dfrac{d}{dz} \\ \dfrac{d}{d\xi}, & \dfrac{d}{d\eta}, & \dfrac{d}{d\zeta} \\ \dfrac{d}{d\xi}, & \dfrac{d}{d\eta}, & \dfrac{d}{d\zeta} \end{matrix}\right| \text{ upon } \overline{H}(U) + \lambda(x\xi + y\eta + z\zeta)^2,$$

which gives a result of the form $m\{\lambda^2 + S\lambda + T\}$, m being a number, and I conjecture that if

$$\begin{matrix} \dfrac{d}{dx}, & \dfrac{d}{dy}, & \dfrac{d}{dz}, & \dfrac{d}{dt} \\ \dfrac{d}{dx}, & \dfrac{d}{dy}, & \dfrac{d}{dz}, & \dfrac{d}{dt} \\ \dfrac{d}{d\xi}, & \dfrac{d}{d\eta}, & \dfrac{d}{d\zeta}, & \dfrac{d}{d\theta} \\ \dfrac{d}{d\xi}, & \dfrac{d}{d\eta}, & \dfrac{d}{d\zeta}, & \dfrac{d}{d\theta} \end{matrix}$$

be made to operate upon

$$\overline{\overline{H}}\Omega + \lambda(x\xi + y\eta + z\zeta + t\theta)^2,$$

and the result be put under the form

$$m(\lambda^4 + A\lambda^3 + B\lambda^2 + C\lambda + D),$$

that A will be zero, B and C will be respectively Σ and ϑ, and perhaps ontravariant, if it effectively exist, of

8 dimensions in the coefficients of Ω, and of a like number in the contragredients ξ', η', ζ', θ'), also zero. But of the evanescence of D I do not speak with any degree of assurance.

Mr. Salmon has made an excellent observation to the effect that if we call (σ) what σ becomes when ξ', η', ζ' are replaced by $\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{d}{dz}$, $(\sigma)\,h(\omega)$ will represent a covariant to ω of $3+2$, that is, 5 dimensions in the coefficients, and of $6-4$, that is, of 2 dimensions in x, y, z, $h(\omega)$ being of 3 and 6 dimensions in these respectively, and σ of 2 and 4 dimensions respectively in the same. Now these resulting dimensions 5 and 2 precisely agree with the form especially noticed by me in Note (2) of the Appendix, where it was derived as one of a group by the method of unravelment.* There can be little doubt that these two conics each of them indissolubly connected with every curve of the fourth degree are identical. The form $(\sigma)\,h(\omega)$ enables us to prove readily (thanks to Mr. Salmon's calculation of σ, given in his *Higher Plane Curves*, p. 101, under the name of S) that this is a *bonâ fide* existent conic.

For if we take a particular case of ω, say

$$\omega = a_1x^4 + b_2y^4 + c_3z^4 + 6dy^2z^2,$$

we find

$$h(\omega) = \begin{vmatrix} a_1x^2 & 0 & 0 \\ 0 & b_2y^2 + dz^2 & dyz \\ 0 & dyz & c_3z^2 + dy^2 \end{vmatrix}$$

$$= a_1(b_2c_3 + d^2)\,x^2y^2z^2 + a_1b_2.d.x^2y^4 + a_1c_3dx^2z^4,$$

and σ becomes $\qquad a_1d\eta'^2\zeta'^2,$

and consequently (σ) is

$$a_1d\left(\frac{d}{dy}\right)^2.\left(\frac{d}{dz}\right)^2,$$

and therefore $\quad (\sigma)\,h(\omega) = 4a_1^2d(b^2c^3 + d^2)\,x^2,$

the conic here reducing to a pair of coincident straight lines. This example demonstrates that the conic is in general actually existent.

As I have said so much upon S and T it may not be irrelevant to state in this place how I obtained the ditions for U, the characteristic of the curve of th

* Vide Note (2).

degree becoming the characteristic of a conic and a straight line, *i.e.* breaking up into a linear and a quadratic factor, which Mr. Salmon has inserted in the notes to his work above referred to. When U is of this form it may obviously by linear transformations be expressed by $ax^3 + bdxyz$, but when starting with the general form,

$$a_1x^3 + b_2y^3 + c_3z^3 + \&c. + 6Dxyz,$$

we form two contravariants from S and T, to wit

$$\left(\xi^3 \frac{d}{da_1} + \eta^3 \frac{d}{db_2} + \zeta^3 \frac{d}{dc_3} + \&c. + \xi\eta\zeta \frac{d}{dD}\right) S, \text{ say } S',$$

$$\left(\xi^3 \frac{d}{da_1} + \eta^3 \frac{d}{db_2} + \zeta^3 \frac{d}{dc_3} + \&c. + \xi\eta\zeta \frac{d}{dD}\right) T, \text{ say } T',$$

and then make $a_1 = a$, $D = d$, and all the other coefficients zero, it will easily be seen on examining the forms of S and T, given by Mr. Salmon (pp. 184, 186), that (S) and (T) (the evectants of S and T) become respectively

$$4d^3\xi\eta\zeta, \quad 31d^3\xi\eta\zeta;$$

we have therefore $(T) + \lambda(S) = 0$: and (T) and (S), although contravariantive to their primitive U, are covariantive with one another, so that $(T) + \lambda(S) = 0$ is a persistent relation unaffected by linear transformations; it follows therefore that when U is of, or reducible to, the form supposed,

$$\frac{dS}{da_1} : \frac{dS}{db_2} : \frac{dS}{dc_3} : \&c. : \frac{dS}{dD}$$

$$= \frac{dT}{da_1} : \frac{dT}{db_2} : \frac{dT}{dc_3} : \&c. : \frac{dT}{dD},$$

which is the criterion given in the note referred to.*

I am also able to obtain these equations more directly by another method found upon a New View of the Theory of Elimination, an account of which, however, I must reserve for another occasion, but which, I may mention, serves to fix not merely the conditions, as in the ordinary restricted theory, that a given set of equations may be simultaneously satisfiable by some one system of values of the variables, but the *conditions* that such set of equations may be simultaneously satisfiable by any given number of distinct systems of variables.

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* Mr. Salmon has remarked that the two evectants (S) and (T) intersect in the nine cuspidal points of the polar reciprocal to the curve.

Mr. Salmon has remarked to me to the effect th
in τ we write $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$, in place of the contragredi
and call τ so altered (τ), then $(\tau)\,h(\omega)$ will be an inv
of 6 dimensions in the coefficients of ω. This sext-inv
I have little doubt is identical with that obtained by oper
upon ω with the commutation symbol

$$\begin{array}{llllll} \left(\frac{d}{dx}\right)^2, & \frac{d}{dx}\cdot\frac{d}{dy}, & \left(\frac{d}{dy}\right)^2, & \frac{d}{dy}\cdot\frac{d}{dz}, & \left(\frac{d}{dz}\right)^2, & \frac{d}{dz}\cdot\frac{d}{dz} \\ \left(\frac{d}{dx}\right)^2, & \frac{d}{dx}\cdot\frac{d}{dy}, & \left(\frac{d}{dy}\right)^2, & \frac{d}{dy}\cdot\frac{d}{dz}, & \left(\frac{d}{dz}\right)^2, & \frac{d}{dz}\cdot\frac{d}{dx} \end{array}$$

This, like every other commutant of 2 lines only, is of co
capable of being expressed under the form of an ordi
determinant, and the remark is not without interest
shewing how the proposition known with respect to q
ratic functions of any number of variables, viz. of every
having an invariantive determinant, lends itself to the gen
case of functions of any even degree of any numbe
variables which also have always an invariantive determin
attached to them, of which the terms are simple coeffici
of such functions. The only peculiarity (if it be one)
quadratic functions in this respect being that they have
but one invariant of such form and no other. In the
before us, if we write

$$\begin{aligned} \omega = {} & a_1x^4 + b_2y^4 + c_3z^4 + 4a_2x^3y + 4a_3x^3z + 4b_1y^3x + 4b_3y^3z \\ & + 4c_1z^3x + 4c_2z^3y + 6dy^2z^2 + 6ez^2x^2 + 6fx^2y^2 + 12lx^2yz \\ & + 12mxy^2z + 12nxyz^2, \end{aligned}$$

the sext-invariant in question becomes representable un
the form of the determinant

$$\begin{array}{cccccc} a_1 & a_2 & f & l & e & a_3 \\ a_2 & f & b_1 & m & n & l \\ f & b_1 & b_2 & b_3 & d & m \\ l & m & b_3 & d & c_2 & n \\ e & n & d & c_2 & c_3 & c_1 \\ a_3 & l & m & n & c_1 & e \end{array}\text{*}$$

* This determinant is identical with the determinant formed by t
the second differential coefficients of the function and arranging in
usual manner the coefficients of the several powers and combinatio
powers of the variables treated as if they were independent quantities.

ore quitting the subject of S and T the two invariants he cubic function of 3 variables, or, as it may be termed, the cubic curve, it may not be amiss to give the complete le which I have formed corresponding to all the singular es which can befall such curve, which will be seen below be eight in number; it is of the highest importance to sh forward the advanced posts of geometry, and for this rpose to obtain the same kind of absolute power and thority over, and clear and absolute knowledge of, the operties and affections of cubic forms as have been already ttained for forms of the second degree.

Let $U = ax^3 + 4bx^2y + 4cx^2z + \&c.$

(1). When U has one double point $S^3 + T^2 = 0$.

(2). When U has two double points, *i.e.* becomes a conic and right line

$$\frac{dS}{da} \cdot \frac{dT}{db} - \frac{dS}{db} \cdot \frac{dT}{da} = 0, \ \&c. \ \&c.$$

(3). When U has a cusp $S = 0, \ T = 0$.

(4). When U has two coincident double points, *i.e.* is a conic and a tangent line thereto, which comprises the two preceding cases in one,

$$\frac{dT}{da} = 0, \quad \frac{dT}{db} = 0, \ \&c.$$

and also *therefore* $\qquad S = 0.$

(5). When U becomes three right lines forming a triangle

$$\frac{d^2S}{da.db} \cdot \frac{d^2T}{dc.de} - \frac{d^2T}{da.db} \cdot \frac{d^2S}{dc.de} = 0, \ \&c.$$

where a, b, c, e each represent any of the coefficients arbitrarily chosen, whether distinct or identical.

Another, and lower in degree system of equations, may be substituted for the above, obtained by affirming the equality of the ratios between the coefficients of U and the corresponding coefficients of its Hessian.

(6). When U represents a pencil of three rays meeting in a point

$$\frac{dS}{da} = 0, \quad \frac{dS}{db} = 0, \ \&c.$$

erefore $\qquad T = 0.$

Also in place of this system may be substituted the system obtained by taking all the coefficients of the Hessian zero.

(7). When (U) becomes a line, and two other coincident lines,

$$\frac{dS}{da} = 0, \quad \frac{dS}{db} = 0, \text{ \&c.}$$

and also

$$\frac{d^2T}{da} = 0, \quad \frac{dT}{da.db} = 0, \text{ \&c.}$$

I have not ascertained whether this second system necessarily implies the first; I rather think that it does not. In the preceding case also it would be interesting to shew the direct algebraical connexion between the system formed by the coefficients of the Hessian and the system consisting of the first derivatives of S.

(8). When U becomes a perfect cube representing three coincident right lines

$$\frac{d^2S}{da^2} = 0, \quad \frac{d^2S}{da.db} = 0, \text{ \&c.}$$

and

$$\frac{d^2T}{da^2} = 0, \quad \frac{d^2T}{da.db} = 0, \text{ \&c.}$$

The first of these systems of equations necessarily implies the equations $\frac{dT}{da} = 0$, $\frac{dT}{db} = 0$, &c., as is obvious from the equation

$$T = \frac{dS}{da} \cdot \frac{d^3H}{dx^3} + \frac{dS}{db} \cdot \frac{d^3H}{dx^2.dy}, \text{ \&c.}$$

but not necessarily the second and lower system $\frac{d^2T}{da^2} = 0$, &c. above written.

So if we take

$$u = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4:$$

when 2 roots are equal $s^3 + t^2 = 0$,

when 2 pairs of roots are equal

$$\frac{ds}{da} \cdot \frac{dt}{db} - \frac{ds}{db} \cdot \frac{dt}{da} = 0, \text{ \&c.}$$

when 3 roots are equal $s = 0$, $t = 0$,

and when all 4 roots are equal

$$\frac{dt}{da} = 0, \quad \frac{dt}{db} = 0, \text{ \&c.}$$

Before closing this Section I may make a remark, in reference to the sextic invariant of ω, which admits of being extended to all commutants formed by operating upon the function with a commutantive symbol obtained by writing over one another lines consisting of powers of $\frac{d}{dx}$, $\frac{d}{dy}$, &c. and their combinations (to which, in the third section, I gave the name of *compound* commutants, a qualification which, for reasons that will hereafter be adduced, I think it advisable to withdraw). The remark I have to make is this, viz. that the invariant obtained by operating upon ω with

$$\left.\begin{matrix} \left(\frac{d}{dx}\right)^2 & \frac{d}{dx}\cdot\frac{d}{dy} & \left(\frac{d}{dy}\right)^2 & \frac{d}{dy}\cdot\frac{d}{dz} & \left(\frac{d}{dz}\right)^2 & \frac{d}{dz}\cdot\frac{d}{dx} \\ \left(\frac{d}{dx}\right)^2 & \frac{d}{dx}\cdot\frac{d}{dy} & \left(\frac{d}{dy}\right)^2 & \frac{d}{dy}\cdot\frac{d}{dz} & \left(\frac{d}{dz}\right)^2 & \frac{d}{dz}\cdot\frac{d}{dx} \end{matrix}\right\}$$

is precisely the same as may be obtained by operating with

$$\left.\begin{matrix} \frac{d}{du}, & \frac{d}{dv}, & \frac{d}{dw}, & \frac{d}{dp}, & \frac{d}{dq}, & \frac{d}{dr} \\ \frac{d}{du}, & \frac{d}{dv}, & \frac{d}{dw}, & \frac{d}{dp}, & \frac{d}{dq}, & \frac{d}{dr} \end{matrix}\right\}$$

upon the concomitant quadratic function to ω obtained by the method of unravelment, as in Note (2) of the Appendix, (p. 93); and so, in general, every commutant obtained by operating upon a function of any number of variables of the degree $2mp$ with a symbol consisting of $2p$ lines in which the m^{th} powers of $\frac{d}{dx}, \frac{d}{dy}$, &c. and their m^{th} combinations occur, will be identical with the commutant obtained by operating with a symbol also of $2p$ lines, in which only the simple powers occur of $\frac{d}{du}$, $\frac{d}{dv}$, &c. (where u, v, &c. are cogredient with x^p, $x^{p-1}y$, &c.) upon a function of u, v, &c., formed by the method of unravelment from the given function.

Finally, before quitting the subject of reciprocity, I may state, it follows from the general statement made at the commencement of this Section, that inasmuch as

$$(x\xi + y\eta + z\zeta, \text{\&c.})^2$$

is a universal concomitant form, so also must

$$\left(\frac{d}{d\xi}\cdot\frac{d}{dx} + \frac{d}{d\eta}\cdot\frac{d}{dy} + \frac{d}{d\zeta}\cdot\frac{d}{dz}, \text{\&c.}\right)^2$$

be an universal concomitant symbol of operation; accordingly it is certain that any concomitant in which x, y, z, &c., ξ, η, ζ, &c. enter, operated upon with this symbol, will remain a concomitant: in several cases which I have examined, the effect of this operation will be to produce an evanescent form, but I see no ground for supposing that this is other than an accidental, or at all events for supposing that it is a necessary and universal consequence of the operation. It may also be observed that in the case of as many cogredient sets of variables as variables in each set, as for instance 3 sets of 3 variables each, the determinant which may be formed by arranging them in regular order, as $\begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ x'', & y'', & z'' \end{vmatrix}$ is evidently a universal concomitant, and moreover an equivocal concomitant, possessing the property of remaining a concomitant when the variables are respectively but simultaneously exchanged for their contragredients ξ, η, ζ; ξ', η', ζ'; ξ'', η'', ζ''; which shews also that in place of the variables may be written the differential operators

$$\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}; \frac{d}{dx'}, \frac{d}{dy'}, \frac{d}{dz'}; \frac{d}{dx''}, \frac{d}{dy''}, \frac{d}{dz''}:$$

a remark which leads us to see the exact place in the general theory occupied by Mr. Cayley's method of generating covariants given in the concluding paragraph of the first section, page 59. I may likewise add, that inasmuch as $(x'\xi + y'\eta + z'\zeta, \text{ \&c.})^r$ is a universal concomitant,

$$\left(x' \frac{d}{dx} + y' \frac{d}{dy} + \text{\&c.}\right)^r$$

will be so too, by virtue of the general law of interchange, which conducts immediately to the theory of emanation, shewing that this last symbol, operating upon any function, furnishes covariants thereunto for any integer value of z.

One additional interesting remark presents itself to be made concerning U, the cubic function of x, y, z, which is, that calling as before T its sextic invariant, and a, $3b$, $3c$, d, &c. the coefficient,

$$\left(\xi^3 \frac{d}{da} + \xi^2\eta \frac{d}{db} + \xi^2\zeta \frac{d}{dc} + \xi\eta\zeta \frac{d}{d.d} \text{ \&c.}\right)^2 . T$$

will give the polar reciprocal, or, as it has been agreed to term it, the reciprocant of U. I believe the remark of the probability of this being the case originated with myself, but Mr. Cayley, first verified it by actual calculation, using for that

purpose the value of T, given by Mr. Salmon in his work *On the Higher Plane Curves*, already frequently alluded to, and which is an indispensable manual equally for the objects of the higher special geometry as for the new or universal algebra, being in fact a common ground where the two sciences meet and render mutual aid.

Mr. Salmon also observed, that the first evect of T, viz. $\left(\xi^3 \frac{d}{da} + \xi^2\eta \frac{d}{db} \text{ \&c.}\right)T$, was identical in form with what may be termed the first devect of the polar reciprocal, *i.e.* the result of operating upon the polar reciprocal with what U becomes when $\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}$, are substituted in the stead of x, y, z. And inasmuch as, by Euler's law,

$$\left\{a\left(\frac{d}{d\xi}\right)^3 + 3b\left(\frac{d}{d\xi}\right)^2 \cdot \frac{d}{d\eta} \text{ \&c.}\right\} \times \left\{\xi^3 \frac{d}{da} + \xi^2.\eta \frac{d}{db} \text{ \&c.}\right\} T$$

$$= 6\left\{a \frac{d}{da} + b \frac{d}{db} + \text{\&c.}\right\} T = 36\,T,$$

it follows that T is the second *devect* of the polar reciprocal, or at least identical with it in point of form. But, since the preceding matter was printed, I have discovered in the course of a most instructive and suggestive correspondence with Mr. Salmon, the principle upon which these and similar identifications depend, thereby dispensing with the necessity for the excessively tedious labour of verification which, even in the simple example before us, would be found to extend over several pages of work.

The theory in which this principle is involved will be given, along with other very important matter, in the next number of the *Journal*.

Supplementary Observations on the Method of Reciprocity.

It has been observed, that ξ, η, &c. may always be inserted in place of $\frac{d}{dx}, \frac{d}{dy}$, &c., and *vice versâ*, in a concomitant form, without destroying its concomitance. Accordingly, instead of the evector symbol

$$\xi^3 \frac{d}{da} + \ldots \frac{d}{\ldots} \text{ \&c.},$$

we may employ

$$\left(\frac{d}{dx}\right)^3 \cdot \frac{d}{da} + \left(\ldots\right.$$

comitant to a respondent is a concomitant to its primitive. When the inverse of any concomitant to the respondent is made to operate upon the same concomitant of the primitive, it will be found that the result is a power of the universal concomitant. If the concomitant to the respondent be an invariant thereof, the rule indicates that on merely replacing in the respondent a, b, c, &c. by $\frac{d}{da}, \frac{d}{db}, \frac{d}{dc}$, &c., the result operating on any invariant or other concomitant of the primitive, leaves it still an invariant or other concomitant. For instance, if we take the function $ax^5 + 5bx^4y + wcx^3y^2 + 10dx^2y^3 + 5ex^1y^4 + fy^5$, which has three invariants L, M, N, of the degrees 4, 8, 12, respectively: and if we call λ, μ, ν what L, M, N become when, in place of a, b, c, d, e, f respectively, we write

$$\frac{d}{da}, \frac{1}{5}\cdot\frac{d}{db}, \frac{1}{w}\cdot\frac{d}{dc}, \frac{1}{10}\cdot\frac{d}{d.d}, \frac{1}{5}\cdot\frac{d}{d.e}\,\frac{d}{d.f},$$

we shall find that $\lambda.M = L$, $\mu N = L$,

and $\qquad \lambda.N =$ a linear function of M and L^2.

Again, if in the case of any function of x, y, z, &c., we take, instead of any other concomitant to the respondent, the respondent itself, its inverse gives the symbol of operation

$$\left(\frac{d}{da}\right)\cdot\left(\frac{d}{dx}\right)^3 + \frac{d}{db}\left(\frac{d}{dx}\right)^2\cdot\left(\frac{d}{dy}\right), \text{ \&c.,}$$

just previously treated of. If again, in the case of a function of x, y, say

$$ax^n + nbx^{n-1}.y + \text{\&c.} + nb'.xy^{n-1} + a'y^n,$$

we take the inverse of the polar reciprocal of the respondent, we get the operator

$$\frac{d}{da}\cdot\left(\frac{d}{d\eta}\right)^n - b\left(\frac{d}{d\eta}\right)^{n-1}\cdot\frac{d}{d\xi} + \text{\&c.};$$

and replacing $\frac{d}{d\eta}, \frac{d}{d\xi}$ by y, x, we find that

$$y^n\cdot\frac{d}{da} - y^{n-1}.x\,\frac{d}{db}, \text{ \&c.,}$$

operating on any concomitant, leave it still a concomitant, which is M. Eisenstein's theorem before adverted to, only generalized by the introduction of any concomitant in lieu of the discriminant.

This extraordinary theorem of respondence will be found on reflection to favour the notion of treating the coefficients of a general function as themselves, a system of variables, in a manner contragradient to the terms to which they are affixed.

Finally, there is yet another mode of applying the principle of reciprocity, which must be carefully distinguished from any previously stated in these pages.

I have said that in place of the quantitative symbols of one alphabet, as x, y, z, &c., we may always substitute the operation symbols $\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}$, &c. of the opposite alphabet. But now I say, in place of the quantitative symbols x, y, z, &c. occurring in the concomitant to any form f, may be substituted the quantities (observe, no longer operative symbols but quantities) $\frac{dF}{d\xi}, \frac{dF}{d\eta}, \frac{dF}{d\zeta}$, &c., F being itself any concomitant to f. Thus, for instance, taking F identical with f, we see that $f\left(\frac{df}{d\xi}, \frac{df}{d\eta}, \frac{df}{d\zeta}, \&c.\right)$ is concomitant to f: or again, if f be a function of x, y only, say $f(x, y)$, taking F the polar reciprocal of f, i.e. $f(-\eta, \xi)$, we see that $f\left(-\frac{df}{dx}, \frac{df}{dy}\right)$ will be a concomitant to f: this concomitant, by the way it may be observed, will always contain f as a factor, because when $f = 0$, $x\frac{df}{dx} + y\frac{df}{dy} = 0$. Possibly it may be true that, when f is a function of any number of variables x, y, z, &c., and $F(\xi, \eta, \zeta, \&c.)$ its polar reciprocal,

$$f\left(\frac{dF(x, y, z, \&c.)}{dx}, \frac{dF(x, y, z, \&c.)}{dy}, \&c.\right),$$

which is a concomitant to f, contains f as a factor; but I have not had time to see how this is. It is rather singular that Mr. Cayley and Professor Borchardt of Berlin have both independently made to me the observation that, when $f(x, y)$ is taken a cubic function of x and y, $f\left(\frac{df}{dy}, \frac{-df}{dx}\right)$ is equal to the product of f by the first evectant of the discriminant of f. The general consideration of the consequences of this new and important application of the idea of reciprocity, must be reserved for a future section.

N.B. The reader is requested to take notice that Note (1) referred to in the marginal note at p. 181 is suppressed, and that Note (2) referred to in the marginal note to p. 184 will be given as Note (9), as it has been thought better to make the numbering of the Appendix notes run on from the former sections.

SECTION V.

Applications and Extension of the Theory of the Plexus.

If $$\phi = ax^4 + 4bx^3y + 6cx^2y^2 - 4dxy^3 + ey^4,$$

we can obtain, by operating catalectically with x', y' upon

$$\left(x'\frac{d}{dx} + y'\frac{d}{dy}\right)^2\phi, \quad \left(x'\frac{d}{dx} + y'\frac{d}{dy}\right)^4\phi,$$

the two concomitants

$$\begin{vmatrix} ax^2 + 2bxy + cy^2, & bx^2 + 2cxy + dy^2 \\ bx^2 + 2cxy + dy^2, & cx^2 + 2dxy + ey^2 \end{vmatrix} \ldots\ldots (1),$$

$$\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

the one in fact being the Hessian, the other the catalecticant of ϕ itself. Again, if

$$\phi = ax^5 + 5bx^4y + 10cx^3y^2 + \&c. + fy^5,$$

by operating catalectically with x', y' upon the second and fourth emanants, as in the last case, we obtain the two covariants

$$\begin{vmatrix} ax^3 + 3bx^2y + 3cxy^2 + dy^3, & bx^3 + 3cx^2y + 3dxy^2 + ey^3 \\ bx^3 + 3cx^2y + 3dxy^2 + ey^3, & cx^3 + 3dx^2y + 3exy^2 + fy^3 \end{vmatrix} \ldots (1),$$

$$\begin{vmatrix} ax + by, & bx + cy, & cx + dy \\ bx + cy, & cx + dy, & dx + ey \\ cx + dy, & dx + ey, & ex + fy \end{vmatrix} \ldots\ldots (2),$$

which are in fact the Hessian and Canonizant respectively of ϕ. So in general, for a function of x, y of the degree 2ι or $2\iota + 1$, we can obtain ι covariantive forms, the first being the Hessian, and the last the catalecticant on the first supposition and the canonizant on the second: calling the index of the function for either case n, the forms appearing in this scale will be of the degree $(r + 1)$ in the constants, and of the degree $(r + 1)(n - 2r)$ in x and y.

In the foot note of page 96 of the present volum *Journal*, I intimated that all these determinants of a remarkable transformation.

This transformation may be expressed more eleg dealing not directly with the covariant forms as above but with their polar reciprocants obtained immedi writing ξ for $-y$ and η for x.

(1). Suppose $\phi = ax^3 + 2x^2y + 3cxy^2 + dy^3$,

$$\begin{array}{ccc} a & 2b & c \\ b & 2c & d \\ \xi^2 & 2\xi\eta & \eta^2 \end{array}$$

will be found to be the reciprocant of its Hessian.

(2). Let $\phi = ax^4 + 4bx^3y + \&c. + ey^4$,

the reciprocant of its Hessian will be found to be

$$\begin{array}{cccc} a & 3b & 3c & d \\ b & 3c & 3d & e \\ \xi^2 & 2\xi\eta & \eta^2 & \\ & \xi^2 & 2\xi\eta & \eta^2 \end{array}$$

(3). Let $\phi = ax^5 + 5bx^4y + \&c. + fy^5$,

the reciprocant of its Hessian will be

$$\begin{array}{ccccc} a & 4b & 6c & 4d & e \\ b & 4c & 6d & 4e & f \\ \xi^2 & 2\xi\eta & \eta^2 & & \\ & \xi^2 & 2\xi\eta & \eta^2 & \\ & & \xi^2 & 2\xi\eta & \eta^2 \end{array}$$

and the reciprocant of its canonizant is

$$\begin{array}{cccc} a & 3b & 3c & d \\ b & 3c & 3d & e \\ c & 3d & 3e & f \\ \xi^3 & 3\xi^2\eta & 3\xi\eta^2 & \eta^3 \end{array}$$

The numerical coefficients in this and in the first c inserted for the sake of uniformity, but it will of co readily observed that when there is but one line of that the numerical coefficients being the same f column may be rejected without affecting the form result.

So again, if $\phi = ax^6 + 6bx^5y + \&c. + gy^6$,
the reciprocant of the Hessian is

$$\begin{array}{cccccc} a & 5b & 10c & 10d & 5e & f \\ b & 5c & 10d & 10e & 5f & g \\ \xi^2 & 2\xi\eta & \eta^2 & & & \\ & \xi^2 & 2\xi\eta & \eta^2 & & \\ & & \xi^2 & 2\xi\eta & \eta^2 & \\ & & & \xi^2 & 2\xi\eta & \eta^2 \end{array}$$

and the reciprocant of the second form in the scale, which comes between the Hessian and the catalecticant will be

$$\begin{array}{cccccc} a & b & c & d & e & \\ b & c & d & e & f & \\ c & d & e & f & g & \\ \xi^3 & \xi^2\eta & \xi\eta^2 & \eta^3 & & \\ & \xi^3 & \xi^2\eta & \xi\eta^2 & \eta^3 & \end{array}$$

and so in general. The rule of formation is sufficiently plain not to need formulizing in general terms. It is easy to see that all these forms are concomitants to the function from which they are formed; for example, take

$$\phi = ax^6 + 6bx^5y + \&c. + gy^6,$$

$$\left(\frac{d}{dx}\right)^2\phi, \quad \frac{d}{dx}\cdot\frac{d}{dy}\,\phi, \quad \left(\frac{d}{dy}\right)^2\phi \text{ form a plexus.}$$

So likewise if we take $\psi = (x\xi + y\eta)^4$,

$$\frac{d\psi}{d\xi}, \ \frac{d\psi}{d\eta} \text{ form a plexus;}$$

but ψ and ϕ are concomitantive, ψ being a universal concomitant. Hence we may combine together these two plexuses, that is

$$\left.\begin{array}{l} ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4 \\ bx^4 + 4cx^3y + 6dx^2y^2 + 4exy^3 + fy^4 \\ cx^4 + 4dx^3y + 6ex^2y^2 + 4fxy^3 + gy^4 \end{array}\right\}$$

$$\left.\begin{array}{l} \xi^3x^4 + 3\xi^2\eta x^3y + 3\xi\eta^2x^2y^2 + \eta^3xy^3 \\ \qquad\quad \xi^3x^3y + 3\xi^2\eta x^2y^2 + 3\xi\eta^2xy^3 + \eta^3y^4 \end{array}\right\}$$

and, by the principle of the plexus, $x^4, x^3y, x^2y^2, xy^3, y^4$ may be eliminated dialytically, and the resultant will be the determinant last given, which is therefore a contravariant to ϕ.

The manner in which I was led to notice this singular transformation is somewhat remarkable.

In the supplemental part of my essay *On Canonical Forms, &c.*, published by Bell, Fleet Street, my method of solution of the problem of throwing the quintic function of 2 variables under the form $u^5 + v^5 + w^5$, led me to see that u, v, w are the three factors of

$$\begin{array}{ccc} ax + by & bx + cy & cx + dy \\ bx + cy & cx + dy & dx + ey \\ cx + dy & dx + ey & ex + fy \end{array}$$

the more simple mode of the solution of the same problem, given by me in the *Philosophical Magazine* for the month of November last, led to

$$\begin{array}{cccc} a & b & c & d \\ b & c & d & e \\ c & d & e & f \\ y^3 & -xy^2 & x^2y & -x^3 \end{array}$$

as the product of the same three factors; whence the identity of the two forms become manifest. In the paper last named I gave two proofs, one my own, the other Mr. Cayley's, of a like kind of identity for the canonizant of any odd-degreed function of x, y in general. The proof of the identity of the corresponding forms in the much more general proposition above indicated must be reserved until more pressing and important matters are disposed of. In the foot note referred to, p. 96, line 4, there is an erratum, $2r$ being written in place of $2n$; I ought also to have added, in order to make the sense more clear, that the degree of the catalecticant there referred to in respect of the coefficients would be n.

In note 8 (7th line) ξ, η, ζ in the first G are written in place of $\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}$. I regret to think that there are many other typographical errors in the earlier Sections; the most unfortunate of these is in the note at page 87, in the values of P and Q belonging to the cubic commutant dodecadic function of x and y, the corrected values of which will be given in my next communication. I ought also to observe, in correction of the remark made in the foot note to page 72, that it follows as a consequence of a recent paper by Dr. Hesse in *Crelle's Journal*, that the method given by me in the text applied (according to what I have there termed the 1st process for obtaining an invariant

resembling the resultant) to a system of three cubic equation (in which application only the 1st powers of $\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{d}{d}$ enter) produces for that case also, as well as for the case specified in the note, not a counterfeit resemblance of, bu the actual resultant itself.

Returning to the theory of the plexus of which I an about to enunciate a most important extension, I beg t refer my readers to the fourth paragraph, p. 60, in the las number of the *Journal*, where I have shewn how to form under certain conditions, a determinant by combining together various concomitants and eliminating dialytically one set o the variables, which determinant will be concomitantive t the concomitants out of which it is formed, and of course also therefore to their common original.

Now the extension of this theorem, to which I wish to call attention, is this, that not only such determinant as whole is a concomitant to such origin, but every mino system of determinants that can be formed out of it wil form a concomitantive plexus complete within itself to the same original. But, much more generally, it should be observed that there is no occasion to begin with a square determinant; it is sufficient to have a rectangular array of terms formed by taking the several terms of one plexus or of several plexuses combined, provided that they are of the same degree in respect to the variables (or to the selected system of variables, if there be several systems), and forming out of such rectangular array any minor system of determinants at will. Every such system will be a concomitantive plexus. The simple illustrations which follow will make my meaning clear.

Suppose

$$\phi = ax^6 + 6bx^5y + 15cx^4y^2 + 21dx^3y^3 + 15ex^2y^4 + 6fxy^5 + gy^6.$$

I have previously remarked, in the foregoing sections, that a, b, c, d, e, f, g, the coefficients form an invariantive plexus to ϕ; so also we know that the catalecticant

$$\begin{matrix} a & b & c & d \\ b & c & d & e \\ c & d & e & f \\ d & e & f & g \end{matrix}$$

is an invariant to ϕ. But we are now able to couple togeth these facts and see the law which is contained between them

for if we take

$$\left(\frac{d}{dx}\right)^\iota\phi,\ \left(\frac{d}{dx}\right)^{\iota-1}\cdot\frac{d}{dy}\phi\ \ldots\ldots\ \left(\frac{d}{dy}\right)^\iota\phi,$$

ι being any number; as for instance, if we take
shall have as a plexus

$$\begin{array}{l} ax^3 + 3bx^2y + 3cxy^2 + dy^3, \\ bx^3 + 3cx^2y + 3dxy^2 + ey^3, \\ cx^3 + 3dx^2y + 3exy^2 + fy^3, \\ dx^3 + 3ex^2y + 3fxy^2 + gy^3; \end{array}$$

accordingly not only is the determinant

$$\begin{array}{cccc} a & b & c & d \\ b & c & d & e \\ c & d & e & f \\ d & e & f & g \end{array}$$

an invariant, but also the system obtained by stri
any one line and one column, being what I term
minors, will be an invariantive plexus: so too
system of second minors

$$ac - b^2,\ bd - c^2,\ ce - d^2,\ ad - bc,\ ae - bd,\ be - c$$

form an invariantive plexus, as well as the last
i.e. the simple terms a, b, c, d, e, f, g. Again, w
have taken the plexus

$$\left(\frac{d}{dx}\right)^2\phi\quad \frac{d}{dx}\cdot\frac{d}{dy}\phi\quad \left(\frac{d}{dy}\right)^2\phi,$$

which would give the array

$$\begin{array}{ccccc} a & b & c & d & e \\ b & c & d & e & f \\ c & d & e & f & g; \end{array}$$

but the minor systems of determinants herein c
will be found to be identical with those last co
with the exception that the highest system, cont
single determinant only, will now be wanting. So in
it will easily be seen that a similar method in gener
ϕ is of 2ι dimensions, will lead to $\iota + 1$ invariantive
comprising the given coefficients grouped together
extremity of the scale, and the catalecticant alone
other; and if ϕ is of $2\iota + 1$ dimensions, there will

$\iota+1$ such plexuses, commencing with the coefficients as one group and ending with a system of combinations of the $(\iota+1)^{\text{th}}$ degree in regard of the coefficients, which system accordingly takes the place of the catalecticant of the former case, which for this case is nonexistent.

As a profitable example of the application of this law of synthesis, in its present extended form, let it be required to determine the conditions that a function of x, y of the fifth degree may have three equal roots. In general, let $\phi = ax^5 + 5bx^4y + 10cx^3y^2 + 10dx^2y^3 + 5exy^4 + ey^5$, then ϕ has a quadratic and cubic covariant of which I have written at large in my supplemental essay above referred to, being in fact the s and t (*i.e.* the quadrinvariant and cubinvariant) in respect to x', y' (x, y being treated as constants) of

$$\left(x'\frac{d}{dx} + y'\frac{d}{dy}\right)^4 \phi.$$

Let these covariants respectively be called

$$Ax^2 + 2Bxy + Cy^2 = u,$$
$$\alpha x^3 + 3\beta x^2y + 3\gamma xy^2 + \delta y^3 = v;$$

then
$$\left.\begin{matrix} Ax + By \\ Bx + Cy \end{matrix}\right\} \text{ forms a plexus,}$$

and
$$\left.\begin{matrix} \alpha x^2 + 2\beta xy + \gamma yz \\ \beta x^2 + 2\gamma xy + \gamma yz \end{matrix}\right\} \text{ will form another.}$$

Now when $a=0$, $b=0$, $c=0$, ϕ will have 3 equal roots, and $\left(x'\frac{d}{dx} + y'\frac{d}{dy}\right)^4 \phi$ becomes

$$6dy.x'^2y'^2 + 4(dx + ey)\,x'y'^3 + (ex + fy)\,y'^4,$$

the quadrinvariant in respect to x', y' of which is easily seen to be d^2y^2 and the cubicinvariant d^3y^3. Accordingly the grouping $\begin{matrix} A & B \\ B & C \end{matrix}$ becomes $\left.\begin{matrix} 0 & 0 \\ 0 & d^2 \end{matrix}\right\}$, and the grouping $\left.\begin{matrix} \alpha & \beta & \gamma \\ \beta & \gamma & \delta \end{matrix}\right\}$ becomes $\left.\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & d^3 \end{matrix}\right\}$. Accordingly, we see that the determinant $\left.\begin{matrix} A & B \\ B & C \end{matrix}\right\}$ and all the first minors of $\left\{\begin{matrix} \alpha & \beta & \gamma \\ \beta & \gamma & \delta \end{matrix}\right.$

i.e. $\qquad \alpha\gamma - \beta^2, \quad \beta\delta - \gamma^2, \quad \alpha\delta - \beta\gamma,$

become zero; but the former single quantity $\left.\begin{matrix} A & B \\ B & C \end{matrix}\right\}$ being an invariant, and this last system being an invariantive

Consequently we shall still have all the first minors of

$$\left.\begin{matrix} A & B & C \\ a & \beta & \gamma \\ \beta & \gamma & \delta \end{matrix}\right\},$$

although there is not even so much as a pair of equal roots in ϕ; $AC - B^2$ however, it will be observed, is not zero in this supposition.

The theory of Hessians, simple or bordered, may be regarded as one among the infinite diversity of applications of the principle of the plexus. Let U, V, W, &c. be any number of concomitants having the common system of variables $x, y \ldots z$. Let χ represent

$$x' \frac{d}{dx} + y' \frac{d}{dy} + \ldots + z \frac{d}{dz},$$

and take $$\chi^2 U + \lambda\chi V + \text{\&c.} + \mu\chi W = S,$$

then $$\frac{dS}{dx'}, \frac{dS}{dy'}, \ldots \frac{dS}{dz'} \text{ forms a plexus;}$$

and this, combined with χV, &c. ... χW, enables us to eliminate dialytically $x', y', z', \lambda \ldots \mu$. The result is a Hessian of U, bordered with

$$\frac{dV}{dx}, \frac{dV}{dy}, \ldots \frac{dV}{dz}$$

horizontally and vertically, and also with

$$\frac{dW}{dx}, \frac{dW}{dy}, \ldots \frac{dW}{dz}$$

&c. &c.

similarly dispersed; which Hessian, so bordered, is thus seen to be a concomitant to $U, V, \ldots W$. The Hessian, as ordinarily bordered with $\xi, \eta, \ldots \zeta$, is derived by taking for V the universal concomitant $x\xi + y\eta + \ldots + z\zeta$, and for W (if there be a double border)

$$x\xi' + y\eta' + \ldots + z\zeta',$$

and so forth.

If V be taken identical with U, the resulting form, con-

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and of course when ϕ is of the form $y^2(dx^2+fy^2)$, $L=0$, $M=0$, $N=0$; it being obviously true in general, as remarked by Mr. Cayley, that when not less than half the roots of a function of two variables are equal, all its invariants must vanish together.

sisting of U bordered with $\frac{dU}{dx}, \frac{dU}{dy}, \ldots \frac{dU}{dz}$, has been shewn in my paper 'On certain general Properties of Homogeneous Functions,' in this *Journal*, to be equal the product of the simple Hessian of U and of U itself multiplied by a numerical factor. The theory of the bordered Hessian may be profitably extended by taking

$$S = \chi^{2r}.U + \lambda\chi^{r}.V + \ldots + \mu\chi^{r}.W,$$

and combining with $\chi^r.V \ldots\ldots \chi^r.W$, the plexus obtained by operating upon S with the r^e powers and products of $\frac{d}{dx}, \frac{d}{dy}, \ldots \frac{d}{dz}$, and eliminating dialytically the r^e powers and products of $x', y' \ldots z'$. Thus if

$$U = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4 \text{ and } V = (x\xi + y\eta)^2,$$

we obtain, by taking $S = \chi^4.U + \lambda\chi^2.V$, and proceeding as indicated in the preceding,

$$\begin{array}{cccc} a & b & c & \xi^2 \\ b & c & d & \xi\eta \\ c & d & e & \eta^2 \\ \xi^2 & \xi\eta & \eta^2 & \end{array}$$

as a concomitant to U. So again, if

$$U = ax^5 + 5bx^4y + \text{\&c.} + fy^5,$$

we find

$$\begin{array}{cccc} ax + by & bx + cy & cx + dy & \xi^2 \\ bx + cy & cx + dy & dx + ey & \xi\eta \\ cx + dy & dx + ey & ex + fy & \eta^2 \\ \xi^2 & \xi\eta & \eta^2 & \end{array}$$

a concomitant to U.

These extensions of the ordinary theory of Hessians will be found to be of considerable practical importance in the treatment of forms, for which reason they are here introduced.

SECTION VI.

On the partial Differential Equations to Concomitants, Orthogonal and Plagioyonal Invariants, &c.

In the 7th note of the Appendix to the three preceding sections I alluded to the partial differential equations by which every invariant may be defined.

This method may also be extended to concomitants generally. M. Aronhold, as I collect from private information, was the first to think of the application of this method to the subject; but it was Mr. Cayley who communicated to me the equations which define the invariants of functions of two variables.* The method by which I obtain these equations and prove their sufficiency is my own, but I believe has been adopted by Mr. Cayley in a memoir about to appear in *Crelle's Journal*. I have also recently been informed of a paper about to appear in *Liouville's Journal* from the pen of M. Eisenstein, where it appears the same idea and mode of treatment have been made use of. Mr. Cayley's communication to me was made in the early part of December last, and my method (the result of a remark made long before) of obtaining these and the more general equations, and of demonstrating their sufficiency, imparted a few weeks subsequently—I believe between January and February of the present year.

The method which I employ, in fact, springs from the very conception of what an invariant means, and does but throw this conception into a concise analytical form.

Suppose, to fix the ideas,

$$\phi = ax^n + nbx^{n-1}.y + n.\tfrac{1}{2}(n-1).cx^{n-2}.y^2 + \&c. \ldots + ly^n,$$

and let $I(a, b, c \ldots l)$ be any invariant to ϕ.

Now suppose x to become $x + ey$, but y to remain unchanged, the modulus of the transformation $\left.\begin{matrix}1 & e\\ 0 & 1\end{matrix}\right\}$ being unity, I cannot alter in consequence of this substitution; but the effect of this substitution is to convert ϕ into the form

$$\alpha x^n + n\beta x^{n-1}.y + n.\tfrac{1}{2}(n-1).\gamma x^{n-2}.y^2 + \&c. + \lambda y^n,$$

where $\alpha = a, \quad \beta = b + ae, \quad \gamma = c + 2be + ae^2$, &c. &c.

$$\lambda = l + \&c. \&c. + nbe^{n-1} + ae^n.$$

* It is extremely desirable to know whether M. Aronhold's equations are the same in form as those here subjoined. It is difficult to imagine what else they can be in substance. Should these pages meet the eye of that distinguished mathematician he will confer a great obligation on the author and be rendering a service to the theory by communicating with him on the subject: and I take this opportunity of adding that I shall feel grateful for the communication of any ideas or suggestions relating to this new Calculus from any quarter and in any of the ordinary mediums of language—French, Italian, Latin, or German, provided that it be in the Latin character. My address in London will be found appended at the end of this paper.

hence

$$\left\{\left(a\frac{d}{db}+2b\frac{d}{dc}+3c\frac{d}{dd}\,\&c.\right)\right\}\left\{\left(a\frac{d}{db}+2b\frac{d}{dc}+3c\frac{d}{dd}\,\&c.\right)\right\}I$$
$$+\left\{a\frac{d}{db}+2b\frac{d}{dc}+3c\frac{d}{dd}\,\&c.\right\}^2 I=0;$$

i. e. $$\left\{2\left(a\frac{d}{dc}+3b\frac{d}{dd}+6c\frac{d}{de}\,\&c.\right)\right.$$
$$\left.+\left(a\frac{d}{db}+2b\frac{d}{dc}+3c\frac{d}{dd}\,\&c.\right)^2\right\}I=0:$$

repeating the application of the symbolic operator

$$\left(a\frac{d}{db}+2b\frac{d}{dc}\,\&c.\right),$$

we obtain

$$\left.\begin{aligned}&1.2.3\left\{a\frac{d}{dd}+4b\frac{d}{de}+10c\frac{d}{df}\,\&c.\right\}\\&+1.2\left\{a\frac{d}{db}+2b\frac{d}{dc}+\&c.\right\}.\left\{a\frac{d}{dc}+3b\frac{d}{dd}\,\&c.\right\}\\&+\left(a\frac{d}{db}+2b\frac{d}{dc}+3c\frac{d}{dd}\,\&c.\right)^3\end{aligned}\right\}I=0,$$

and so on; the numerical multipliers of the terms of the several series within the parentheses forming the regular succession of figurate numbers 1, 2, 3, &c.
1, 3, 6, &c.
1, 4, 10, &c.

It is easy to see that these equations correspond to the results of making the coefficients of the successive powers of e equal to zero.

I may remark, that the first instance as far as I know on record of this, (as some may regard it rather bold) but in point of fact perfectly safe and legitimate method of differentiating conjointly operator and operand, occurs in a paper by myself in this *Journal*, Feb. 1851, "On certain general properties of homogeneous functions;" where I have applied it in operating with

$$\left\{(x_1-a_1e)\frac{d}{da_1}+(x_2-a_2e)\frac{d}{da_2}\,\&c.\right\}$$

upon $$\left\{(x_1-a_1e)\frac{d}{da_1}+(x_2-a_2e)\frac{d}{da_2}\,\&c.\right\}^r.\omega;$$

...stitutions any further, as it is clear that we may satisfy ... equations

$$1 + ef = a, \quad e + g + efg = b,$$
$$f = a', \qquad 1 + fg = b',$$

... all values of a, b, a', b', which satisfy the equation

$$ab' - a'b = 1;$$

...d for none other except such values; hence I remains un-...ered for any unit-modular linear transformation of x, y, ...d is therefore an invariant by definition.

If ϕ be taken a function of three variables, x, y, z, and ... thrown under the form

$$az^n + (a_1x + b_1y)z^{n-1} + (a_2x^2 + 2b_2xy + c_2y^2)z^{n-2} + \&c.,$$

...d I be any invariant of ϕ, by supposing x to become $x + ey$, ...d giving b_1, b_2, c_2, &c., the corresponding variations, and ...king e indefinitely small, we obtain

$$\left\{a_1\frac{d}{db_1} + \left(a_2\frac{d}{db_2} + 2b_2\frac{d}{dc_2}\right)\right.$$
$$\left.+ \left(a_3\frac{d}{db_3} + 2b_3\frac{d}{dc_3} + 3c_3\frac{d}{dd_3}\right) + \&c.\right\} I = 0,$$

...

$$\left\{b_1\frac{d}{da_1} + \left(c_2\frac{d}{db_2} + 2b_2\frac{d}{da_2}\right) + \&c. \; \&c.\right\} I = 0:$$

...d in like manner, by arranging ϕ according to the powers ... y and of x, we obtain two other pairs of equations: it ... clear, however, that three equations (it would seem any three out of the six) would suffice and imply the other three. The method of demonstration will be the same as in the instance of two variables: 1st, It can be shewn by the method of successive accretions, that I remaining invariable when x receives an indefinitely small increment ϵy, or z an indefinitely small increment ϵx, or z an indefinitely small increment ϵx, it will also remain invariable when these increments are taken of any finite magnitude. 2nd, By eight successive transformations, admissible by virtue of the preceding conclusion, x, y, z may be changed into any linear functions of x, y, z, consistent with the modulus of transformation being unity. And in general for a function of m variables, m partial differential equations similarly constructed (but not however arbitrarily selected) will be necessary and sufficient to determine any invariant: and it is clear that all the general

properties of invariants must be contained in and be capable of being educed out of such equations.

The same method enables us also to establish the partial differential equations for any covariant, or indeed any concomitant whatever.

Thus let

$$\phi = ax^n + nbx^{n-1}y + n\,\tfrac{1}{2}(n-1)\,cx^{n-2}y^2 + \&c. + nb'xy^{n-1} + a'y^n = 0,$$

and let $K(a, b, c, \&c.;\ x, y, x', y', \&c.;\ \xi, \eta, \&c.)$ represent any concomitant, x, y; x', y' being cogredient, and ξ, η, &c. contragredient systems; when x, y become $x + ey$, y, any such systems x', y' becomes $x' + ey'$, y'; and any such system as ξ, η become $\xi, \eta - e\xi$; and taking e indefinitely small, the second coefficients a, b, c, &c. become a, $b + ae$, $c + 2be$, &c. as before; hence the equation to the concomitant becomes

$$\left\{a\frac{d}{db} + 2b\frac{d}{dc} + \&c. - y\frac{d}{dx} - y'\frac{d}{dx'} + \&c. + \xi\frac{d}{d\eta} - \&c.\right\} = 0;\text{*}$$

and in like manner, by changing y into $y + ex$, results the corresponding equation

$$\left\{a'\frac{d}{db'} + 2b'\frac{d}{dc'} + \&c. - x\frac{d}{dy} - x'\frac{d}{dy} + \&c. + \eta\frac{d}{d\xi}\,\&c.\right\} K = 0.$$

These two equations define in a perfectly general manner every concomitant (with any given number of cogredient and contragredient systems) to the form ϕ; and the due number of pairs of similarly constituted equations will serve to define the concomitant to a function of any given number of variables.†

In like manner we may proceed to form the equations corresponding to what may be termed *conditional* concomitants, whether *orthogonal* or *plagiogonal*. The concomitants previously considered may be termed absolute, the linear transformations admissible being independent of any but the one general relation, imposed merely for the purpose of convenience, viz. of their modulus being made unity. An orthogonal concomitant is a form which remains invariable, not for arbitrary unit-modular, but for orthogonal transformation, *i.e.* for linear substitutions of $x, y \ldots : z$, which leave unchanged $x^2 + y^2 + \ldots + z^2$: in like manner, a plagiogonal

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* For we have

$$K(a, b + ab, c + ac, \&c.,\ \ x, y, \&c.;\ \ \xi, \eta, \&c.),$$
$$= K(a, b, c, \&c.,\ \ x, x + ey, \&c.,\ \ \xi, \eta - e\xi, \&c.,\ \ \&c.)$$

† Vide Note (10)

concomitant may be defined of a form which remains invariable for all linear substitutions of $x, y \ldots z$, which leave unaltered any given quadratic function of $x, y \ldots z$. Thus, let it be required to express the condition of $Q(a, b, c \ldots x, y; \xi, \eta)$, being a conditional concomitant to the form

$$ax^n + nbx^{n-1}y + \&c. + nb'xy^{n-1} + a'y^n.$$

Let x become $x + ey$, e being indefinitely small, then y must become $y - ex$, and the variations of $a, b, \ldots b', a'$ will be the sum of the variations produced by taking separately $x + ey$ for x and $y - ex$ for y. Hence the one sole condition for Q being of the required form becomes

$$\left\{\begin{array}{l} a\dfrac{d}{db} + 2b\dfrac{d}{dc} + \&c. - y\dfrac{d}{dx} + \xi\dfrac{d}{d\eta} \\ a'\dfrac{d}{db'} + 2b'\dfrac{d}{dc'} + \&c. - x\dfrac{d}{dy} + \eta\dfrac{d}{d\xi} \end{array}\right\} Q = 0;$$

or, as it may be written, $\theta.Q - \omega.Q = 0$ where $\theta.Q = 0$, $\omega.Q = 0$ are the two equations expressing the conditions of Q, being an unconditional or absolute concomitant; and so in general if ϕ be a function of m variables, we may obtain $\frac{1}{2}m(m-1)$ equations of the form $L - M = 0$ for the concomitant, of which however $(m-1)$ only will be independent.

Supposing, again, the substitutions to which x, y are subject to be conditioned by $lx^2 + 2mxy + xy^2$ remaining unalterable, or which is a more convenient and only in appearance less general supposition by $x^2 + 2mxy + xy^2$ remaining unalterable, the general type of an infinitesimal system of substitutions will be rendered by supposing x, y to become $(1 + me)x + ey$, $-ex + (1 - me)y$, respectively, for then $x^2 + 2mxy + y^2$ becomes

$$(1 - m^2e^2)x^2 + \{2m + (2m - 2m^3)e^2\}xy + (1 - m^2e^2)y^2,$$

which differs from $x^2 + 2mxy + y^2$ only by quantities of the second order of smallness which may be neglected, and ξ and η will therefore become $(1 - me)\xi - e\eta$, $-ex + (1 + me)y$, respectively: then, as to the coefficients of ϕ in addition to the variations which they undergo when m is zero, there will be the variations consequent upon x, assuming the increment mex and y the increment $-mey$: but by making x become $x + mex$, a, b, c, &c. b', a' assume respectively the variations

$$n.mea,\ (n-1)meb,\ \&c.\ meb',\ 0,\ \text{respectively},$$

and by making y become $y - mey$, the corresponding variations become

$$0, \; - meb, \; \dots - (n-1)\, meb', \; na', \text{ respectively.}$$

Hence the equation becomes

$$\theta . Q - \omega Q + m(\lambda Q - \mu Q) = 0,$$

where θ and ω have the same signification as before, and where λ denotes

$$na \frac{d}{da} + (n-1)\, b \frac{d}{db} + \dots\dots + b' \frac{d}{db'} + x \frac{d}{dx} - \xi \frac{d}{d\xi},$$

and μ denotes

$$b \frac{d}{db} + 2c \frac{d}{dc} + \dots\dots\dots\dots + na' \frac{d}{da'} - y \frac{d}{dy} + \eta \frac{d}{d\eta}.$$

If there be several systems of x, y or of ξ, η, or of both, the only difference in the equation of condition will consist in putting

$$\Sigma\left(y \frac{d}{dx}\right), \quad \Sigma\left(x \frac{d}{dy}\right), \quad \Sigma\left(x \frac{d}{dx}\right), \quad \Sigma\left(y \frac{d}{dy}\right),$$

$$\Sigma\left(\eta \frac{d}{d\xi}\right), \quad \Sigma\left(\xi \frac{d}{d\eta}\right), \quad \Sigma\left(\xi \frac{d}{d\xi}\right), \quad \Sigma\left(\eta \frac{d}{d\eta}\right),$$

instead of the single quantities included within the sign of definite summation.

Fearing to encroach too much on the limited space of the *Journal*, I must conclude for the present with shewing how to integrate the general equation to the orthogonal invariant of ϕ the general function of x, y.

Beginning with $\phi = ax^2 + 2bxy + cy^2$, the equation becomes

$$\left\{- 2b \frac{d}{da} + (a-c) \frac{d}{db} + 2b \frac{d}{dc} + y \frac{d}{dx} - x \frac{d}{dy}\right\} Q = 0.$$

Write now
$$\begin{aligned} da &= -\, 2b d\theta, & dx &= y d\theta, \\ db &= (a-c)\, d\theta, & dy &= -\, x d\theta, \\ dc &= +\, 2b d\theta\,; \end{aligned}$$

we have then

$$\lambda da + \mu db + \nu dc = d\theta \left\{\mu a + 2(\nu - \lambda)\, b - \mu c\right\}.$$

Let $\quad \mu = \kappa\lambda\,; \quad 2(\nu - \lambda) = \kappa\mu\,; \quad -\mu = \kappa\nu\,;$

then
$$d \log (\lambda a + \mu b + \nu c) = \kappa d\theta\,;$$

$$\lambda a + \mu b + \nu c = b e^{\kappa\theta}.$$

To find κ we have the determinant:

$$\begin{vmatrix} \kappa & -1 & 0 \\ 2 & \kappa & -2 \\ 0 & 1 & \kappa \end{vmatrix} = 0,$$

that is, $$\kappa^3 + 4\kappa = 0,$$

and calling the 3 roots of this equation $\kappa_1, \kappa_2, \kappa_3$, we have

$$\kappa_1 = 0, \quad \kappa_2 = 2\iota, \quad \kappa_3 = -2\iota;$$

accordingly we may put

$$\kappa = 0, \quad \lambda = 1, \quad \mu = 0, \quad \nu = 1,$$

or $$\kappa = 2\iota, \quad \lambda = 1, \quad \mu = 2\iota, \quad \nu = -1,$$

or $$\kappa = -2\iota, \quad \lambda = 1, \quad \mu = -2\iota, \quad \nu = -1.$$

Again, $$pdx + qdy = (py - qx)\, d\theta;$$

and putting $-q = e.p$, $p = e.q$, so that $px + qy = E.e^{e\theta}$,

$$e^2 = -1, \quad e_1 = \iota, \quad e_2 = -\iota;$$

and we may put

$$e = \iota, \quad p = 1, \quad q = -\iota,$$

or $$e = -\iota, \quad p = 1, \quad q = +\iota.$$

Consequently the complete integral of the given partial differential equation is found by writing

$$a + c = l, \qquad x - \iota y = E.e^{\iota\theta},$$
$$a + 2\iota b - c = l'e^{2\iota\theta}, \quad x + \iota y = E'.e^{-\iota\theta},$$
$$a - 2\iota b - c = l''e^{-2\iota\theta}.$$

By means of these 5 equations, after eliminating θ, we may obtain 4 independent equations between $a, b, c; x, y$. Suppose

$$Q_1 = 0, \quad Q_2 = 0, \quad Q_3 = 0, \quad Q_4 = 0;$$

then $Q = F(Q_1, Q_2, Q_3, Q_4)$ is the complete integral required.

Pursuing precisely the same method for the general case, it will be found that, calling the degree of the given function n when n is even, the equation in κ to be solved will be

$$\kappa(\kappa^2 + 4)(\kappa^2 + 9) \ldots (\kappa^2 + n^2) = 0;$$

and when n is odd (say $2m + 1$), the equation in κ to solve will be

$$(\kappa + 1)(\kappa^2 + 9) \ldots (\kappa^2 + n^2) = 0;$$

and performing the necessary reductions, and calling the roots of the equation, arranged in order of magnitude, $\kappa_1{}^{\iota}, \kappa_2{}^{\iota} \ldots \kappa_n{}^{\iota}$, respectively, it will be found that the equations

containing the integral become

$$\left.\begin{array}{l} L_1 = l_1 e^{\kappa_1 \iota\theta} \\ L_2 = l_2 e^{\kappa_2 \iota\theta} \\ L_3 = l_3 e^{\kappa_3 \iota\theta} \\ \&c. \quad \&c. \\ L_{n+1} = l_{n+1} e^{\kappa_{n-1} \iota\theta} \end{array}\right| \qquad \left.\begin{array}{l} x - \iota y = Ee^{\iota\theta} \\ x + \iota y = E'e^{-\iota\theta} \end{array}\right\}$$

where $l_1, l_2 \ldots l_{n+1}$; E, E' are arbitrary constants, and where $L_1, L_2 \ldots L_{n+1}$ are the values assumed by the 1st, 2nd...$(n+1)$th coefficients of the given function ϕ, or

$$ax + nbx^{n-1} + \&c. + nb'xy^{n-1} + by^n,$$

when it is transformed by writing $x + \iota y$ in place of x, and $y + \iota x$ in place of y. ι is of course employed in the foregoing according to the usual annotation to represent $\sqrt{-1}$. The same method applies to the general theory of plagiogonal concomitants, where the linear substitutions are supposed such as to leave $lx^2 + 2mxy + xy^2$ unaltered in form, and the equations in θ which contain the integral present themselves under a similar aspect. But a more full discussion of these interesting integrals must be reserved until the ensuing number of the *Journal.*

[*To be continued.*]

26, Lincoln's Inn Fields,
April 1, 1852.

NOTES IN APPENDIX.

(9) The scale of covariants to a function of (x, y) obtained by the method of unravelment (*Journal*, No. XXVIII. p. 66) may be otherwise deduced in a form more closely analogous to that in which the corresponding theorems for the corresponding invariantive scale (No. XXVIII. p. 65) by a method which has the advantage of exhibiting the scale equally well for the case of functions of the degree $4\iota + 2$ or $4\iota + 4$, the only difference being that in the latter case the coefficients of the odd powers of λ will be found all to vanish, so that the degrees of the covariants will rise by steps of 4 instead of by steps of 2, just conversely to what happens in the invariantive scale; whereas in the invariantive scale alluded to the forms containing odd powers of λ vanish when the degree of the function is of the form $4\iota + 2$, but do not vanish when it is of the form 4ι. This method in the form here subjoined is a slight modification of one suggested to me by my friend Mr. Cayley.

Let F be the given function of x, y of the degree $2n$; take the systems x', y'; x_1, y_1 cogredient with one another and with x, y. Then form the concomitant

$$K = \left(x' \frac{d}{dx} + y' \frac{d}{dy}\right)^n F + \lambda\,(x'y - y'x)^{n-1}\,(x'y_1 - y'x_1)\,(xy_1 - yx_1).$$

Then (by what may be termed the Divellent method, and which has been previously applied by me in the *Philosophical Magazine* for Nov. 1851) calling $\theta_0, \theta_1, \theta_2 \ldots \theta_n$, the coefficients of

$$x'^n, x'^{n-1}, y', \ldots y'^n \text{ in } K,$$

we shall have

$$\theta_0 = A_0 x^n + B_0 x^{n-1} y + \ldots L_0 y^n,$$
$$\theta_1 = A_1 x^n + B_1 x^{n-1} y + \ldots L_1 y^n,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\theta_n = A_n x^n + B_n x^{n-1} y + \ldots L_n y^n,$$

the coefficients being functions of the coefficients of f and of quadratic combinations of x_1, y_1, affected with the multiplier λ, and the determinant

$$\begin{array}{c} A_0, \; B_0 \ldots L_0, \\ A_1, \; B_1 \ldots L_1, \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ A_n, \; B_n \ldots L_n, \end{array}$$

will give a function of λ in which the coefficients of the several powers of λ will be all zero or covariants of F.

The actual form of this determinant is not here given for want of space and time, but will be exhibited hereafter. Precisely an analogous method applies to obtain the scale to $(x, y, z)^4$ given in Note (2). Calling $F = (x, y, z)^4$, let the systems x', y', z'; x_1, y_1, z_1, be taken cogredient with one another and with x, y, z. Then, using R to express the determinant

$$\begin{array}{ccc} x', & y', & z', \\ x, & y, & z, \\ x_1, & y_1, & z_1, \end{array}$$

and making

$$K = \left(x' \frac{d}{dx} + y' \frac{d}{dy} + z' \frac{d}{dz}\right)^2 F + \lambda R,$$

and proceeding as above by the divellent method, we obtain the scale required.

(10). It is obvious that these defining equations ought to give the means of discovering and verifying all the properties of concomitants; but it is very difficult to see how in the present state of analysis many of the general theorems that have been stated, readily admit of being deduced from them.

The comparatively simple but eminently important theory of the evector symbol does however admit of a very pretty verification by aid of these equations. Thus, suppose θ any concomitant; suppose a contravariant to a function F of x, y, say

$$ax^n + nbx^{n-1}y + \ldots + nb'xy^{n-1} + a'yn.$$

Then θ must satisfy the two equations

$$\left(L + \xi \frac{d}{d\eta}\right)\theta = 0, \quad \left(L' + \eta \frac{d}{d\xi}\right)\theta = 0,$$

where
$$L = a\frac{d}{db} + 2b\frac{d}{dc} + \ldots + nb'\frac{d}{da'},$$

$$L' = a'\frac{d}{db'} + 2b'\frac{d}{dc'} + \ldots + nb\frac{d}{da}.$$

Now let $\phi = \chi(\theta)$ where

$$\chi = \xi^n \frac{d}{da} + \xi^{n-1}\eta \frac{d}{db} + \xi^{n-2}\eta^2 \frac{d}{dc} + \ldots + \eta^n \frac{d}{da};$$

then $$L(\chi\theta) = \chi(L\theta) - (\chi L)\theta,$$

$$= \chi(L\theta) - \left(\xi^n \frac{d}{db} + 2\xi^{n-1}\eta \frac{d}{dc} + \ldots + n\xi\eta^{n-1} \frac{d}{da}\right)\theta,$$

$$\xi \frac{d}{d\eta}(\chi\theta) = \chi\left(\xi \frac{d}{d\eta}\theta\right) + \left(\xi \frac{d}{d\eta}\chi\right)\theta,$$

$$= \chi\left(\xi \frac{d}{d\eta}\theta\right) + \left(\xi^n \frac{d}{db} + 2\xi^{n-1}\eta \frac{d}{dc} + \ldots + n\xi\eta^{n-1} \frac{d}{da}\right)\theta.$$

Hence $$\left(L + \xi \frac{d}{d\eta}\right)\chi(\theta) = \chi\left\{\left(L + \xi \frac{d}{d\eta}\right)\theta\right\} = \chi 0 = 0.$$

Similarly $$\left(L' + \eta \frac{d}{d\xi}\right)\chi\theta = 0.$$

Hence if θ is an integral of the two conditioning equations, so also $\chi(\theta)$. In like manner, if θ be a covariant or any other kind of concomitant of F, it may be proved that its evectant $\chi(\theta)$ is the same.

(11). Very much akin with the supposed equations (p. 182) is the following most remarkable equation, which can be proved to exist. Let ϕ be a function of x and y of the 5th degree. Let P and Q be the quadratic and cubic covariants of ϕ. [P is of two dimensions in the coefficients and also in the variables, and Q of three dimensions in both. They are in fact the s and t (in respect to x' and y') of $\left(x'\frac{d}{dx} + y'\frac{d}{dy}\right)^4\phi$;] then, giving P and Q proper numerical factors, it will be found that

$$H_2\phi + PH\phi + Q\phi = 0.$$

I believe that a similar equation connects any function of x and y above the 3rd degree with its first and second Hessians. The proof will be given in a subsequent section, where also I shall give a complete proof, which occurred to me immediately after sending the preceding notes to the press, of the complete Theory of the Respondent by means of the general equations of concomitance.

POSTSCRIPT.

Since the preceding was in type, I have ascertained the existence and sufficiency of a general method for forming the polar reciprocal and probably also the discriminant to functions of any degree of three variables by an explicit process of permutation and differentiation. In particular I am enabled to give the actual rule for constructing the polar reciprocal and the discriminant curves of the 4th and 5th degrees. So far as regards the polar reciprocal of curves of the 4th degree, Mr. Hesse has already given a method of obtaining it, but mine is entirely unlike to this, and rests upon certain extremely simple and universal principles of the calculus of

[redacted]

orms. The only thing necessary to be done in order to arry on the process to curves of the 6th or higher degrees, to ascertain the relation of the discriminants of functions f 2 variables of those respective degrees to such of the undamental invariants as are of an inferior order to the iscriminant.

The theory applies equally well to surfaces and to functions of any number of variables, and may, I believe, withut any serious difficulty be extended so as to reduce to an xplicit process the general problem of effecting the eliminaon between functions of any degree and of any number of ariables. The method above adverted to will appear in subsequent Section.

[*To be continued.*]

ON THE LAWS OF ELASTICITY.

y William John Macquorn Rankine, C.E., F.R.S.E., F.R.S.S.A., etc.

Sect. VI.—*On the Application of the Method of Virtual Velocities to the Theory of Elasticity.*

25. Lagrange's method of Virtual Velocities having been pplied to the problems of the equilibrium and motion of lastic media by Mr. Green (*Camb. Trans.*, VII.), and by Mr. Haughton (*Trans. Royal Irish Acad.*, XXI., XXII.), ; is my purpose in this and the following Section to point ut the mutual correspondence between the coefficients in he formulæ arrived at by these gentlemen, and the coefficients of elasticity which form the subject of the previous ortion of this paper, and also to shew how far the laws f relation between the nine coefficients of elasticity of a omogeneous body, which I originally proved by a method hiefly geometrical, are capable of being deduced symbolially from equations found according to Lagrange's method.

26. The principle of Virtual Velocities, as applied to nolecular action, is as follows. Let X, Y, Z, denote the otal accelerative forces applied to any particle whose mass s m, of an elastic medium, through agencies distinct from nolecular action (such as the attraction of gravitation): let u, v, w be the components of the velocity of m; let δx, δy, δz lenote indefinitely small virtual variations of x, y, z; let S e the total accelerative molecular force applied to m, δs an ndefinitely small virtual variation of the line along whick

it acts; then the following equation

$$\Sigma\left[m\left\{\left(X-\frac{du}{dt}\right)\delta x+\left(Y-\frac{dv}{dt}\right)\delta y+\left(Z-\frac{dw}{dt}\right)\delta z\right\}\right]$$

$$+\ \Sigma(mS\delta s)=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16),$$

(the summation Σ being extended to all the particles of the medium), expresses at once all the conditions of equilibrium and motion of every particle of the medium.

27. In applying this principle to the theory of elastic media, both Mr. Green and Mr. Haughton assume the following postulates:

First, that in calculation we may treat each particle m as if it were a small rectangular space $dx\,dy\,dz$, filled with matter of a certain density ρ: so that for the symbol Σm we may substitute that of a triple integration

$$\iiint \rho\, dx\, dy\, dz.$$

Secondly, that the virtual moment $mS\delta s$ of the total molecular force acting on any particle m, is capable of being expressed by the product of the small rectangular space $dx\,dy\,dz$ into the variation, δV, of a certain function V of the relative position of m and the other particles of the body.*

Equation 16 is thus transformed into the following:

$$\iiint \rho\left\{\left(X-\frac{du}{dt}\right)\delta x+\left(Y-\frac{dv}{dt}\right)\delta y+\left(Z-\frac{dw}{dt}\right)\delta z\right\} dx\,dy\,dz$$

$$+\iiint \delta V\, dx\,dy\,dz=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17).$$

28. The term *elasticity* properly comprehends those molecular forces only whose variations are produced by, and tend to produce, variations in the volume and figure of bodies. There are, therefore, conceivable kinds of molecular force, which are not included in the term *elasticity.* For example, let us take the forces which Mr. MacCullagh ascribed to the particles of the medium which transmits light.

Let ξ, η, ζ denote displacements of a point in the medium, parallel respectively to x, y, z. Then Mr. MacCullagh supposes the molecular forces to be functions of

$$\frac{d\eta}{dz}-\frac{d\zeta}{dy};\quad \frac{d\zeta}{dx}-\frac{d\xi}{dz};\quad \frac{d\xi}{dy}-\frac{d\eta}{dx};$$

* This amounts in fact to the assumption that no part of the power developed by a variation of the relative positions of the particles is permanently converted into heat, or any other agency: in other words, that the body is perfectly elastic.

which are proportional to the rotations of an element $dx\,dy\,dz$ from its position of equilibrium about the three axes respectively. This amounts to ascribing to the particles of the medium a species of *polarity*, tending to place three orthogonal axes in each particle parallel respectively to the three corresponding axes in each of the other particles: the rotative force acting between the corresponding axes in each pair of particles being a function of the projection of the relative angular displacement of the axes on the plane passing through them, of the position of that plane, and of the distance between the particles.

A portion of a medium endowed with such molecular forces only, would transmit oscillations; but it would not tend to preserve any definite bulk or figure, nor would it resist any change of bulk or figure. It would be a *medium* or *system*, but not a *body*. Molecular forces of this kind, therefore, are not comprehended under the term *elasticity;* and the limits of the present investigation exclude those forms of the function V which represent the laws of their action.

29. The inquiry being thus restricted to molecular forces dependent on the variations of the bulk and figure of bodies, there is to be introduced a

Third Postulate: That supposing the body to be divided mentally into small parts, which, in the undisturbed state of the body are rectangular and of equal size, those parts, in the disturbed state, continue to be *sensibly* of equal bulk and similar figure, throughout a distance round each point at least equal to the greatest extent of appreciable molecular action.

This assumption has been made in all previous investigations, except those respecting the dispersion of light; and it seems, indeed, to be perfectly consistent with the real state of tangible bodies.

Its advantage in calculation is, that it enables us to treat the variations of the molecular forces acting on a given particle, as functions simply of the variations of bulk and figure of an originally rectangular element situated at that particle: seeing that the adjoining elements throughout the extent of appreciable molecular action continue always to undergo sensibly the same variations of bulk and figure as the element under consideration.

Let x_0, y_0, z_0 be the coordinates of any physical point in a homogeneous body in equilibrio, and whose particles

are not operated upon by any extraneous forces X, Y, Z. In this condition it is evident that

$$\delta V = 0$$

at every point, and that we may also make

$$V = 0.$$

In the disturbed condition, let ξ, η, ζ be the displacements of the point whose undisturbed position is (x_0, y_0, z_0), so that

$$x = x_0 + \xi, \text{ \&c.}$$

Then all the variations of bulk and figure which can be undergone by an originally rectangular element, consistently with the third postulate, may be expressed by means of the following six quantities, which I have elsewhere called ***strains:***

$$\frac{d\xi}{dx} = \alpha; \quad \frac{d\eta}{dy} = \beta; \quad \frac{d\zeta}{dz} = \gamma;$$

$$\frac{d\eta}{dz} + \frac{d\zeta}{dy} = \lambda; \quad \frac{d\zeta}{dx} + \frac{d\xi}{dz} = \mu; \quad \frac{d\xi}{dy} + \frac{d\eta}{dx} = \nu;^*$$

of which α, β, γ, are longitudinal *extensions* if positive, *compressions* if negative, and λ, μ, ν, are *distortions* in the planes perpendicular to x, y, z, respectively.

Hence it appears that

$$V = \phi(\alpha, \beta, \gamma, \lambda, \mu, \nu) \ldots\ldots\ldots\ldots (18).$$

30. The first assumption, that we may treat the body in calculation as composed of rectangular elements $\rho\, dx\, dy\, dz$, involves the consequence, that we may express all the molecular forces which act on each such element, by means of pressures, normal and tangential, exerted on its six faces. Taking yz, zx, xy, to denote the position of the faces of such an element, P to denote generally a normal pressure expressed in units of force per unit of area, and Q a tangential pressure similarly expressed, let the nine component pressures on unity of area of those faces be thus denoted:

Position of Face.	*Direction of Pressure.* x	y	z
yz	P_1	Q_3	Q_2'
zx	Q_3'	P_2	Q_1
xy	Q_2	Q_1'	P_3

* This notation is substituted for

$$N_1,\ N_2,\ N_3,\ 2T_1,\ 2T_2,\ 2T_3,$$

as being more convenient.

By the definition of elasticity all pressures are excluded except those whose variations produce and are produced by variations of volume and figure of the parts of the body. Hence the pressures

$$Q_1 - Q_1'; \quad Q_2 - Q_2'; \quad Q_3 - Q_3';$$

whose tendency is to make the element $dx\,dy\,dz$ rotate about its three axes respectively, without change of form, must be null; and therefore

$$Q_1 = Q_1'; \quad Q_2 = Q_2'; \quad Q_3 = Q_3'. \ldots\ldots. (19).$$

Mr. Haughton correctly remarks that this often quoted theorem of Cauchy is not true for all conceivable media. It is not true, for instance, for a medium such as that which Mr. MacCullagh assumed to be the means of transmitting light. It is true, nevertheless, for all molecular pressures which properly fall under the definition of elasticity, if that term be confined to the forces which preserve the figure and volume of bodies.

Let us now express the sum of the virtual moments of the molecular forces acting on the element $dx\,dy\,dz$, in terms of the pressures P_1, &c.; to do which, we must multiply each pressure by the virtual variation of the effect which it tends to produce in its own direction. Thus we obtain the following result:

$$\delta V = P_1\delta\alpha + P_2\delta\beta + P_3\delta\gamma + Q_1\delta\lambda + Q_2\delta\mu + Q_3\delta\nu \ldots (20).$$

Hence the function V bears the following relations to the normal and tangential pressures at the faces of a rectangular element:

$$\left.\begin{array}{lll} P_1 = \dfrac{dV}{d\alpha}; & P_2 = \dfrac{dV}{d\beta}; & P_3 = \dfrac{dV}{d\gamma} \\ Q_1 = \dfrac{dV}{d\lambda}; & Q_2 = \dfrac{dV}{d\mu}; & Q_3 = \dfrac{dV}{d\nu} \end{array}\right\} \ldots . (21).$$

31. A *Fourth Postulate*, generally assumed in investigations of this kind, is that the pressures are sensibly proportional simply to the strains with which they are connected. This assumption must be approximately true of any law of molecular action, when the pressures and strains are sufficiently small. It is known to be sensibly true for almost all bodies, so long as the pressures and strains a great as to impair their power of recovering th volume and figure.*

According to this postulate, the pressures algebraic functions of the first order of the str

and consequently V is an algebraic function of the second order of those strains. The constant part of V, as we have already seen (Art. 29), is null.

Following the notation adopted by Mr. Haughton, let (a) denote the coefficient of a in V, (a^2) that of $-\frac{a^2}{2}$, $(\beta\gamma)$ that of $-\beta\gamma$, &c. Then

$$\begin{aligned} V = {} & (a)\,a + (\beta)\,\beta + (\gamma)\,\gamma + (\lambda)\,\lambda + (\mu)\,\mu + (\nu)\,\nu \\ & - (a^2)\,\frac{a^2}{2} - (\beta^2)\,\frac{\beta^2}{2} - (\gamma^2)\,\frac{\gamma^2}{2} - (\lambda^2)\,\frac{\lambda^2}{2} - (\mu^2)\,\frac{\mu^2}{2} - (\nu^2)\,\frac{\nu^2}{2} \\ & - (\beta\gamma)\,\beta\gamma - (\gamma a)\,\gamma a - (a\beta)\,a\beta - (\mu\nu)\,\mu\nu - (\nu\lambda)\,\nu\lambda - (\lambda\mu)\,\lambda\mu \\ & \qquad - (a\lambda)\,a\lambda - (a\mu)\,a\mu - (a\nu)\,a\nu \\ & \qquad - (\beta\lambda)\,\beta\lambda - (\beta\mu)\,\beta\mu - (\beta\nu)\,\beta\nu \\ & \qquad - (\gamma\lambda)\,\gamma\lambda - (\gamma\mu)\,\gamma\mu - (\gamma\nu)\,\gamma\nu \ldots\ldots\ldots\ldots (22). \end{aligned}$$

The six coefficients of the terms of the first order in this equation, obviously represent the pressures, uniform throughout the whole extent of the body, to which it is subjected when its particles are in those positions from which the displacements are reckoned: that is to say, when

$$\xi = 0; \quad \eta = 0; \quad \zeta = 0.$$

Let $P_{1,0}$ &c. denote those pressures. Then

$$\left.\begin{array}{lll} (a) = P_{1,0}; & (\beta) = P_{2,0}; & (\gamma) = P_{3,0} \\ (\lambda) = Q_{1,0}; & (\mu) = Q_{2,0}; & (\nu) = Q_{3,0} \end{array}\right\} \ldots\ldots (23).$$

The *twenty-one* coefficients of the terms of the second order are the *coefficients of elasticity* of the body, as referred to the three axes selected. The negative sign is prefixed to each, because it is essential to the stability of a body, that molecular pressures should be opposite in direction to the strains producing them.

The transformation of the quantities in equation (22) for any set of rectangular axes, is effected by means of equation (2) of § I. Art. 9, by making the following substitutions:

for	$P_1,\ P_2,\ P_3,\ 2Q_1,\ 2Q_2,\ 2Q_3,$
substitute	$a,\ \beta,\ \gamma,\ \lambda,\ \mu,\ \nu,$

and make similar substitutions for the accented symbols. By multiplying the six equations referred to together by pairs, twenty-one equations are obtained, serving to transform the squares and products of $a, \beta, \gamma, \lambda, \mu, \nu$. Formulæ similar

to those which transform the strains α, &c. and their half-squares and products, serve also to transform the respective coefficients of those quantities in equation (22).*

It is shewn by Mr. Haughton, that by properly selecting the axes of co-ordinates, the number of independent coefficients of elasticity may always be reduced to three less than when the axes are indefinite; and by Mr. Haughton and Mr. Green, that when the body has orthogonal axes of elasticity at each point, then, if those axes be taken as the axes of co ordinates, the coefficients of elasticity are reduced to the first nine. The latter proposition is obvious, because if molecular action be symmetrical about three orthogonal planes, and those be taken for coordinate planes, then the value of that part of the function V which is of the second order cannot be altered by a change in the *sign* of either of the distortions λ, μ, ν; so that the coefficients of the last twelve terms of equation (22) must each be null

The nine coefficients of elasticity of a body in those circumstances have the following values, in terms of the notation of the previous sections:

$$\left.\begin{array}{c}
\textit{Coefficients of Longitudinal Elasticity.}\\
(\alpha^2) = A_1; \quad (\beta^2) = A_2; \quad (\gamma^2) = A_3.\\
\textit{Coefficients of Lateral Elasticity.}\\
(\beta\gamma) = B_1; \quad (\gamma\alpha) = B_2; \quad (\alpha\beta) = B_3.\\
\textit{Coefficients of Rigidity}\\
(\lambda^2) = C_1; \quad (\mu^2) = C_2; \quad (\nu^2) = C_3.
\end{array}\right\}\ldots\ldots(24).$$

32. DEF. *Let the term* PERFECT FLUID *be used to denote the state of a body, which under a given uniform normal pressure, and at a given temperature, tends to preserve, and if disturbed to recover, a certain bulk; but offers no resistance to change of figure.*

In such a body, if the element whose original bulk was $dx\,dy\,dz$, becomes of the bulk $(1+\sigma)\,dx\,dy\,dz$ (σ being a small fraction), we shall have

$$\sigma = \alpha + \beta + \gamma,$$

and the function V must be of the form

$$V = P_0\sigma - (\sigma^2)\frac{\sigma^2}{2}\ \ldots\ldots\ldots\ldots(25),$$

where P_0 is the uniform normal [pres]sure when the particles are not displaced, and (σ^2) [...] [e]lasticity, whose

* See the note a[...]

[redacted]

force which act on each other, but merely of their distances apart, so that if the actions of the several equal wedges into which a body may be conceived to be divided, round a given particle, are different, this does not arise directly from the angular positions of the wedges, but from the different distribution of their centres of force as to *distance* from those of the particle operated upon.

(I have proved this, in a manner slightly different in form, in a supplementary paper to § III., Art. 17).

This further shews, that the mutual action of two centres of force in a solid must be directed along the line joining them; for otherwise it would tend to bring that line into some definite angular position, and would be a function of the direction of the line.

It finally results, from what has been stated, that the action of an indefinitely slender pyramid of a solid body upon a particle at its apex must be a direct attraction or repulsion along the axis of the pyramid, which is a function of the several distances of the centres of force in the pyramid from those in the particle at the apex, and which, in the unstrained condition of the body, must be null.

The principles stated above have to a greater or less extent been taken for granted in previous investigations, but have not hitherto been demonstrated. They may all be regarded as the necessary consequences of the following:

Def. *Let the term* Elastic Solid *be used to denote the condition of a body, which, when acted upon by any given system of pressures, or by none, and at a given temperature, tends to preserve, and if disturbed, to recover, a definite bulk and figure; and such that, if while in an unstrained condition it be cut into parts of any figure, those parts, when separate, will tend to preserve the same bulk and figure as they did when they formed one body.*

Experience informs us that bodies *sensibly* agreeing with this definition exist; its consequences are therefore applicable to them in practice.

35. Let r denote the distance apart of two centres of force in an unstrained solid, and let ϕr be proportional to their mutual action. Then

$$dx\,dy\,dz\,d^2\omega.\Sigma\phi r = 0$$

may be taken to represent the total action of an indefinitely slender pyramid which subtends the element of angular space $d^2\omega$ upon a particle at its apex $dx\,dy\,dz$.

In consequence of a strain, let each of the distances r become

$$(1+\epsilon)r,$$

ϵ being a very small fraction. Then the total action of the pyramid becomes

$$dx\,dy\,dz\,d^2\omega.\Sigma(\phi r + \epsilon r\phi' r) = dx\,dy\,dz\,d^2\omega.\epsilon\Sigma r\phi' r\ ;$$

for by the third postulate, ϵ is uniform throughout the extent of appreciable molecular action.

The quantity which the force acting between two centres of force tends to vary, is their relative displacement along the line joining them, or ϵr. Hence the sum of the virtual moments of the actions of all the slender pyramids into which the solid is conceived to be divided, that is to say, the total virtual moment of its molecular action, upon the particle $dx\,dy\,dz$, is

$$\delta V dx\,dy\,dz = dx\,dy\,dz \iint \epsilon\delta\epsilon.\Sigma(r^2\phi' r).d^2\omega,$$

the double integration extending to all angular space. Consequently, we obtain as a new value of the function V,

$$V = \iint \tfrac{1}{2}\epsilon^2\ \Sigma(r^2\phi' r).d^2\omega \ \ldots\ldots\ldots\ldots (28).$$

Let a, b, c be the direction-cosines of the axis of a given slender pyramid. Then it is easily seen that the strain ϵ along that axis has the following value in terms of the six strains as referred to the axes of coordinates

$$\epsilon = \alpha a^2 + \beta b^2 + \gamma c^2 + \lambda bc + \mu ca + \nu ab,$$

and consequently that

$$\begin{aligned}\frac{\epsilon^2}{2} = \frac{\alpha^2}{2}a^4 + \frac{\beta^2}{2}b^4 + \frac{\gamma^2}{2}c^4 + \frac{\lambda^2}{2}b^2c^2 + \frac{\mu^2}{2}c^2a^2 + \frac{\nu^2}{2}a^2b^2 \\ + \beta\gamma b^2c^2 + \gamma\alpha c^2a^2 + \alpha\beta a^2b^2 \\ + \mu\nu a^2bc + \nu\lambda ab^2c + \lambda\mu abc^2 \\ + \alpha\lambda a^2bc + \alpha\mu a^3c + \alpha\nu a^3b \\ + \beta\lambda b^3c + \beta\mu ab^2c + \beta\nu ab^3 \\ + \gamma\lambda bc^3 + \gamma\mu ac^3 + \gamma\nu abc^2\end{aligned}$$

If this value of $\frac{1}{2}\epsilon^2$ be substituted in equation (28), and the result compared with equation (22), it is at once obvious that the twenty-one coefficients of elasticity have the following values (putting $\Sigma(r^2\phi' r) = -R$, which is negative th[at] equilibrium may be stable):

$$
\begin{aligned}
(\alpha^2) &= \iint a^4 R d^2\omega \\
(\beta^2) &= \iint b^4 R d^2\omega \\
(\gamma^2) &= \iint c^4 R d^2\omega \\
(\lambda^2) &= (\beta\gamma) = \iint b^2 c^2 R d^2\omega \\
(\mu^2) &= (\gamma\alpha) = \iint c^2 a^2 R d^2\omega \\
(\nu^2) &= (\alpha\beta) = \iint a^2 b^2 R d^2\omega \\
(\mu\nu) &= (\alpha\lambda) = \iint a^2 bc R d^2\omega \\
(\nu\lambda) &= (\beta\mu) = \iint ab^2 c R d^2\omega \\
(\lambda\mu) &= (\gamma\nu) = \iint abc^2 R d^2\omega \\
(\beta\lambda) &= \iint b^3 c R d^2\omega \\
(\gamma\lambda) &= \iint bc^3 R d^2\omega \\
(\gamma\mu) &= \iint c^3 a R d^2\omega \\
(\alpha\mu) &= \iint ca^3 R d^2\omega \\
(\alpha\nu) &= \iint a^3 b R d^2\omega \\
(\beta\nu) &= \iint ab^3 R d^2\omega
\end{aligned}
\quad \Bigg| \;\ldots\ldots
$$

In the above equations, which agree with those g Mr. Haughton, the number of independent coeff *fifteen*.

36. Their reduction to a smaller number arises nature of the function

$$R = -\Sigma(r^2\phi' r).$$

This quantity is a function of the distances of th of force in a given indefinitely slender pyramid fro in a particle at its apex, and can vary with the di cosines a, b, c of the axis of the pyramid, solely those distances vary with them. Now in a homo solid, that is, one composed of a succession of sim regularly placed groups of centres of force, those depend upon a quantity which may be called th *interval* between the centres of force in a given di a quantity of such a nature that the product of values for any three orthogonal directions is a quantity; being the space occupied by a centre of forc a definite group of such centres. To have this prop mean interval must be a quantity of this form:

$$i = e^{f + ga^2 + hb^2 + kc^2 + lbc + mca + nab} \ldots\ldots\ldots$$

that is to say, its logarithm must be proportiona

l of the square of the radius of an ellipsoid, whose those of molecular arrangement, and therefore of r action, and of elasticity.

e axes of this ellipsoid be taken as axes of co-. Then $l = 0$, $m = 0$, $n = 0$; and the above equa-educed to

$$i = e^{f+ga^2+hb^2+kc^2} \dots\dots\dots\dots (30A),$$

use the quantity

$$R = F(i) = \psi(f + ga^2 + hb^2 + kc^2) \dots\dots (31)$$

change its sign or value by any change of the signs sines a, b, c, it follows that all the coefficients in (29) g odd powers of those cosines, that is to say, all he *first six*, disappear when the axes of molecular nent are taken for axes of coordinates.

six, for all known homogeneous substances, are to *three*, by the following reasoning:

assume as a *Fifth Postulate*, what experience shews nsibly true of all known homogeneous substances, their elasticity varies very little in different direc-Those substances, such as timber, whose elasticity in directions varies much, are not homogeneous, but l of fibres, layers, and tubes of different substances. be assumed, it follows that, in the expression (31), uantity R, the variable terms

$$ga^2 + hb^2 + kc^2$$

small compared with the constant term f, and that e developed in the form

$$R = \psi(f) + \psi'(f).(ga^2 + hb^2 + kc^2) + \&c.$$

alue of R be introduced into equation (29), and if ntities of the second order be neglected, it is easily performing the integrations, that the following rela-st amongst the six coefficients already specified:

$$(\beta^2) + (\gamma^2) = 6(\lambda^2) = 6(\beta\gamma)$$
$$(\gamma^2) + (\alpha^2) = 6(\mu^2) = 6(\gamma\alpha)$$
$$(\alpha^2) + (\beta^2) = 6(\nu^2) = 6(\alpha\beta)$$

ansformation, ...(32).

$$(\alpha^2) = 3\{(\mu^2) + (\nu^2) - (\lambda^2)\}$$
$$(\beta^2) = 3\{(\nu^2) + (\lambda^2) - (\mu^2)\}$$
$$(\gamma^2) = 3\{(\lambda^2) + (\mu^2) - (\nu^2)\}$$

These equations reduce the number of independent coefficients of elasticity arising from the actions of centres of force, to *three*. They are identical with the equations (7) and (8), embodied in the fifth theorem in § III., although arrived at by a different process.

37. Let us suppose the solid under consideration to possess a portion of fluid elasticity, represented by the coefficient J. Then the coefficients of elasticity have evidently the following relations:

$$\left.\begin{aligned}
(\alpha^2) &= 3\{(\mu^2) + (\nu^2) - (\lambda^2)\} + J \\
(\beta^2) &= 3\{(\nu^2) + (\lambda^2) - (\mu^2)\} + J \\
(\gamma^2) &= 3\{(\lambda^2) + (\mu^2) - (\nu^2)\} + J \\
(\beta\gamma) &= (\lambda^2) + J \\
(\gamma\alpha) &= (\mu^2) + J \\
(\alpha\beta) &= (\nu^2) + J
\end{aligned}\right\} \quad \ldots\ldots(33),$$

which are identical with the six equations (9) comprehended under the sixth theorem, in § IV.

38. The Laws of Elasticity stated in this paper, are the necessary consequences of the definitions of elasticity and of fluid and solid bodies, given in Arts. 28, 32, and 34, respectively, when taken in conjunction with five postulates or assumptions, which however may be summed up in two, viz.

First, That the variations of molecular force concerned in producing elasticity are sufficiently small to be represented by functions of the first order of the quantities on which they depend: and

Secondly, That the integral calculus and the calculus of variations are applicable to the theory of molecular action. It is thus apparent that the science of elasticity is to a great extent one of deduction *à priori*.

The functions of perceptive experience in connexion with it are twofold: first, by observation, to inform us of the existence of substances, agreeing to a greater or less degree of approximation with the definitions and postulates; and secondly, by experiment, to ascertain the numerical values of the coefficients of elasticity of each substance.

59, St. Vincent Street, Glasgow,
March 8, 1852.

NOTE TO SECTIONS VI. AND VII. OF PRECEDING PAPER.

By W. J. M. Rankine.

the Transformation of Coefficients of Elasticity, by the aid of a Surface of the Fourth Order.

(The following note contains no original principle, and is designed rely to put upon record, for the sake of convenient reference, a series equations which will be found useful in future investigations.)

It has been pointed out by Mr. Haughton, in his first paper, that if take into consideration that part only of the elasticity of a solid ich arises from the mutual actions of centres of force, so that the ction V shall contain at most but fifteen unequal coefficients (viz. se whose values are given in equation 29), and if, with those fifteen fficients, we construct a surface of the fourth order, whose equation the following,

$$\begin{aligned} U = {} & (\alpha^4)\,x^4 + (\beta^4)\,y^4 + (\gamma^4)\,z^4 \\ & + 6\,(\lambda^2)\,y^2z^2 + 6\,(\mu^2)\,z^2x^2 + 6\,(\nu^2)\,x^2y^2 \\ & + 12\,(\alpha\lambda)\,x^2yz + 12\,(\beta\mu)\,xy^2z + 12\,(\gamma\nu)\,xyz^2 \\ & + 4\,(\alpha\nu)\,x^3y + 4\,(\alpha\mu)\,x^3z \\ & + 4\,(\beta\lambda)\,y^3z + 4\,(\beta\nu)\,y^3x \\ & + 4\,(\gamma\mu)\,z^3x + 4\,(\gamma\lambda)\,z^3y \\ = {} & 1 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A); \end{aligned}$$

n will U be the same function of the six quantities

$$x^2, \quad y^2, \quad z^2, \quad 2yz, \quad 2zx, \quad 2xy,$$

t $-2V$ is of the six strains

$$\alpha, \quad \beta, \quad \gamma, \quad \lambda, \quad \mu, \quad \nu,$$

ich are known to be transformed by the same equations with the ve functions of the second order of x, y, z; and consequently the ne equations which serve to transform the coefficients of the surface $= 1$, into those suitable for a new set of rectangular coordinates, will o serve to transform the coefficients of elasticity in the function V.

Now it is obvious that if the equation

$$\phi(x, y, z) = \psi(x', y', z')$$

true for two sets of rectangular coordinates having the same origin, n must the equation

$$\phi\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right) = \psi\left(\frac{d}{dx'}, \frac{d}{dy'}, \frac{d}{dz'}\right) \ldots\ldots\ \ldots\ldots (B)$$

true also.

It follows that the fifteen coefficients of elasticity (α^4) &c., which are oportional to the differential coefficients of U of the fourth order with spect to x, y, z, are transformable by means of the same equations hich serve to transform the fifteen algebraical functions of the fourth der of x, y, z, by which they are respectively multiplied in the value U.

The following is the investigation of those fifteen equations of th urth order, as well as of the six equations of the second order, fro hich they are formed by multiplication.

Let the relative direction-cosines of the two sets of rectangular [illegible] be expressed as follows:

$$\begin{array}{c|ccc|l} \text{Original Axes.} & & \text{New Axes.} & & \\ & x & y & z & \\ \hline x & a_1 & b_1 & c_1 & \\ y & a_2 & b_2 & c_2 & \text{cosines.} \\ z & a_3 & b_3 & c_3 & \end{array}$$

Let the following notation be used for functions of those [illegible] (It is the same which is employed by Mr. Haughton):

$$p_1 = b_1c_1; \quad p_2 = b_2c_2; \quad p_3 = b_3c_3;$$
$$q_1 = c_1a_1; \quad q_2 = c_2a_2; \quad q_3 = c_3a_3;$$
$$r_1 = a_1b_1; \quad r_2 = a_2b_2; \quad r_3 = a_3b_3;$$

$$l_1 = b_2c_3 + b_3c_2; \quad l_2 = b_3c_1 + b_1c_3; \quad l_3 = b_1c_2 + b_2c_1;$$
$$m_1 = c_2a_3 + c_3a_2; \quad m_2 = c_3a_1 + c_1a_3; \quad m_3 = c_1a_2 + c_2a_1;$$
$$n_1 = a_2b_3 + a_3b_2; \quad n_2 = a_3b_1 + a_1b_3; \quad n_3 = a_1b_2 + a_2b_1;$$

then the following are the six equations of transformation of the [illegible] order for the surface $U' = 1$,

$$x'^2 = x^2a_1^2 + y^2a_2^2 + z^2a_3^2 + 2yza_2a_3 + 2zxa_3a_1 + 2xya_1a_2,$$

(for y'^2, z'^2, similar equations in b, c, respectively).

$$y'z' = x^2p_1 + y^2p_2 + z^2p_3 + yzl_1 + zxl_2 + xyl_3,$$

(for $z'x'$, a similar equation in q and m),

(for $x'y'$, a similar equation in r and n) ....(

Those equations are made applicable to the transformation of [illegible] by the following substitutions:

for x^2, y^2, z^2, $2yz$, $2zx$, $2xy$,

substitute α, β, γ, λ, μ, ν;

and to that of pressures, by the following:

for x^2, y^2, z^2, yz, zx, xy,

substitute P_1, P_2, P_3, Q_1, Q_2, Q_3;

and similar substitutions for the accented symbols.

The following are the fifteen equations of transformation of the [illegible] order:

$$x'^4 = x^4a_1^4 + y^4a_2^4 + z^4a_3^4 + 6y^2z^2a_2^2a_3^2 + 6z^2x^2a_3^2a_1^2 + 6x^2y^2a_1^2a_2^2$$
$$+ 12x^2yza_1^2a_2a_3 + 12xy^2za_1a_2^2a_3 + 12xyz^2a_1a_2a_3^2$$
$$+ 4x^3ya_1^3a_2 + 4x^3za_1^3a_3 + 4y^3za_2^3a_3 + 4y^3xa_2^3a_1 + 4z^3xa_3^3a_1 + 4z^3ya_3^3 \ldots$$

(for y'^4, z'^4, similar equations in b, c, respectively).

$$y'^2z'^2 = x^4p_1^2 + y^4p_2^2 + z^4p_3^2 + y^2z^2(l_1^2 + 2p_2p_3) + z^2x^2(l_2^2 + 2p_3p_1) + x^2y^2(l_3^2 + \ldots$$
$$+ 2x^2yz(p_1l_1 + l_2l_3) + 2xy^2z(p_2l_2 + l_3l_1) + 2xyz^2(p_3l_3 + l_1l_2)$$
$$+ 2x^3yp_1l_3 + 2x^3zp_1l_2 + 2y^3zp_2l_1 + 2y^3xp_2l_3 + 2z^3xp_3l_2 + 2z^3yp_3l_1;$$

(for $z'^2x'^2$, a similar equation in q and m);

(for $x'^2y'^2$, a similar equ[illegible] and n);

$$
\begin{aligned}
x'^2 y'z' = {} & x^4 a_1^2 p_1 + y^4 a_2^2 p_2 + z^4 a_3^2 p_3 \\
& + y^2z^2(a_3^2 p_2 + a_2^2 p_3 + 2a_2 a_3 l_1) \\
& + z^2x^2(a_3^2 p_1 + a_1^2 p_3 + 2a_3 a_1 l_2) \\
& + x^2y^2(a_1^2 p_2 + a_2^2 p_1 + 2a_1 a_2 l_3) \\
& + x^2yz(a_1^2 l_1 + 2a_2 a_3 p_1 + 2a_3 a_1 l_3 + 2a_1 a_2 l_2) \\
& + xy^2z(a_2^2 l_2 + 2a_3 a_1 p_2 + 2a_1 a_2 l_1 + 2a_2 a_3 l_3) \\
& + xyz^2(a_3^2 l_3 + 2a_1 a_2 p_3 + 2a_2 a_3 l_2 + 2a_3 a_1 l_1) \\
& + x^3y(a_1^2 l_3 + 2a_1 a_2 p_1) + x^3z(a_1^2 l_2 + 2a_1 a_3 p_1) \\
& + y^3z(a_2^2 l_1 + 2a_2 a_3 p_2) + y^3x(a_2^2 l_3 + 2a_2 a_1 a_2) \\
& + z^3x(a_3^2 l_2 + 2a_3 a_1 p_3) + z^3y(a_3^2 l_1 + 2a_3 a_2 p_3);
\end{aligned}
$$

(for $x'y'^2z'$, a similar equation in b, q, m);

(for $x'y'z'^2$, a similar equation in c, r, n).

$$
\begin{aligned}
x'^3 y' = {} & x^4 a_1^2 r_1 + y^4 a_2^2 r_2 + z^4 a_3^2 r_3 \\
& + 3y^2z^2 a_2 a_3 n_1 \\
& + 3z^2x^2 a_3 a_1 n_2 \\
& + 3x^2y^2 a_1 a_2 n_3 \\
& + 3x^2yz(a_1^2 n_1 + 2a_2 a_3 r_1) \\
& + 3xy^2z(a_2^2 n_2 + 2a_3 a_1 r_2) \\
& + 3xyz^2(a_3^2 n_3 + 2a_1 a_2 r_3) \\
& + x^3y(a_1^2 n_3 + 2a_1 a_2 r_1) + x^3z(a_1^2 n_2 + 2a_3 a_1 r_1) \\
& + y^3z(a_2^2 n_1 + 2a_2 a_3 r_2) + y^3x(a_2^2 n_3 + 2a_1 a_2 r_2) \\
& + z^3x(a_3^2 n_2 + 2a_3 a_1 r_3) + z^3y(a_3^2 n_1 + 2a_2 a_3 r_3);
\end{aligned}
$$

(for x'^3z', a similar equation in a, q, and m);

(for y'^3z', a similar equation in b, p, and l);

(for y'^3x', a similar equation in b, r, and n);

(for z'^3x', a similar equation in c, q, and m);

(for z'^3y', a similar equation in c, p, and l) .... (D).

...ve equations are made applicable to the transformation of the ... of elasticity arising from the mutual actions of centres of ..., by the following substitutions:

	x^4,	y^4,	z^4,	y^2z^2,	z^2x^2,	x^2y^2,	
	(α^2),	(β^2),	(γ^2),	$(\beta\gamma) = (\lambda^2)$,	$(\gamma\alpha) = (\mu^2)$,	$(\alpha\beta)$	(ν^2);

for $\quad x^2yz, \quad xy^2z, \quad xyz^2,$

...stitute $(\alpha\lambda) = (\mu\nu)$, $(\beta\mu) = (\nu\lambda)$, $(\gamma\nu) = (\lambda\mu)$;

for $\quad x^3y, \quad x^3z, \quad y^3z, \quad y^3x, \quad z^3x, \quad z^3y,$

...stitute $(\alpha\nu)$, $(\alpha\mu)$, $(\beta\lambda)$, $(\beta\nu)$, $(\gamma\mu)$, $(\gamma\lambda)$;

... substitutions for the accented symbols.

... the substance under consideration be endowed with a portion ...sticity in addition to that which arises from the ... of force, the coefficient of that fluid elasticity ... the coefficients into which it enters; viz.

..., (γ^2), $(\beta\gamma) = (\lambda^2) + J$, $(\gamma\alpha) = (\mu^2) + J$, ...

...g the transformation.

The results of the transformation for those six coefficients, bein increased by the same quantity J which was previously subtracte will give their entire values for the new axes.

If the original axes of coordinates are those of elasticity, each of th fifteen equations of transformation is reduced to its first six terms, i which the following substitutions are to be made for the unaccent symbols :

for x^4, y^4, z^4, y^2z^2, z^2x^2, x^2y^2,
substitute $A_1 - J$, $A_2 - J$, $A_3 - J$, $B_1 - J = C_1$, $B_2 - J = C_2$, $B_3 - J = C_3$

59, St. Vincent Street, Glasgow,
April 7, 1852.

ON THE DOCTRINE OF IMPOSSIBLES IN ALGEBRAIC GEOMETR

By William Walton.

In a work of great value *On the Higher Plane Curves*, b Mr. Salmon, which has recently appeared, may be see a general critique on the theory of Impossibles, in whic he has pronounced an unfavourable opinion in regard t the special views propounded some years ago by the lat Mr. Gregory and by myself on the geometrical interpreta tion of impossibles. As I cannot, after a careful perusal o his remarks, assent to the validity of the grounds of hi objections, I feel that I have no alternative but to expres in this *Journal*, where the theory was originally published a few observations on the points at issue. I may perhap be allowed to observe that, from the very high regard which I entertain for the genius of this writer, I do not enter upon this question without duly considering the weight of his authority on matters of geometrical science.

The papers in the *Cambridge Mathematical Journal* which appear to form the subject of Mr. Salmon's critique, are the following, the first of them being Mr. Gregory's:

"On the Existence of Branches of Curves in several Planes." *May*, 1839.

"On the General Theory of the Loci of Curvilinear Inter section." *February*, 1840.

"On the General Interpretation of Equations betwee two variables in Algebraic Geometry." *May*, 1840.

"On the General Theory of Multiple Points." *November*, 1840.

"On the Existence of Possible Asymptotes to Impossibl Branches of Curves." February, 1841.

I will now endeavour to state, as concisely as possible, Mr. Salmon's objections to the theory advocated in these papers, together with some of his inferences and remarks. Lest I should not have given an adequate abstract of his opinions, the reader is requested, if convenient, to consult his own work. I will commence, however, with transcribing a portion of Mr. Salmon's statement of Mr. Gregory's views, in order to render the observations of Mr. Salmon intelligible.

"It is to a certain extent arbitrary what interpretation we give to our algebraical equations; but the greatest advantage is gained when we adopt the most general methods, and when every algebraical symbol has its appropriate geometrical representation. Thus the inventors of analytic geometry might, if they pleased, have left the sign $-$ uninterpreted, and, confining their attention to the positive values of the variables, have only considered those branches of a curve which lie in the upper right-hand angle between the axes. It was soon seen, however, how much generality might be gained by interpreting the line $-a$ as of equal length, but opposite direction, to the line $+a$; and no curve is now considered as completely traced unless the negative, as well as the positive, values of the variables be taken into account. This, however, is merely a matter of convention, and we might, if we pleased, have restricted ourselves to the positive values of the coordinates."..."This system of interpretation" (viz. that given by the theory of impossibles developed in the articles named) "is quite as legitimate an extension as that of the negative values of the variables, and is as necessary to the thorough understanding the course of a curve."

The following is a list of Mr. Salmon's objections and some of his remarks:

(1). "Our method of interpretation is not so wholly conventional as Mr. Gregory represents. It is necessary, in the first place, that our interpretations should be consistent; thus, had we commenced by rejecting the negative values of the variables, we should have discovered the incompleteness of our method, by finding that on transformation of coordinates our equations had no longer the same geometrical signification."

(2). "We do not obtain our first knowledge of geometrical figures from interpreting the equations of analytic geometry; but, on the contrary, our interpretations of analytical equa-

tions must be made to coincide with our previous geometrical knowledge."......" We know what a circle is before we know anything about the equation $x^2 + y^2 = a^2$, and any interpretation of this equation differing either by defect or excess from our previous geometrical conception, must be rejected. We discover that we should be wrong in leaving the sign $-$ uninterpreted, because then the equation $x^2 + y^2 = a^2$ would only represent a fragment of a circle; and we may in the same manner discover that it is objectionable to give a real interpretation to the symbol $\sqrt{(-1)}$ in the equations of analytic geometry, because then $x^2 + y^2 = a^2$ represents not only a circle but an irrelevant curve besides."

(3). " In Mr. Gregory's first papers, when he came to an imaginary value, he usually contented himself with saying that the curve left the plane of reference, without much troubling himself to inquire where it went to."

(4). "This omission was supplied in a paper by Mr. Walton, which I cannot but regard as an able *reductio ad absurdum* of Mr. Gregory's theory. He finds that, according to Mr. Gregory's principles, properly generalized, the equation $x^2 + y^2 = a^2$ represents not only a circle, but also a curve of the fourth degree in space, whose ordinary equations would be written

$$x^2 + y^2 - x^2 \tan^2 2r\pi - (z - x \tan 2r\pi)^2 = a^2 \cos 4m\pi,$$
$$2x(x - y) \tan 2r\pi + 2yz = a^2 \sin 4m\pi,$$

where, however, r is arbitrary; and we are as much at liberty to choose which of an infinity of such curves we are to regard as the companion of the circle, as in M. Poncelet's method we were left to choose between an infinity of equilateral hyperbolæ." (Mr. Salmon has given an account of M. Poncelet's method). " Now if it can be shewn that our ordinary conceptions of a circle are defective, and that one or all of these curves possess the same properties, and are entitled to be regarded as a portion of the same curve, then Mr. Gregory's mode of interpretation must be hailed as a great discovery. But if these curves differ from a circle in form and properties, then it is an abuse of language to speak of them as branches of a circle, merely because they can be represented by the same equation."

(5). "To talk of plane curves having branches out of their plane appears to me calculated to confound all a student's ideas, to make him likely to lose his hold of the principle

at a curve of the n^{th} degree is always cut by a right line n points, and to make him fancy that he has distinct nceptions, which he cannot possibly have, of imaginary ints and lines."

(6). "The explanation of conjugate points on this theory es not shew why these points should be always double ints, and at any rate the relation of such a point (conlered as the limit of an oval) to the curve, seems sufficiently telligible."

(7). "The same theory has been applied to the explanation an impossible branch of a curve having a real asymptote; t this is nothing more than a conjugate point at infinity."

Having now stated Mr. Salmon's observations, I will proed to reply to them seriatim.

(1). Mr. Gregory's remarks on the use of the sign − e I conceive based on the supposition that our object to interpret the geometrical signification of equations in gebraic geometry in accordance with the laws of comnation of algebraic symbols and with the geometrical eanings primarily assigned to symbols of quantity and fection. The ancient geometricians, unacquainted with gebraic analysis, had no alternative but to discuss the roperties of curves by geometrical reasonings. On the vention of the Cartesian method of geometry the primary bject of geometricians was to express in an algebraic form e known geometrical properties of curves. A species of verse investigation was yet open to the inquiries of mathematicians, viz. to define curves by algebraic formulæ, and ence to infer their forms by interpretation subjected to y or no restriction. Mr. Gregory's observations are I ink obviously based upon the latter hypothesis, and I do ot see how any objection can be properly urged against e assertion, that we are entitled, if we choose, to restrict rselves to the exclusive use of the sign +, it following course thereby that our equations have a more limited ometrical signification than they would have were the sign also accepted. If we first of all define a circle geometrilly, and then, using the equation $x^2 + y^2 = a^2$, confine ourlves to the sign +, calling the locus of the equation a circle, e no doubt contradict ourselves by creating incompatible finitions. But if we define a curve as the "locus of the uation $x^2 + y^2 = a^2$, on the supposition that the sign − is jected," we certainly are guilty of no inconsistency. It may

[redacted]

would not be inconsistent with this hypothesis, but would add something to it. In fact, branches of the curve emanating from different points of the finite oval would intersect at the point of the oval's evanescence. If however we adopt the theory of the vanishing oval to the exclusion of all other theory, how are we to explain the fact that $\frac{dy}{dx}$ is sometimes possible and sometimes impossible at a conjugate point? For examples, at the conjugate point $x = a$, $y = a^3c$, of the curve

$$(y - cx^3)^2 = (x - b)^5 (x - a)^6,$$

where a is supposed to be less than b, we see that $\frac{dy}{dx} = 3a^2c$; while, at the conjugate point $x = a$, $y = b$, of the curve

$$(y - b)^2 = (x - a)^2 (x - c),$$

where a is less than c,

$$\frac{dy}{dx} = \pm (a - c)^{\frac{1}{2}},$$

impossible values. A geometrical explanation of these analytical phenomena flows naturally from the doctrine of impossibles.

(7). Supposing that what I have called an asymptote to an impossible branch of a curve resolves itself into nothing more than a conjugate point at infinity, my reply to the preceding division of Mr. Salmon's remarks will be available as far as a vindication of the fundamental principles of the theory of impossibles is concerned. I do not however concur with Mr. Salmon in considering it to be nothing more than a conjugate point at infinity. I admit that it may be regarded, if we please, as a conjugate point of a peculiar kind. It may in fact be considered as a conjugate point at an infinite distance in a straight line, dependent in position upon the parameters of the curve, which passes within a finite distance from the origin, and of which the ordinates at an infinite distance differ by zero from those of the curve. Such lines are, under ordinary circumstances, called rectilinear asymptotes. In saying that we are at liberty, if we please, to take this view of possible asymptotes to impossible branches, I am virtually concurring in the opinion expressed by Mr. Gregory in an analogous question, that the inventors of analytic geometry might, if they pleased,

have left the sign − uninterpreted, thereby forfeiting the advantages of a more enlarged interpretation.

I will avail myself of this opportunity of developing the principles of the transformation of coordinates applicable either to possible or impossible values of the coordinates. I shall adopt the relations given in my paper of May, 1840, which connect the quantitive and affectional coordinates. Like principles would be applicable for the transformation of coordinates on Mr. Gregory's or any other legitimate method of constructing the impossible axes.

Let x, y, be the affectional coordinates of any point referred to any rectangular axes; x_1, y_1, those of the same point referred to any other such axes in the plane of reference. Let

$$x = (+)^r\alpha, \quad y = (+)^s\beta,$$
$$x_1 = (+)^{r_1}\alpha_1, \quad y = (+)^{s_1}\beta_1.$$

Let x', y', z', be the quantitive coordinates of the same point in relation to the former, and x_1', y_1', z_1', in relation to the latter axes. Then, θ being the angle between the axes of x' and x_1', we have

$$x' = x_1' \cos\theta - y_1' \sin\theta, \quad y' = x_1 \sin\theta + y_1' \cos\theta, \quad z' = z_1';$$

and therefore

$$\alpha \cos 2r\pi = \alpha_1 \cos 2r_1\pi \cos\theta - \beta_1 \cos 2s_1\pi \sin\theta \ \ldots (1),$$
$$\beta \cos 2s\pi = \alpha_1 \cos 2r_1\pi \sin\theta + \beta_1 \cos 2s_1\pi \cos\theta \ \ldots (2),$$
$$\alpha \sin 2r\pi + \beta \sin 2s\pi = \alpha_1 \sin 2r_1\pi + \beta_1 \sin 2s_1\pi.$$

From these three equations we see that

$$\tan 2r\pi(\alpha_1 \cos 2r_1\pi \cos\theta - \beta_1 \cos 2s_1\pi \sin\theta)$$
$$+ \tan 2s\pi(\alpha_1 \cos 2r_1\pi \sin\theta + \beta_1 \cos 2s_1\pi \cos\theta)$$
$$= \alpha_1 \sin 2r_1\pi + \beta_1 \sin 2s_1\pi.$$

This equation must not establish any relation between α_1 and β_1: hence we must have

$$\tan 2r\pi \cos\theta + \tan 2s\pi \sin\theta = \tan 2r_1\pi \ldots\ldots\ldots (3),$$

and $$-\tan 2r\pi \sin\theta + \tan 2s\pi \cos\theta = \tan 2s_1\pi \ldots\ldots\ldots (4).$$

The equations (1) and (2) are equivalent to

$$(+)^{-r}.x.\cos 2r\pi = (+)^{-r_1}.x_1.\cos 2r_1\pi.\cos\theta - (+)^{-s_1}.y_1.\cos 2s_1\pi.\sin\theta \ldots (5),$$
$$(+)^{-s}.y.\cos 2s\pi = (+)^{-r_1}.x_1.\cos 2r_1\pi.\sin\theta + (+)^{-s_1}.y_1.\cos 2s_1\pi.\cos\theta \ldots (6).$$

Thus (5) and (6) are the proper formulæ of transformation,

r_1 and s_1 being determinable, by the equations (3) and (4) in r, s, θ.

If $r = 0$, $s = 0$, then $r_1 = 0$, $s_1 = 0$, and the formulæ of transformation are reduced to

$$x = x_1 \cos\theta - y_1 \sin\theta, \quad y = x_1 \sin\theta + y_1 \cos\theta.$$

From the formulæ here given it is evident that neither the curvilinear loci nor the superficial locus of an equation between x and y is affected by the transformation of co-ordinates, and that accordingly no such objection, whatever may be its weight, as that urged by Mr. Salmon against Poncelet's method is available against Mr. Gregory's or mine.

I may here briefly remark, that the principles of the general interpretation of the equation $f(x, y) = 0$ by the doctrine of impossibles may be easily applied to the interpretation of the equation $f(x, y, z) = 0$. I may sometime enter more fully into this question.

Cambridge, March 17, 1852.

ON THE SIGNS + AND − IN GEOMETRY (*continued*), AND ON THE INTERPRETATION OF THE EQUATION OF A CURVE.

By Professor De Morgan.

Having recently had the advantage of looking over Mr. Salmon's two excellent works, the second edition of the *Conic Sections*, and that on the *Higher Plane Curves*, I beg to offer a few remarks on two points connected with the subject of those works: the general signification of an equation of two variables, first started, I believe, by D. F. Gregory; and the extension of the interpretation of the signs + and −, which I commenced in my *Differential Calculus*, and continued in this *Journal* (vol. vi. p. 156).

The first matter now stands thus: Gregory (*Camb. Math. Journal*, vol. i. p. 259) assigned to the equation $\phi(x, y) = 0$, the ordinary plane curve, and an infinity of curves not in the plane, derived from imaginary values of x and y; and from the first reading of his paper, I was satisfied that this infinity of curves constituted a surface. Mr. Walton (*Camb. Math. Journal*, vol. ii. p. 103, a paper which I did not know of until I saw Mr. Salmon's reference) maintains that the equation is always that of a surface, developes Gregory's mode of interpreting the imaginaries, and draws on the

surface an infinite number of *companions* to the plane curve. Mr. Salmon (*Higher Plane Curves*, p. 301) objects to the extension, in a manner which most readers of this *Journal* will see for themselves.

It seems to me that a distinction is required between algebraic geometry and geometrical algebra; using the substantive to denote the object-matter, the adjective to denote the auxiliary. We use algebra in aid of geometry to assist in gaining knowledge of forms: we use geometry in aid of algebra to assist in gaining representation of functions. In the first case, a circle, for instance, is our *datum*, and must not be associated with other curves at the bidding of algebra: in the second, a circle is the representation of the relations of real values in $x^2 + y^2 =$ const., and we are bound to select some extension which, while it includes this mode of representing the real values, shall also include more. The question, 'Given algebraical phenomena, to substitute geometrical ones for them,' is not answered until all relations of value are represented: and the geometer, in his search after the properties of defined curves, has no more right to limit the algebraist in his representation of defined functions, than the latter has a right to compel the former to enlarge his definition. Accordingly, I agree with Mr. Salmon in one capacity, and differ from him in the other.

The object of the *algebraist* is representation: and he adapts his coordinates accordingly. If, having to discuss $x^2 + y^2 = a^2$, he adopt rectilinear coordinates, but prefer polar ones for $y \cos x = a$, it is because each adaptation is the best for the case: he does not deny that, for some purposes, the character of the coordinates might be advantageously transposed. If a point be determined by three coordinates, and not more, it is because three is the *minimum* number; of which it is one convenience that each point has one set of coordinates only. But let there be $3 + m$ coordinates, then a surface will require $1 + m$ equations, and a curve $2 + m$. A curve, the geometrical intersection of surfaces, would indeed not be necessarily the result of all the equations of two surfaces, unless either $m = 0$, or the $2 + 2m$ equations be reducible to $2 + m$, m of them being deducible from the rest. When m is not $= 0$, a point on the common curve will generally be derived from different values of coordinates in the two surfaces.

Extending the number of coordinates, we preserve the Cartesian definition, namely, that each coordinate is in a line

of determinate direction chosen from among those which pass through a given point, and that the point determined by coordinates is at the extremity of their resultant. If there be six coordinates, $x, y, z, \xi, \eta, \zeta$, we must have four equations for a surface, say $V_1 = 0$, $V_2 = 0$, $V_3 = 0$, $V_4 = 0$. If we choose three axes of ordinary coordinates for ultimate reference, we have three equations of the form

$$X = ax + by + cz + \alpha\xi + \beta\eta + \gamma\zeta;$$

from which, with the four equations of the surface, we eliminate the six quantities x, ξ, &c., and produce an ordinary equation between X, Y, Z.

Generally, we can only assign two of the six x, ξ, &c.: but if we assign more than two values or relations, in such manner as to satisfy some of the four equations identically, we may thus be able to point out a curve upon the surface. Let, for instance, $\xi = 0$, $\eta = 0$, $\zeta = 0$, identically satisfy $V_3 = 0$, $V_4 = 0$; there remain two equations of the four, and the three $X = ax + by + cz$, &c., from which we may eliminate x, y, z, and thus produce two equations between X, Y, Z, belonging to a certain curve upon the surface. All this happens if we derive V_1 &c. from the two equations

$$\phi(x + \xi\sqrt{-1},\quad y + \eta\sqrt{-1},\quad z + \zeta\sqrt{-1}) = V_1 + V_2\sqrt{-1},$$
$$\psi(x + \xi\sqrt{-1},\quad y + \eta\sqrt{-1},\quad z + \zeta\sqrt{-1}) = V_3 + V_4\sqrt{-1}.$$

If then we take the ordinary curve

$$\phi(x, y, z) = 0, \quad \psi(x, y, z) = 0,$$

and ask how we may give representation to the solutions of these equations when imaginary values are used, one answer, among an infinite number, is as follows. Let the real parts of the coordinates be measured on the original axes, and the imaginary parts on three other axes chosen at pleasure through the same origin: let the point determined be at the extremity of the resultant of the six coordinates. Then the equations $\phi = 0$, $\psi = 0$, belong to a surface, and the curve of real values is on that surface. It does not follow that every surface can be so represented: for V_1 and V_2 are related functions, as are V_3 and V_4.

It will be observed that $\sqrt{-1}$ is not here assumed as the sign of perpendicularity: the three imaginary axes may be at any angles to the real axes and to one another. The attainment of full representation no more depends on x and ξ being at right angles to one another, than upon x and y being so related. It is not pretended that any

generalization of *operations* can take place: and, in so far as Mr. Gregory and Mr. Walton assume, if they do at all assume, that x and ξ must necessarily be at right angles to each other, I differ from them with Mr. Salmon, until further cause is shewn. With respect, however, to the transformation of coordinates, the case seems to stand as follows. This transformation can be made if we can assume three pairs of equations with the same coefficients, such as

$$x = ax_1 + by_1 + cz_1, \quad \xi = a\xi_1 + b\eta_1 + c\zeta_1;$$

x_1, ξ_1, &c. being the new coordinates. In this case, $x + \xi\sqrt{-1}$ becomes

$$a(x_1 + \xi_1\sqrt{-1}) + b(y_1 + \eta_1\sqrt{-1}) + c(z_1 + \zeta_1\sqrt{-1}),$$

and we clearly have, after substitution in $\phi = 0$, and $\psi = 0$, a pair of equations to the same absolute surface, with the same absolute curve in space for the real values. But this requires that a, b, c, &c., nine quantities in number, should satisfy twelve equations; that is, our supposition demands three relations between the twelve angles

$$xy,\ yz,\ zx,\quad x_1y_1, \text{ \&c.},\quad \xi\eta, \text{ \&c.},\quad \xi_1\eta_1, \text{ \&c.}$$

The resultant of both sets of real coordinates is the same, and of both sets of imaginary coordinates. If the axes of x, y, z, make the same angles with one another which are made by the axes of ξ, η, ζ, and if the same be true of $x_1y_1z_1$, and $\xi_1\eta_1\zeta_1$, all conditions are satisfied; and the position of $x_1y_1z_1$ with respect to xyz is the same as that of $\xi_1\eta_1\zeta_1$ with respect to $\xi\eta\zeta$. But the permanence in space of the surface, under transformation of coordinates, concerns the algebraic geometer, not the geometrical algebraist.

The most symmetrical of all easy cases, is that in which the real axes of x, y, z, are the imaginary axes of ζ, ξ, η. Taking the axes of x, y, z, as the ultimate axes, we have then

$$X = x + \zeta, \quad Y = y + \xi, \quad Z = z + \eta.$$

If the real curve be $x^2 + y^2 = a^2$, $z = 0$, the surface is

$$(X - Z)(X^2 - Z^2 - a^2) + (X + Z)Y^2 = 0.$$

This surface is made by an hyperbola of variable minor axis revolving about the axis of z. The major semiaxis is always a: the minor varies from a to 0. The points requiring a moment's attention are the mode in which the axis of y belongs to the surface, and also the mode in which one asymptote of the equilateral hyperbola belongs to the surface at an infinite distance.

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I am not sure I understand the particular way in which D. F. Gregory proposed to interpret the imaginary values. Mr. Walton makes the axis of z to be that of ξ and η both, so that the planes of xz and yz are separate planes of interpretation for x and y. This, whatever may be thought of it when $z + \zeta\sqrt{-1}$ also takes finite values, is the most striking method when the real curve is plane. It gives in that case

$$X = x, \quad Y = y, \quad Z = \xi + \eta.$$

And $x^2 + y^2 = a^2$ gives the surface

$$(X^2 + Y^2)\,Z^2 = (X^2 + Y^2 - a^2)\,(Y - X)^2.$$

The preceding *representations*, looked on as nothing more, should, I think, remain at one boundary of the subject, as indicating a direction in which further inquiry should be made. The particular cases in which one coordinate becomes imaginary, the other continuing real, are those which most want illustration, and which most decidedly receive it.

The second matter of this paper consists in some further applications of the method of interpreting + and − in plane geometry. They are no more than I have found to be necessary in making demonstrations coextensive with their enunciations. The numbering is continued from the paper already cited, and I have not thought it worth while to insert demonstrations. It is to be remembered that every equation here given is asserted to be universally true, under the previous conventions as to sign.

11. Let AB, BC, CA, be in the lines R, P, Q. Then

$$\begin{aligned} AC\sin Q^\circ R &= BC\sin P^\circ R = CB\sin R^\circ P, \\ AB &= AC\cos Q^\circ R + CB\cos R^\circ P, \\ AB^2 &= AC^2 + CB^2 + 2AC.CB\cos P^\circ Q, \\ &= AC^2 + BC^2 - 2AC.BC\cos P^\circ Q. \end{aligned}$$

12. If four concurrent lines A, B, C, D, contain the four colinear points a, b, c, d, then

$$\frac{ab}{bc}\cdot\frac{cd}{da} = \frac{\sin A^\circ B}{\sin B^\circ C}\cdot\frac{\sin C^\circ D}{\sin D^\circ A},$$

each side of which is − 1 when the pencil is harmonic.

13. Let every line drawn through the origin be called an *original*, and *the* original of all parallels to it. Again, every line divides the whole plane of reference into two half-planes. Let these half-planes take signs, that which contains the

rigin taking sign from the original perpendicular *to* the straight line. But when the line is original, let its half-planes take sign from the parts of the original perpendicular with which the straight line makes a right angle. Inversion of the straight line is also inversion of the half-planes.

14. Parallels drawn through two points of the same original line have the same signature of translation, and make the same angles with any third line. And if P and Q be parallels of the same length and direction, the indefinite equation $P = \pm Q$ may be settled from the equation

$$P \sin P^{\circ}R = Q \sin Q^{\circ}R,$$

R being any third line: $P^{\circ}R$ and $Q^{\circ}R$ being either equivalents or *opponents*.*

15. Let the origin O be removed to $O'(m, n)$, and let new rectangular axes be taken, $x'^{\circ}x$ being ϕ. Let OO' be of the same sign with reference to both the new axes, and let the new coordinates (measured *on*, not *parallel to*, their axes) take the signs determined by the system, $y'^{\circ}x'$ being $\frac{1}{2}\pi$ or $\frac{3}{2}\pi$, as it shall happen. Then

$$x = m + x' \cos\phi - y' \sin\phi . \sin y'^{\circ}x',$$
$$y = n + x' \sin\phi + y' \cos\phi . \sin y'^{\circ}x',$$
$$x' = x \cos\phi + y \sin\phi - (m \cos\phi + n \sin\phi).$$

Let the axis of x' be original: and let $O\dot{O}' = p$. Then

$$O'O = -(m \cos\phi + n \sin\phi),$$

and
$$x' = x \cos\phi + y \sin\phi - p.$$

That is to say, if the original perpendicular drawn *to* any straight line be p, and if $p^{\circ}x = \phi$, then the perpendicular drawn *from* the straight line to any point (x, y), projected upon the original perpendicular, is $x \cos\phi + y \sin\phi - p$: but the projected perpendicular drawn *to* the straight line is $p - x \cos\phi - y \sin\phi$; and this last has the sign of the half-plane in which (x, y) lies.

Let the above line be denoted by (p, ϕ): then $(p, \phi)^{\circ} x$ is always $\phi + \frac{1}{2}\pi$, and $(p, \phi)^{\circ}(q, \psi)$ is $(\phi + \frac{1}{2}\pi) - (\psi + \frac{1}{2}\pi)$ or $\phi - \psi$.

16. Two concurrent straight lines make four *Euclidean* angles, which may be named the $++$, $+-$, $-+$, $--$, angles after the signs of translation of the bounding lines drawn

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* I call $\pi + \theta$ the *opponent* of θ, and $2\pi - \theta$ the *completion of* θ.

The second formula does not change if all the tangents be augmented or multiplied by the same, or changed into their reciprocals, or subjected to any combination of these changes. One such combination is that which changes t into $(m + nt) : (m' + n't)$. Hence, U and V being any straight lines, the anharmonic ratio of $a_1 U + b_1 V$, &c. is the third of the preceding formulæ.

22. The radius of curvature is to be the line drawn *from* the centre of curvature *to* the curve. The formula $\dfrac{ds^3}{dx\,d^2y}$ always gives the projection of this radius on the original perpendicular to the tangent, if ds take sign on the tangent.

23. When a polygon is taken in the ordinary sense, all the original radii drawn to angular points must have one sign. Anything in contravention of this would amount to using a line and its inversion in determining two adjacent angles, not the same line in both.

24. Theorems relative to non-original lines are for convenience, not of necessity, often constructed with reference to their original parallels. Passage of a line through the origin either inverts that line, or its original radii: and theorems which are true of lines on one side of their original parallel, are often inverted for lines on the other.

I cannot find any instance in which the preceding extensions are either contradictory of each other, or insufficient either for demonstration or interpretation. And I think I can undertake to remove any difficulty in these respects, which they may present to any reader who will trouble himself to master their details.

March 25, 1852.

Though any case of the method given in the first part of this paper succeeds in representing all values of x and y, and illustrating all phenomena of curves, yet I am inclined to maintain that no one of them is the direct extension of the Cartesian method of coordinates. This direct extension I take to be as follows:

Let $X = (x, \xi)$ signify that X is a line of x units of length, inclined at the angle ξ to the unit-line. Then, Y being (y, η), X and Y are said to be *coordinates* of the other extremity of $X + Y$, when one extremity is the origin. That

is, X and Y are coordinates of a point whenever x,
are so taken as to satisfy two *equations of coordination*,

$$\phi(x, y, \xi, \eta) = 0, \quad \psi(x, y, \xi, \eta) = 0,$$

which define a *system* of coordinates. In the common
the equations of coordination are $\xi = 0$, $\eta = \frac{1}{2}\pi$. If
a third equation $$F(x, y, \xi, \eta) = 0,$$

we have the definition of a curve, of which this th
be called the equation. Any one of the four bein
the other three are determined. It can easily be sh
even if we allow imaginary values for x, y, &c., we
thereby gain any extension, but only conversion
system of the above kind into another.

To this it will perhaps be objected that the third
ought to be $F(X, Y) = 0$; for it will be said that
tial of the Cartesian method is the representation of t
by a relation *between coordinates*. To this I reply th
we import the Cartesian case into *complete* algebr
not find any equation between coordinates: for th
nates are x and $y\sqrt{-1}$, while the equation is between
What we do find is precisely that which I have
namely, three equations between the four *arithmetic*
sentatives of length and direction by which the tw
nates are determined. One equation between X
equivalent to two equations between x, y, ξ, η. Th
which Gregory first opened, and on which Mr.
Mr. Salmon, and myself, have hitherto written, is
ment of a part of the Cartesian system, extension of
and arbitrary introduction in place of what was ab
Coordination is abandoned: that is to say, all pre
striction on X and Y. The equation of the curve F
which was essentially a relation between lengths, ind
of directions, is extended into $F(X, Y) = 0$, which g
equations between $x\cos\xi$, $x\sin\xi$, $y\cos\eta$, $y\sin\eta$. An
of laying down these last-named lengths are then ch

There is no question that the pure extension, as
to be, does not exhibit any method of interpreting th
nary coordinates of the original system. For ther
such imaginary coordinates; the equations of coor
$\xi = 0$, $\eta = \frac{1}{2}\pi$, cannot be disturbed in that system.
sideration of an equation between coordinates, in th
general form, requires abandonment, and is not co
with pure extension or enlargement of existing
relation.

[redacted]

n the extension, the system of coordination

$$\phi(x, y, \xi, \eta) = 0, \quad \psi(x, y, \xi, \eta) = 0,$$

ibined with the equation of the curve

$$F(x, y, \xi, \eta) = 0,$$

s the ordinary curve obtained by eliminating x, y, ξ, η reen the preceding equations and

$$x_1 = x \cos\xi + y \cos\eta, \quad y_1 = x \sin\xi + y \sin\eta.$$

uly 5, 1852.

CERTAIN SYSTEMS IN SPACE ANALOGOUS TO THE COMPLETE TETRAGON AND COMPLETE QUADRILATERAL.*

By Thomas Weddle.

 a plane figure the straight line that joins two angles is ed a *diagonal*, but so far as I know, no name has been owed on the point in which two sides intersect; such int might, I think, be called a *collateral* with much proty. (It will be observed that 'diagonal' and 'collateral' reciprocal.) In Solid Geometry there are several analos points, lines and planes, and I imagine the following ninology will be found convenient.

. In a solid figure a *diagonal line* is a straight line joining angular points not on the same edge.

. A *diagonal plane* is a plane passing through three ular points not all in one face.

. A *synhedral line* is the intersection of two faces that not pass through the same edge.

. A *synhedral point* is the point of intersection of three s that do not meet in the same angular point.

. When two edges intersect but are not in the same face, do not meet in an angle, the point of intersection is ed a *syngrammatic point*, or simply a *syngram*.

. When two edges are in the same plane, but are in the same face and do not meet in an angle, the ie passing through them is denominated a *diagrammatic* ie.

This paper is intended to form a supplement to that part of my first oir "On the Theorems in Space analogous to those of Pascal and nchon in a Plane," (*Journal*, new series, vol. IV. p. 26), which treats mbilical surfaces.

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... planarity; besides
for the change) we
through three angles
seems to be the app
jected to the term
'diagram' is used in
the adjective only, no
term may therefore be

Again, in Plane Ge
sides, it is a *multilatera*
like manner, we name
faces; we may term it
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triangle and *quadrangle*,
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Moreover, in Plane Ge
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by straight lines. Now the
Geometry to these systems

(1). A *complete polyk*...
(*n*) of pl...

[redacted]

a system, in its most general state, will have $\frac{n(n-1)}{1.2}$ and $\frac{n(n-1)(n-2)}{1.2.3}$ angular points.

). A *complete multangle* is formed by taking a number f points in space for angles, joining every two of them traight lines for edges and passing planes through every for faces. Such a system, in its most general state, have $\frac{n(n-1)}{1.2}$ edges and $\frac{n(n-1)(n-2)}{1.2.3}$ faces.

ese definitions are, as I have already said, the most us analogies in space, but they exclude certain systems n, being analogous to the complete quadrilateral and lete tetragon, seem to have some title to the term plete.' To include these systems it will be necessary odify the two preceding definitions, and after some consi-ation I am inclined to think that the best way of making nodification is by saying that in a *complete polyhedron* point in which *m* of the faces intersect is an angular , and every straight line in which two faces intersect on which are at least $m - 1$ angular points is an edge; hat in a *complete multangle* every plane passing through gular points is a face, and every straight line which two angular points and through which at least $m - 1$ pass is an edge. Whether these definitions will ulti-ly be found convenient may perhaps admit of some t,* but they suit my present purpose better than any s I can think of. It may be here observed that the cases which I shall have to consider are $m = 3$ (when definitions coincide with the former two), and $m = 4$. fore proceeding further I must add two more definitions, thers will be given as wanted.

. If the opposite faces of a hexahedron intersect in straight lines in one plane, the figure will have six ammatic planes and three syngrammatic points; such id figure I shall denominate a *syngrammatic octangular hedron*, or simply a *syngrammatic hexahedron*.

). If the straight lines joining the opposite angles of an edron intersect in a point, the figure will have three

* Would it be preferable to allow the former definitions of a complete edron and complete multangle to stand, and to apply some other term complete' to the systems that create the difficulty?

diagrammatic planes and six syngrammatic points. It would in some respects be convenient to call this solid a *diagrammatic octahedral sexangle;* but to depart as little as possible from the usual denomination, I shall term it a *diagrammatic sexangular octahedron*, or simply a *diagrammatic octahedron.*

It has been shewn in my first memoir "On the Theorems in Space analogous to those of Pascal and Brianchon in a Plane," (*Journal*, new series, vol. IV. p. 35), that the equations to the faces of a syngrammatic hexahedron may be denoted by

$$\left.\begin{array}{ll} T+S=0, & T-S=0 \\ U+S=0, & U-S=0 \\ V+S=0, & V-S=0 \end{array}\right\} \ldots\ldots\ldots\ldots(1),$$

the planes $T+S$ and $T-S$ being opposite faces, &c.; and that the equations to the diagrammatic planes are

$$\left.\begin{array}{ll} T+U=0, & T-U=0 \\ T+V=0, & T-V=0 \\ U+V=0, & U-V=0 \end{array}\right\} \ldots\ldots\ldots\ldots(2).$$

Let $ABCDA'B'C'D'$ be the angular points of the hexahedron, these points being denoted as follows:

$$\left.\begin{array}{ll} A,\ -S=T=U=V; & A',\ S=T=U=V \\ B,\ S=-T=U=V; & B',\ -S=-T=U=V \\ C,\ S=T=-U=V; & C',\ -S=T=-U=V \\ D,\ S=T=U=-V; & D',\ -S=T=U=-V \end{array}\right\} \ldots(3),$$

so that A is opposite to A', &c.

Again, let the diagonal straight lines joining opposite angular points intersect in E; also let the three sets of four edges each intersect in the points F, G, H; so that the points E, F, G, H, are denoted as follows:

$$\left.\begin{array}{l} E,\ T=U=V=0 \\ F,\ U=V=S=0 \\ G,\ V=S=T=0 \\ H,\ S=T=U=0 \end{array}\right\} \ldots\ldots\ldots\ldots(4).$$

The twelve planes (1) and (2) can be taken in four different ways, so that six of the planes shall be the faces of a syngrammatic hexahedron whose diagrammatic planes are the remaining six planes.

The planes (1) are the faces of one such hexahedron whose diagrammatic planes are the planes (2), whose opposite faces

[redacted]

intersect in the plane S, and whose syngrammatic points are F, G, and H.

Also the planes

$$S + T = 0, \quad S - T = 0,$$
$$U + T = 0, \quad U - T = 0,$$
$$V + T = 0, \quad V - T = 0,$$

are the faces of another syngrammatic hexahedron, whose diagrammatic planes are the other six planes, whose opposite faces intersect in the plane T, and whose syngrammatic points are G, H, and E.

Again, the planes

$$S + U = 0, \quad S - U = 0,$$
$$T + U = 0, \quad T - U = 0,$$
$$V + U = 0, \quad V - U = 0,$$

are the faces of a third syngrammatic hexahedron whose diagrammatic planes are the remaining six planes, whose opposite faces intersect in the plane U, and whose syngrammatic points are H, E, and F.

Finally, the planes

$$S + V = 0, \quad S - V = 0,$$
$$T + V = 0, \quad T - V = 0,$$
$$U + V = 0, \quad U - V = 0,$$

are the faces of a fourth syngrammatic hexahedron whose diagrammatic planes are the remaining six planes, whose opposite faces intersect in the plane V, and whose syngrammatic points are E, F, and G.

It will be observed that all these hexahedra have the same angular points $ABCDA'B'C'D'$.

Considering then (1) and (2) as a single system, it will have eight angular points $ABCDA'B'C'D'$, twelve faces (1, 2), sixteen edges and four syngrammatic points E, F, G, H; the faces pass six by six through the angles, six by six through the syngrams, and three by three through the edges. Such a system has much analogy to the complete tetragon, and I shall denominate it a *complete syngrammatic octangle.*

It is easily shewn that the points $ABCDEFGH$ are the angles of a complete syngrammatic octangle whose syngrams are $A'B'C'D'$; and that the points $A'B'C'D'EFGH$ are the angles of another complete syngrammatic octangle whose syngrams are $ABCD$. Hence any two of the three systems

[redacted]

of points $ABCD$, $A'B'C'D'$, $EFGH$, are the angles of a complete syngrammatic octangle of which the other four points are the syngrams.

A syngrammatic hexahedron may be viewed also as a complete system in another manner, as will appear further on.

Again, writing S for $2S$ in the equations at p. 32, vol. IV. new series, of this *Journal*, the faces of a diagrammatic octahedron may be denoted as follows:

$$\left.\begin{array}{ll} T+U+V+S=0, & T+U+V-S=0 \\ -T+U+V-S=0, & -T+U+V+S=0 \\ T-U+V-S=0, & T-U+V+S=0 \\ T+U-V-S=0, & T+U-V+S=0 \end{array}\right\}\ldots(5);$$

and the angles of this octahedron are easily found to be denoted as follows (they will be designated by the letters prefixed):

$$\left.\begin{array}{ll} a,\ U=V=T+S=0; & a',\ U=V=T-S=0 \\ b,\ T=V=U+S=0; & b',\ T=V=U-S=0 \\ c,\ T=U=V+S=0; & c',\ T=U=V-S=0 \end{array}\right\}\ldots(6);$$

and the syngrammatic points (which are all situated in the plane S), are

$$\left.\begin{array}{ll} \alpha,\ S=T=U+V=0; & \alpha',\ S=T=U-V=0 \\ \beta,\ S=U=T+V=0; & \beta',\ S=U=T-V=0 \\ \gamma,\ S=V=T+U=0; & \gamma',\ S=V=T-U=0 \end{array}\right\}\ldots(7).$$

The planes (5) may be taken opposite in four different ways, so as to form the faces of a diagrammatic octahedron. If the planes in the same horizontal line of (5) be considered opposite, then (5) will be the faces of a diagrammatic octahedron whose angular points are $aa'bb'cc'$, syngrammatic points $\alpha\alpha'\beta\beta'\gamma\gamma'$, diagrammatic planes T, U, and V, and whose opposite faces intersect in the plane S.

Again, taking the planes (5) opposite in the following manner,

$$\begin{array}{ll} U+V+S+T=0, & U+V+S-T=0, \\ -U+V+S-T=0, & -U+V+S+T=0, \\ U-V+S-T=0, & U-V+S+T=0, \\ U+V-S-T=0, & U+V-S+T=0, \end{array}$$

we have the faces of another diagrammatic octahedron whose angular points are $aa'\beta\beta'\gamma\gamma'$, syngrammatic points $\alpha\alpha'bb'cc'$, diagrammatic planes U, V, S, and whose opposite faces intersect in the plane T.

Also, taking the planes (5) opposite as follows,

$$\begin{array}{rr} V+S+T+U=0, & V+S+T-U=0, \\ -V+S+T-U=0, & -V+S+T+U=0, \\ V-S+T-U=0, & V-S+T+U=0, \\ V+S-T-U=0, & V+S-T+U=0, \end{array}$$

we get the faces of a diagrammatic octahedron whose angular points are $\alpha\alpha'bb'\gamma\gamma'$, syngrammatic points $aa'\beta\beta'cc'$, diagrammatic planes V, S, T, and whose opposite faces intersect in the plane U.

Finally, taking the planes (5) opposite in the following manner,

$$\begin{array}{rr} S+T+U+V=0, & S+T+U-V=0, \\ -S+T+U-V=0, & -S+T+U+V=0, \\ S-T+U-V=0, & S-T+U+V=0, \\ S+T-U-V=0, & S+T-U+V=0, \end{array}$$

we have the faces of a diagrammatic octahedron whose angular points are $\alpha\alpha'\beta\beta'cc'$, syngrammatic points $aa'bb'\gamma\gamma'$, diagrammatic planes S, T, U, and whose opposite faces intersect in the plane V.

Considering then the eight planes (5), and the twelve points (6, 7) as a single system, we shall have a system that has eight faces (5), twelve angular points (6, 7), sixteen edges, and four diagrammatic planes S, T, U, V; the angles lie six by six on the faces, six by six on the diagrammatic planes, and three by three on the edges. This system has considerable analogy to the complete quadrilateral, and I shall denominate it a *complete diagrammatic octahedron.*

It is not difficult to see that, of the three systems of planes

$$S=0, \quad T=0, \quad U=0, \quad V=0,$$

$$\left.\begin{array}{r} -S+T+U+V=0 \\ S-T+U+V=0 \\ S+T-U+V=0 \\ S+T+U-V=0 \end{array}\right\}$$

and

$$\left.\begin{array}{r} S+T+U+V=0 \\ -S-T+U+V=0 \\ -S+T-U+V=0 \\ -S+T+U-V=0 \end{array}\right\}$$

any two will be the faces of a complete diagrammatic octahedron of which the other four planes are the diagrammatic planes.

A diagrammatic octahedron may be viewed also as a complete system, in a different manner, as will be seen presently.

The general equation to surfaces of the second degree circumscribed about a syngrammatic hexahedron is (*Journal*, new series, vol. v. p. 66),

$$l(T+S)(T-S)+m(U+S)(U-S)+n(V+S)(V-S)=0,$$

or, $$-(l+m+n)S^2+lT^2+mU^2+nV^2=0,$$

that is, where $$\left.\begin{aligned} kS^2+lT^2+mU^2+nV^2&=0\\ k+l+m+n&=0\end{aligned}\right\}\ldots\ldots\ldots(8).$$

This is also of course the general equation to surfaces of the second degree circumscribed about (that is, passing though the angular points of) a complete syngrammatic octangle; and from (4) and (8) we infer the following theorem, which is perfectly analogous to a property of the complete tetragon.

I. *In a complete syngrammatic octangle, each of the four syngrammatic points is, relative to every surface of the second degree circumscribed about the octangle, the pole of the plane passing through the other three.*

Again (*Journal*, new series, vol. IV. p. 37), the general equation to surfaces of the second degree touching the faces (5) of a complete diagrammatic octahedron (recollecting that S must be written for $2S$), is

$$lT^2+mU^2+nV^2=S^2,$$

where $$l^{-1}+m^{-1}+n^{-1}=1.$$

For symmetry write $-k^{-1}l$, $-k^{-1}m$, and $-k^{-1}n$, for l, m, and n, and the equation becomes

where $$\left.\begin{aligned} kS^2+lT^2+mU^2+nV^2&=0\\ k^{-1}+l^{-1}+m^{-1}+n^{-1}&=0\end{aligned}\right\}\ldots\ldots\ldots(9).$$

Hence, since S, T, U, and V are the diagrammatic planes, we have the following theorem, which is the reciprocal of (I) and is perfectly analogous to a property of the complete quadrilateral.

II. *In a complete diagrammatic octahedron, each of the four diagrammatic planes is, relative to every surface of the second degree inscribed in the octahedron, the polar of the intersection of the other three.*

Again (*ibid.* pp. 30, 35), the equations to the four surfaces of the second degree touching the edges of the four syn-

grammatic hexahedra which compose the complete syngrammatic octangle (1, 2) are

$$\left.\begin{aligned} T^2 + U^2 + V^2 &= 2S^2 \\ U^2 + V^2 + S^2 &= 2T^2 \\ V^2 + S^2 + T^2 &= 2U^2 \\ S^2 + T^2 + U^2 &= 2V^2 \end{aligned}\right\} \dots\dots\dots\dots (10).$$

Hence

III. *In a complete syngrammatic octangle, each of the four syngrammatic points is, relative to the umbilical surface of the second degree touching the edges of any of the four syngrammatic hexahedra, the pole of the plane passing through the other three.*

Also (*ibid.* pp. 30, 36, recollecting to write S for $2S$ in equation (22), p. 30), the equations to the four surfaces of the second degree touching the edges of the four diagrammatic octahedra that can be formed out of the complete diagrammatic octahedron (5), are

$$\left.\begin{aligned} T^2 + U^2 + V^2 &= \tfrac{1}{2}S^2 \\ U^2 + V^2 + S^2 &= \tfrac{1}{2}T^2 \\ V^2 + S^2 + T^2 &= \tfrac{1}{2}U^2 \\ S^2 + T^2 + U^2 &= \tfrac{1}{2}V^2 \end{aligned}\right\} \dots\dots\dots\dots (11).$$

Hence,

IV. *In a complete diagrammatic octahedron, each of the four diagrammatic planes is, relative to the umbilical surface of the second degree touching the edges of any of the four simple diagrammatic octahedra, the polar of the intersection of the other three.*

The general equation to surfaces of the second degree touching the faces of the syngrammatic hexahedron (1) is (*ibid.* p. 36)

$$\begin{aligned} &\sin^2\theta . T^2 + \sin^2\phi . U^2 + \sin^2\psi . V^2 + 2\cos\theta . \sin\phi . \sin\psi . UV \\ &\quad + 2\cos\phi . \sin\psi . \sin\theta . TV + 2\cos\psi \sin\theta \sin\phi . TU \\ &\quad = (1 - \cos^2\theta - \cos^2\phi - \cos^2\psi + 2\cos\theta . \cos\phi . \cos\psi) S^2 \dots (12), \end{aligned}$$

so that the point $T = U = V = 0$ is the pole of the plane S. Hence,

V. *In a syngrammatic hexahedron, the point in which the diagonal straight lines joining opposite angles intersect is, relative to any surface of the second degree inscribed in the hexahedron, the pole of the plane in which the opposite faces intersect.*

Also, if $t = 0$, $u = 0$, $v = 0$, $w = 0$, $t' = 0$, $u' = 0$, $v' = 0$, $w' = 0$, be the equations to the faces of any octahedron, t and t' being opposite, &c., then (*Journal*, new series, vol. v. p. 66, foot-

note), the general equation to surfaces of the second degree circumscribed about the octahedron will be

$$ftt' + guu' + hvv' + kww' = 0.$$

When the octahedron is diagrammatic, we shall have

$$t = T + U + V + S, \quad t' = T + U + V - S, \text{ \&c.},$$

as in (5); and then the last equation reduces to

$$T^2 + U^2 + V^2 + 2lUV + 2mTV + 2nTU = S^2 \ldots (13),$$

where $$l = \frac{f+g-h-k}{f+g+h+k}, \text{ \&c.}$$

Hence the point $T = U = V = 0$ being the pole of the plane $S = 0$, we see that

VI. *In a diagrammatic octahedron the point in which the diagonal straight lines intersect is, relative to every surface of the second degree circumscribed about the octahedron, the pole of the plane in which the opposite faces intersect.*

In order that the products UV, VT, TU should disappear from (12) and that the other terms should not disappear, we must have $\cos\theta = \cos\phi = \cos\psi = 0$, giving $\sin^2\theta = \sin^2\phi = \sin^2\psi = 1$, when the equation reduces to $T^2 + U^2 + V^2 = S^2$; now it will be found that this surface (which is evidently umbilical) touches each face in the point in which the diagonals of that face intersect; and it may easily be shewn conversely, that if a surface of the second degree touch the faces at these points, its equation will reduce to the preceding form. Hence the equations to the four surfaces touching (in the manner just mentioned) the faces of the four syngrammatic hexahedra which compose the complete syngrammatic octangle, are

$$\left.\begin{aligned} T^2 + U^2 + V^2 &= S^2 \\ U^2 + V^2 + S^2 &= T^2 \\ V^2 + S^2 + T^2 &= U^2 \\ S^2 + T^2 + U^2 &= V^2 \end{aligned}\right\} \ldots\ldots\ldots\ldots (14).$$

Hence,

VII. *In a complete syngrammatic octangle, each of the four syngrammatic points is the pole of the plane passing through the other three, relative to the umbilical surface of the second degree touching the faces of any of the four syngrammatic hexahedra at the intersections of the diagonals of these faces.*

Also, that the products UV, VT, TU should disappear from (13), we must have $l = m = n = 0$, and it will then be

and that the surface touches the twelve synhedral lines that pass through the angles of the octahedron; thus the umbilical surface $T^2 + U^2 + V^2 = S^2$ touches the synhedral line in which the faces $-T+U+V-S=0$ and $T-U+V-S=0$ intersect. Conversely, it is easy to shew that if a surface of the second degree circumscribed about the octahedron touch these twelve lines, its equation must be of the preceding form. Hence (14) are the equations to the four umbilical surfaces circumscribed about the four simple diagrammatic octahedra that can be formed by the faces of a complete diagrammatic octahedron, these surfaces moreover touching the synhedral lines that pass through the angles. Hence,

VIII. *In a complete diagrammatic octahedron each of the diagrammatic planes is the polar of the intersection of the other three, relative to the umbilical surface of the second degree which is circumscribed about any of the four simple diagrammatic octahedra, and which at the same time touches the twelve synhedral lines that pass through its angles.*

Other theorems somewhat similar to those already given might be noticed, but as they are of less interest I shall omit them.

The twelve faces (1, 2) of a complete syngrammatic octangle form six pairs, and the planes of each pair intersect in an edge of the tetrahedron whose angles coincide with the syngrammatic points (so that the equations to the faces of the tetrahedron are $S=0$, $T=0$, $U=0$, $V=0$). Thus the two faces $U+V=0$ and $U-V=0$ form one pair, and these intersect in the straight line $U=V=0$, which is one of the edges of the tetrahedron. Also it will be observed that the four planes U, V, $U-V$, and $U+V$ form a harmonic system;* hence the following theorem:

IX. *The twelve faces of a complete syngrammatic octangle form six pairs, the planes of each pair intersecting in an edge of the tetrahedron whose angles are at the four syngrammatic points; also the two faces of the tetrahedron and the two faces of the octangle that intersect in any edge of the tetrahedron, form a harmonic system.*

Again, in a complete diagrammatic octahedron, the twelve angular points form six pairs (aa', bb', cc', $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$), the two points of each pair being situated on an edge of the tetrahedron whose faces are the four diagrammatic planes S, T, U, V.

* When I say that four points or planes form a harmonic system, I mean that the first two are harmonic conjugates with respect to the last two.

nine edges may be divided into three sets—a convergent set AF, AG, AH intersecting in A, a second convergent set $A'F$, $A'G$, $A'H$ intersecting in A', and a non-convergent set GH, HF, FG situated in the unique diagonal plane. Also the first, second, and third of each set may be denominated *corresponding* edges, and each pair of planes AGH, $A'GH$; AHF, $A'HF$; AFG, $A'FG$, *corresponding* faces.

Let $t = 0$, $u = 0$, $v = 0$ denote the faces AGH, AHF, and AFG respectively; also let $S = 0$ denote the unique diagonal plane: supposing t, u, and v to have been multiplied by the proper constants, the corresponding faces may be denoted by

$$t - 2S = 0, \quad u - 2S = 0, \quad v - 2S = 0.$$

Substituting $T + S$, $U + S$, and $V + S$ for t, u, and v, we see that the faces of a pentangular hexahedron may be denoted as follows:

$$\left.\begin{array}{ll} T + S = 0, & T - S = 0 \\ U + S = 0, & U - S = 0 \\ V + S = 0, & V - S = 0 \end{array}\right\} \ldots\ldots\ldots (15),$$

in which the equations to the corresponding faces are placed in the same horizontal line. The intersections of the corresponding faces form the non-convergent edges, the mutual intersections of the three planes on the left one set of convergent edges, and the mutual intersections of the three planes on the right form the other set of convergent edges.

It is evident that the equation to the unique diagonal plane is $S = 0$, the equations to the other three diagonal planes

$$U - V = 0, \quad V - T = 0, \quad T - U = 0,$$

and those of the diagonal line $T = U = V$.

The equation to the surface of the second degree touching the edges of the pentangular hexahedron is easily seen to be

$$4S^2 + T^2 + U^2 + V^2 - 2UV - 2TV - 2TU = 0 \ldots (16);$$

and since this equation may be put under the form

$$4S^2 + (-T + U + V)^2 + (U - V)^2 = (U + V)^2,$$

the surface is umbilical.

From (16) it is easily shewn that the equations to the straight lines joining the points of contact of the corresponding edges of the two convergent sets, are

$$\left.\begin{array}{l} \tfrac{1}{2} T = U = V \\ \tfrac{1}{2} U = T = V \\ \tfrac{1}{2} V = T = U \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (17),$$

and these intersect in the point $T = U = V = 0$ on the diagonal line. Let E be this point.

The equations to the straight lines joining the points of contact of the corresponding edges of the non-convergent set and the convergent set intersecting in A, are

$$\left.\begin{array}{l} T + 2S = U - V = 0 \\ U + 2S = V - T = 0 \\ V + 2S = T - U = 0 \end{array}\right\} \ldots\ldots\ldots\ldots (18),$$

and these intersect in the point $T = U = V = -2S$ on the diagonal line.

Finally, the equations to the straight lines joining the points of contact of the corresponding edges of the non-convergent set and the convergent set intersecting in A', are

$$\left.\begin{array}{l} T - 2S = U - V = 0 \\ U - 2S = V - T = 0 \\ V - 2S = T - U = 0 \end{array}\right\} \ldots\ldots\ldots\ldots (19),$$

and these intersect in the point $T = U = V = 2S$ on the diagonal line.

It is evident that the corresponding lines of (17), (18), and (19) are situated in the diagonal planes $U - V = 0$, $V - T = 0$, and $T - U = 0$ respectively, and hence these planes intersect the edges in the unique diagonal plane in their points of contact.

Let the diagonal line and the unique diagonal plane intersect in L. Since the planes S, T, $T + S$, and $T - S$ pass through the points L, E, A, A' on the diagonal line, these points form a harmonic system.

It is at once seen from (16) that

$$T = 0, \quad U = 0, \quad V = 0 \ldots\ldots\ldots\ldots (20)$$

are the equations to the tangent planes passing through the edges GH, HF, and FG in the unique diagonal plane, and these planes intersect on the diagonal line in the same point (E) as the lines (17).

The equations to the tangent planes through the other edges are

$$\left.\begin{array}{l} U + V + 2S = 0 \\ V + T + 2S = 0 \\ T + U + 2S = 0 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (21),$$

and

$$\left.\begin{array}{l} U + V - 2S = 0 \\ V + T - 2S = 0 \\ T + U - 2S = 0 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (22).$$

Since each set of four planes $T+S$, $T-S$, S, T, and $U\pm S$, $V\pm S$, $U-V\,\{=(U\pm S)-(V\pm S)\}$, $U+V\pm 2S\{=(U\pm S)+(V\pm S)\}$, forms a harmonic system, it follows that the two faces, the diagonal plane, and the tangent plane, that pass through any edge, form a harmonic system.

Again, since each set of four planes S, $T+S$, $T\{(=T+S)-S\}$, $T+2S\{=(T+S)+S\}$, and S, $T-S$, $T\{=(T-S)+S\}$, $T-2S\{=(T-S)-S\}$ forms a harmonic system, it is easy to infer that the two angles on any convergent edge, its point of contact, and the point in which the said edge is intersected by the tangent plane through the corresponding non-convergent edge, form a harmonic system.

It is evident that the point E or $T=U=V=0$ is the pole, relative to (16), of the unique diagonal plane. Hence

XII. *An umbilical surface of the second degree can be drawn to touch the edges of a pentangular hexahedron; the three diagonal planes intersect the non-convergent edges in their points of contact; the three straight lines joining the points of contact of the corresponding edges of any two of the three sets intersect in a point on the diagonal line; the tangent planes to the surface drawn through the non-convergent edges intersect in that point* (E) *on the diagonal line in which intersect the three lines joining the points of contact of the corresponding edges of the two convergent sets; and this point is the pole, relative to the surface, of the unique diagonal plane. The two faces, the diagonal plane, and the tangent plane, that pass through any edge, form a harmonic system, the two angles on any convergent edge, its point of contact, and the point in which the said edge is intersected by the tangent plane through the corresponding non-convergent edge, form a second; and the two angles that are on the diagonal line, the point of intersection of the diagonal planes, and the point E, form a third harmonic system.**

Since the equations (15) coincide with the equations (1), it is evident that (12) will be the general equation to surfaces of the second degree inscribed in the pentangular hexahedron; hence also (*Journal*, new series, vol. IV. p. 37) the equations to the straight lines joining the points of contact of corresponding faces are

$$\left.\begin{aligned} -T &= lU = kV \\ -U &= lT = hV \\ -V &= kT = hU \end{aligned}\right\}\ldots\ldots\ldots\ldots(23),$$

* By aid of some of the properties (XII.) we can find the tangent planes through, and the points of contact on, the edges, when the solid alone is given.

convergent edges passes through the synhedral line: and the uniq[ue] synhedral point is the pole, relative to the surface, of this pla[ne]. The two angles, the synhedral point and the point of contact, [o]f any edge, form a harmonic system; the two triangular faces, th[e] plane of the synhedral points, and the plane S, a second; an[d] the two faces and the tangent plane through any edge, togeth[er] with the plane passing through the said edge, and the point [of] contact of the corresponding edge, form a third harmonic system.

It is easily shewn from (31) that the equations to th[e] angular points of the sexangular pentahedron are denote[d] as in (6); hence these points will be the angles of a dia[-] grammatic octahedron, whose faces are denoted by (5). Consequently (13) is the general equation to surfaces of th[e] second degree circumscribed about the sexangular penta[-] hedron. The equations to the tangent planes touching (13) at the angles of the pentahedron, will be found to be

$$\left.\begin{array}{ll} mV+nU+T+S=0, & mV+nU+T-S=0 \\ lV+nT+U+S=0, & lV+nT+U-S=0 \\ lU+mT+V+S=0, & lU+mT+V-S=0 \end{array}\right\} \ldots(41);$$

and the opposite planes always intersect in the fixed plane $S=0$; also it is evident that this plane is, relative to (13), the polar of the unique synhedral point $T=U=V=0$.

XVI. *If any surface of the second degree be circumscribed abou[t] a sexangular pentahedron, the tangent planes at the corresponding angles will intersect in three straight lines in a plane passing through the synhedral line. This plane is the same for all circumscribed surfaces, and is, relative to any of them, the polar of the unique synhedral point.*

Since (as has been observed) the equations (5) to the faces of a diagrammatic octahedron, and the equations (31) to those of a sexangular pentahedron, both give the equations (6) for the angles, we see that the angular points of both solids form precisely the same system, one of the chief differences being, that in the octahedron the corresponding angles are *opposite*, while in the pentahedron they are *adjacent*, that is, are on the same edge. Indeed, if the three straight lines joining three pairs of points pass through the same point, these six points can be taken so as to form the angles of one diagrammatic octahedron and of four sexangular pentahedra.

It has been virtually shown in my first memoir (repeatedly [re]ferred to) on the "Theorems in Space analogous to those

Hence we can find the points of contact on, and the tangent planes [thro]ugh, the edges, when the solid alone is given.

By arranging the planes as follows,

$$\left.\begin{array}{ll} T+S=0, & T-S=0 \\ U-S=0, & U+S=0 \\ V+S=0, & V-S=0 \end{array}\right\} \ldots\ldots\ldots\ldots (26),$$

and

$$\left.\begin{array}{ll} T+S=0, & T-S=0 \\ U+S=0, & U-S=0 \\ V-S=0, & V+S=0 \end{array}\right\} \ldots\ldots\ldots\ldots (27),$$

we shall get other two pentangular hexahedra. Denoting the angular points by the letters in (3) and (4), the angular points of the syngrammatic octangular hexahedron are $ABCDA'B'C'D'$, and those of the pentangular hexahedra $AA'FGH$, $BB'FGH$, $CC'FGH$, and $DD'FGH$, respectively. Hence the pentangular hexahedra have three angular points F, G, H and one diagonal plane FGH (or $S=0$) common; also their other angles coincide with opposite angles of the octangular hexahedron, so that the diagonal lines AA', BB', CC', DD' of the pentangular hexahedra are the diagonal lines joining opposite angles of the octangular hexahedron. The entire system thus consists of six faces, fifteen edges, and eleven angles; it has a good deal of analogy to the complete quadrilateral, and may be denominated a *complete undecangular** *hexahedron*.

The equation to the surface of the second degree touching the edges of the octangular hexahedron is of course

$$T^2+U^2+V^2=2S^2 \ldots\ldots\ldots\ldots\ldots\ldots (28);$$

also the equations to the surfaces of the second degree touching the edges of the pentangular hexahedra (taken in the preceding order), are

$$\left.\begin{array}{l} 4S^2+T^2+U^2+V^2-2UV-2TV-2TU=0 \\ 4S^2+T^2+U^2+V^2-2UV+2TV+2TU=0 \\ 4S^2+T^2+U^2+V^2+2UV-2TV+2TU=0 \\ 4S^2+T^2+U^2+V^2+2UV+2TV-2TU=0 \end{array}\right\} \ldots (29).$$

Equation (12) is of course the general equation to surfaces of the second degree touching the faces of the complete undecangular hexahedron, and (8) the general equation to

* The adjective 'undecangular' is added to distinguish the system from the complete hexahedron *in its general state*, which has *twenty* angular points.

The equation to the surface of the second degree touching the edges of the diagrammatic octahedron, of course is

$$T^2 + U^2 + V^2 = \tfrac{1}{2}S^2 \ldots\ldots\ldots\ldots (46);$$

also the equations to the surfaces of the second degree touching the edges of the sexangular pentahedra, are

$$\left.\begin{aligned} S^2 &= 4UV + 4VT + 4TU \\ S^2 &= 4UV - 4VT - 4TU \\ S^2 &= -4UV + 4VT - 4TU \\ S^2 &= -4UV - 4VT + 4TU \end{aligned}\right\} \ldots\ldots\ldots (47).$$

Equation (13) is of course the general equation to surfaces of the second degree circumscribed about the hendecahedral sexangle, and (9) the general equation to surfaces of the second degree inscribed in the diagrammatic octahedron. It is easily seen that such of the surfaces (46) and (47) as touch the same edge have a common point of contact. The equations to the points of contact of the various edges will be found to be as follows:

$$\left.\begin{array}{lllll} aa', & U = V = S = 0, & & & \\ bb', & V = T = S = 0, & & & \\ cc', & T = U = S = 0, & & & \\ bc, & T = 0, & U = & V = & -\tfrac{1}{2}S, \\ b'c', & T = 0, & U = & V = & \tfrac{1}{2}S, \\ ca, & U = 0, & V = & T = & -\tfrac{1}{2}S, \\ c'a', & U = 0, & V = & T = & \tfrac{1}{2}S, \\ ab, & V = 0, & T = & U = & -\tfrac{1}{2}S, \\ a'b', & V = 0, & T = & U = & \tfrac{1}{2}S, \\ ab', & V = 0, & -T = & U = & \tfrac{1}{2}S, \\ a'b, & V = 0, & T = & -U = & \tfrac{1}{2}S, \\ ac', & U = 0, & -T = & V = & \tfrac{1}{2}S, \\ a'c, & U = 0, & T = & -V = & \tfrac{1}{2}S, \\ bc', & T = 0, & -U = & V = & \tfrac{1}{2}S, \\ b'c, & T = 0, & U = & -V = & \tfrac{1}{2}S, \end{array}\right\} \ldots\ldots (48)$$

Many other relations of the system might be given, and most of the preceding theorems that have reference to octahedra and pentahedra might be enunciated as properties of the complete hendecahedral sexangle, but I hasten to conclude, and shall therefore merely insert the following theorem.

XVII. *In a complete hendecahedral sexangle the synhedral point common to the pentahedra is the pole of the plane of their fo*

nhedral lines relative to any of the following surfaces of the second gree—the surface touching the edges of any of the five solid figures, ny surface circumscribed about the system, and any surface inscribed the octahedron. If any surface of the second degree be circumribed about the system, the tangent planes drawn at corresponding ngles will intersect in three straight lines in the plane of the synedral lines. Such of the five surfaces touching the edges of the five lid figures of the system, as touch the same edge, have a common oint of contact on that edge; and, omitting the three edges common the pentahedra, the six straight lines joining the points of contact f corresponding edges intersect in the synhedral point common to the ntahedra.

To the properties given in this paper must be added the rst ten theorems of my first memoir on the "Theorems in pace analogous to those of Pascal and Brianchon in a Plane," *ournal*, vol. IV., New Series, pp. 26–38), remembering in he latter to read 'diagrammatic' instead of 'diagonal' plane, nd in the equations that relate to the octahedron to write S for S.

York Town, near Bagshot,
Jan. 26th, 1852.

N THE CALCULUS OF DISTANCES, AREAS, AND VOLUMES, AND ITS RELATIONS TO THE OTHER FORMS OF SPACE.

By Thomas Cotterill.

Suppose we take any number of points in a plane, and hrough them draw all the possible right lines, which will gain intersect in a certain number of points, which in reerence to the original we may call derived points. The eometry of this figure would be known when we had liscovered all the relations between the distances of these oints, the angles between the lines, and the areas of the riangles. The investigation of the two first classes seems o form the object of Trigonometry. But the relations of he areas, even of the original points, when their number is reater than four, has not (that I am aware of) been xamined; and yet, besides its own intrinsic interest, when we consider the importance, both in the Geometry of Position and the Theory of Curves, of certain limiting cases, he subject is evidently one of considerable interest.

On taking the figure thus formed from six points, and ndeavouring to express the area of a Pascal triangle in erms of the original areas, I was fortunate enough to hit pon a method by which this could be done with consider-

able facility and equally applicable to all such cases. The result was, that that peculiar function of the six points, which we may call the conic, could be expressed in fifteen binomials each containing the original areas, and in sixty monomials each involving a Pascal triangle as a factor, in such a way that if one of the original points moved so that a Pascal triangle was constant in area, the point described a *conic section*.

Further examination shewed that the means and instruments by which this was done were equally effective in discovering the relations arising from introducing a fresh point or line; and that besides, by different operations, they could be employed with almost equal advantage to obtain trigonometrical properties. The expressions can also be dualized, either independently or by the method of polar reciprocals.

The question then arose, Can the method be applied to general space? The answer is, Every relation in space gives relations in all other spaces, but sometimes in forms which cannot be called analogues. But there are certain functions whose forms are the same in each space, but differ in their number of terms, and sometimes in the value of a constant.

Amongst these, by far the most important is one which in a different shape is well known, but does not appear to have met with the attention it deserves in the form in which it will here be put.

The explanation of the rule of signs of areas and volumes is given by Mœbius in his *Lehrbuch der Statik*, but admits of simplification. The signs of the trigonometrical functions are made to depend on those already given, and are perfectly free from ambiguity.

Taking space of three dimensions and using a notation which will be easily understood, we have the equation of addition of volumes

$$V = P\alpha + P\beta + P\gamma + P\delta,$$

which in a slightly different form has a pretty enough geometry of its own, instances of which in space of two dimensions are given by Mr. Moon in vol. v. p. 131 of this Journal, and can easily be increased.

I am not aware however that it has ever been observed (though it is strange if it has not), that by substituting the expressions for the volumes in terms of the altitudes and bases, and supposing the variable point to be in one of the planes, we obtain the equation to the plane, and also the

perpendicular on this plane from any point in a form which explains the origin of the double sign in the common expression. Again, taking another point, subtracting and substituting trigonometrically, we obtain an equation which gives the properties of the tetrahedron.

Now take the equation $V = \Sigma Pa$, and the theorem that triangles and tetrahedrons of the same altitude are to one another as their bases. From these two properties we deduce the equation

$$V . P\tau = \Sigma P\alpha \,.\, A\tau,$$

where $P\tau$ is either the shortest distance between the point P and the plane τ, or the tetrahedral volume subtended at P by the triangle τ. It is scarcely possible to overrate the importance of this equation, as I believe that there is no general geometrical relation which cannot be proved from it, and one simple definition founded on the principle of similarity.

This expression is endowed with complete duality. In space of two dimensions, the limiting case (when τ is any line through P) is discussed by Mr. Salmon in the 1st chapter of his *Higher Plane Curves.**

My present object is to shew its use in explaining a point in the theory of curves and curved surfaces considered as envelopes, which requires some explanation. I shall give the formulæ in space of three dimensions, though it must be remembered that they may be increased in space of two dimensions. But first I must explain one point: suppose by means of the foregoing formula we have transformed any expression homogeneous or not involving the coordinates of a point with regard to any number of planes, into another involving the coordinates of the same point with regard to other planes. Let the line ps through the point p meet the plane σ in s; then if $p\sigma$ be the ordinate of p, $p\sigma = ps$, $\sin(ps, \sigma)$. For each ordinate substitute a similar expression, and take p at infinity on the line ps; the terms involving the highest powers of the coordinates will alone remain with sines of angles substituted for ordinates. We have in fact obtained *a porismatic asymptotic hyperdeterminant.*

Let, as before, Greek letters denote planes: then $3V \cos \pi\tau = \Sigma \alpha A_\tau \cos \alpha\pi$ and $3V \cos \alpha\pi = \alpha A_\pi + \beta B_\pi \cos \beta\alpha + \gamma C_\pi \cos \gamma\alpha + \delta D_\pi \cos \delta\alpha$. Substituting in the same way for β, we have

$$gV^2 \cos \pi\tau = \Sigma \alpha^2 A_\pi A_\tau + \alpha\beta (A_\pi B_\tau + A_\tau B_\pi) \cos \alpha\beta;$$

therefore $\quad = -\Sigma (A_\pi - B_\pi)(A_\tau - B_\tau)\, \alpha\beta \cos \alpha\beta.$

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* It is impossible to mention this work, whilst fresh from its perusal, without expressing the pleasure and instruction received.

Hence $gV^2 = -\Sigma(A_\pi - B^\pi)^2\, \alpha\beta \cos\alpha\beta = \Omega$ suppose;*

therefore $$2\Omega \cos\pi\tau = \Sigma A_\pi \frac{d\Omega}{dA_\tau}.$$

Now suppose we have an expression involving tangen[tial] coordinates, and have transformed it into another expres[sion] by means of a formula such as $VP_\tau = \Sigma P\alpha A_\tau$. Then if n be t[he] dimensions of the expression which we had better suppo[se] homogeneous, the transformed expression will consist o[f] terms such as $\Omega^r u_{n-2r}$; u_{n-2r} being a homogeneous function o[f] the coordinates of the degree $n-2r$. Now equations o[f] either point or tangent coordinates are porismatized (using [the] word and idea of Mr. Gompertz, in his ingenious work on Imaginary Quantities), by substituting for instance for A_π $A_\pi + \lambda A_\tau$, and equating coefficients of like powers of λ. Taking the coefficients of the first power of λ, and substitu[-] ting, we shall at last obtain an equation involving n porismati[c] points or planes. Let us see the effect of one such substitu[-] tion on the term $\Omega^r u_{n-2r}$. The coefficient of λ evidently is

$$\Omega^r \Sigma D_\pi \frac{dU_{n-2r}}{dD^\tau} + 2rU_{n-2r}\Omega^{r-1}\Sigma A_\pi \frac{d\Omega}{dA_\tau}$$

$$= \Omega^r\left(\Sigma D_\pi \frac{dU_{n-2r}}{dD_\tau} + 2r\cos\pi\tau\, U_{n-2r}\right).$$

Thus in space of two dimensions the polar of $A_tB_t - b^2$ i[s] $A_tB_r + B_tA_r = 2b^2\cos_r t$.

But taking again our tangential equation, by supposing th[e] plane to be at infinity, we must have the coefficients of th[e] highest dimensions of the coordinates to be equal. Now in Plucker's *Canonical Form*, the highest dimension consists of one term. If then infinity is a tangent, and therefore the sum of the coefficients nothing, this term would seem to disappear. But remember that the perpendicular from an infinite point in a given direction (in fact its ordinate) wit[h] respect to a given plane, varies as the sine of the angl[e] between the two.

One example is the parabola; another is the curv[e] $\Sigma A_t(B_t - C_t)^2 \cot A = 0$, which, by means of the equatio[n] $2K = \Sigma(B_t - C_t)^2 \cot A$, can be put under the form

$$A_t = -2p \sin ct \sin bt, \cos at.$$

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* Compare Salmon, p. 11, and the beautiful explanation there giv[en] of $\Omega = 0$ in space of two dimensions.

Here $\sin at = \frac{B_{,} - C_{,}}{a}$; and we see that the sum of the coefficients of the highest dimensions is nothing.

This is a very remarkable curve. Suppose we call the circle bisecting the six distances joining the three points ABC, and the intersection of the perpendiculars the bisecting circle. The intercepted portion of any tangent to the curve is equal to the diameter of the circumscribing circle, and its middle point is on the bisecting circle: the tangents at its extremities are at right angles to one another, and also meet in the bisecting circle and cut off equal segments of the sides. The sides of this right-angled triangle are as the curvatures at the points at which they touch. The curve is of the fourth order, and is that touched by the asymptotes of the rectangular hyperbolas passing through the points ABC.

Writing from recollection, I stated in the February No. of this Journal, that the tangents at the cusps met in the intersection of the perpendiculars. This is incorrect.

I have not alluded to the relations of distances on the same line. They are nothing but algebra, and the reader can form one of the functions employed by taking any of the expressions which are known of four points in harmonic proportion, and seeing how they can be deduced from each other. When he has thus formed an equation of perhaps sixteen sides, let him see that he has been working nothing but algebra, and that a great portion of the expressions can be derived from a single porismatic one. And so for the anharmonic and involution functions.*

49, Coleshill Street, April 3rd, 1852.

ON A PROBLEM IN COMBINATIONS.

By R. R. Anstice.

The object of this paper is to prove the following proposition. If n is odd and $6n+1$ is a prime number, $12n+3$ symbols can be arranged into $6n+1$ series of triads, $4n+1$ triads in each, each containing all the symbols; and in such a manner that every possible duad shall occur once, and once only, throughout the whole. Also all the series can be deduced

* Since the date of this paper, M. Chasles's invaluable work on Geometry has appeared. The functions here alluded to in their cases of ease form the basis of his system, and are made to depend on each other by the geometrical properties of the anharmonic ratio. I think, however, the knowledge that the expressions employed are only algebraic transformations of the same form, obtained for the most part by given rules, will be found to throw great light on these remarkable functions.

from any one of them by help of three cycles; one of a single term, the other two of $6n+1$ terms each.

As my reasoning will be based on Mr. Kirkman's conclusions, I would ask the reader to have at hand for reference two papers by that author in the Journal (new series, No. x. p. 191, and No. xxiv. p. 256).

Mr. Kirkman has shown in his first paper, that the number of symbols, all the duads of which can be exhibited but not repeated in a system of triads, must be of form $6n+1$ or $6n+3$. Of these two forms, $6n+3$ being divisible by 3, is the only one of course available for the purpose in view.

Next he shows that if this be true, not only for a number N of symbols, but also for $\frac{1}{2}(N-1)$, then the triplets of N may be exhibited in terms of the triplets of $\frac{1}{2}(N-1)$ and the duads of $\frac{1}{2}(N+1)$ symbols. It is to this case that I confine myself in the present paper.

Now N being of form $6n+3$, $\frac{1}{2}(N-1)$ will be of form $3n+1$, which number of symbols cannot be arranged into triads exhausting all the duads unless n is even. Therefore our number must be of form $12n+3$.

Here, when $n=1$ or the symbols are 15, the arrangement has been already made; but not I believe by cyclic changes. I will give the simplest rule I can for this. Put the symbols into 5 triads in any way you like for a primary arrangement. Then to form the first cycle take in succession one term from the first triad, one form the second, one from the third, one from the fourth, one from the fifth, and the remaining two from the fourth. To form the second cycle, take in succession one from the first triad, one from the third, one from the fifth, one from the second, another from the second, another from the fifth, another from the third. There still remains one term of the first cycle, which will form the cycle of one, or will remain unchanged. Thus for a primary arrangement take

$$| \; a_1a_2a_3 \; | \; b_1b_2b_3 \; | \; c_1c_2c_3 \; | \; d_1d_2d_3 \; | \; e_1e_2e_3 \; |$$

And we may take for a first cycle $a_1b_1c_1d_1e_1d_2d_3$, and for a second $a_2c_2e_2b_2b_3e_3c_3$, while a_3 remains unchanged; and by help of these cycles form all the arrangements as follows:

$a_1a_2a_3$	$b_1b_2b_3$	$c_1c_2c_3$	$d_1d_2d_3$	$e_1e_2e_3$
$b_1c_2a_3$	$c_1b_3e_3$	$d_1e_2a_2$	$e_1d_3a_1$	$d_2b_2c_3$
$c_1e_2a_3$	$d_1e_3c_3$	$e_1b_2c_2$	$d_2a_1b_1$	$d_3b_3a_2$
$d_1b_2a_3$	$e_1c_3a_2$	$d_2b_3e_2$	$d_3b_1c_1$	$a_1e_3c_2$
$e_1b_3a_3$	$d_2a_2c_2$	$d_3e_3b_2$	$a_1c_1d_1$	$b_1c_3e_2$
$d_2e_3a_3$	$d_3c_2e_2$	$a_1c_3b_3$	$b_1d_1e_1$	$c_1a_2b_2$
$d_3c_3a_3$	$a_1e_2b_2$	$b_1a_2e_3$	$c_1e_1d_2$	$d_1c_2b_3$

Now to prove that these cycles will be efficient, I observe that, by the principle of cyclic changes, the third arrangement is derived from the second by the same operation as the second from the first: the fourth from the second by the same operation as the third from the first, and so on. Therefore, if we can prove that no duad of the *first* arrangement can recur until the cycle is repeated, we prove at the same time that no duad of *any* of the arrangements will recur. But in the first arrangement the duads agree in letter and only differ in subscript. Suppressing the subscripts, our two cycles become *abcdedd* and *acebbec.*

Now write down either of these cycles, and underneath it write either the same cycle or the other, beginning at any term; and we shall find the letters agree in never more than one place: of course excluding the case where the same identical cycle is written underneath, beginning at the same term. A little thought will show that this proves their efficiency. Or we may do it mechanically thus. Take two circular discs of equal diameter and at equal distances round the circumference of the one and the other respectively: write the two cycles. Do the same in the case of two other equal circular discs of different diameter to the former. Apply two of the discs of unequal diameter concentrically to each other; and in any position we shall find them agree in letter in never more than one place, excepting only when two identical cycles are made to agree in *every* place.

The rule, that two cycles should be efficient *in combination* seems difficult to give. But a simple rule may be given, that a cycle shall be efficient *alone.* For suppose the cycle inefficient, then the interval between two letters, b and c suppose, must be the same as that between another b and another c measured in the same direction. From which it follows, that the interval between the b's and c's themselves must be the same. Consequently, that a cycle may be efficient by itself, the intervals between like letters (neglecting subscripts) must produce a series of numbers all different. But (between the two b's suppose) there are *two* intervals whose sum is the number of terms of the cycle. Thus that a cycle may be efficient alone, we have twice as many conditions as there are repetitions of letter in the cycle.

Now look at the second of the cycles. The letters to be repeated are written in a certain order, then in the reverse order. The intervals between like letters, then, are the odd numbers 1 3 5. The complements of the same are 6 4 2; all different numbers.

ka_0 | a_1a_3 | a_2a_6 | a_4a_5 | ; and using the same cycle for both, we get the two systems

$$\left|\begin{array}{c|c|c|c} ka_0 & a_1a_6 & a_2a_5 & a_3a_4 \\ ka_1 & a_2a_0 & a_3a_6 & a_4a_5 \\ ka_2 & a_3a_1 & a_4a_0 & a_5a_6 \\ ka_3 & a_4a_2 & a_5a_1 & a_6a_0 \\ ka_4 & a_5a_3 & a_6a_2 & a_0a_1 \\ ka_5 & a_6a_4 & a_0a_3 & a_1a_2 \\ ka_6 & a_0a_5 & a_1a_4 & a_2a_3 \end{array}\right| \qquad \left|\begin{array}{c|c|c|c} ka_0 & a_1a_3 & a_2a_6 & a_4a_5 \\ ka_1 & a_2a_4 & a_3a_0 & a_5a_6 \\ ka_2 & a_3a_5 & a_4a_1 & a_6a_0 \\ ka_3 & a_4a_6 & a_5a_2 & a_0a_1 \\ ka_4 & a_5a_0 & a_6a_3 & a_1a_2 \\ ka_5 & a_6a_1 & a_0a_4 & a_2a_3 \\ ka_6 & a_0a_2 & a_1a_5 & a_3a_4 \end{array}\right|$$

which will be found to fulfil the conditions proposed. This twofold arrangement of the duads of 8 symbols has been given by Mr. Kirkman (No. XXIV. p. 261), but he thinks that it would be difficult to give for higher values of $2n$ than 8. However, it may always be done when $2n-1$ is prime and n is even: thus in the case of 12 symbols, let k be the constant term $a_0a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}$, the successive members of the cycle of eleven terms. The primary series of our first system, expressed in terms of these, will be

$$ka_0 \mid a_1a_{10} \mid a_2a_9 \mid a_3a_8 \mid a_4a_7 \mid a_5a_6 \mid .$$

The primary series of our second system, similarly expressed, may be

$$ka_0 \mid a_1a_2 \mid a_4a_8 \mid a_5a_{10} \mid a_9a_7 \mid a_3a_6 \mid .$$

Here, with k is associated a_0 as before. With a_1 is associated a_2, 2 being a primitive root of 11. And from this as base the other duads are derived by successively multiplying the subscripts by the square of two, or four, throwing out elevens. If the same cycle is used to complete the two systems, they will be found on trial, and may readily be proved, to fulfil the conditions proposed. The proof will be involved in that of the proposition which I have proposed to establish.

But to return to the problem of 15: let k be the constant term

$$\left.\begin{array}{l} p_0p_1p_2p_3p_4p_5p_6 \\ q_0q_1q_2q_3q_4q_5q_6 \end{array}\right\} \text{the successive members of the two cycles.}$$

Then the primary arrangement, expressed in terms of these, will be

$$\mid kp_0q_0 \mid p_1q_3q_4 \mid p_2q_1q_6 \mid p_3p_5p_6 \mid p_4q_2q_5 \mid .$$

Here the first triad consists of the constant term associated with the first member of each cycle. Then there are three triads (the 2nd, 3rd, and 5th), where one term of the first cycle is associated with a duad of the second. The formation of the duads is obvious; the sum of their subscripts equals 7.

To find the subscript of the associated term of the first cycle multiply the square of the product of the subscripts of the duads by two, and throw out the sevens, or multiply the fourth power of either of the subscripts by two, and throw out the sevens. The remaining members of the first cycle form the fourth triad. We may complete all the arrangements from this primary one by successively increasing the subscripts by unit, throwing out the sevens.

To prove the efficiency of these cycles, it is sufficient, as I have said, to show that no duad in the primary arrangement can recur till the cycle is repeated. These duads may be arranged into three classes.

The first comprising duads whose terms are taken from the first cycle only; the second comprising those whose terms are taken from the second cycle only; the third comprising those whose terms are taken each from a different cycle. Now the cyclic changes which from any given duad of the first arrangement produce the duads corresponding to it in the others, will affect neither *its class* nor the *difference of its subscripts estimated to modulus* 7.

Therefore it will be sufficient to prove that in the primary arrangement these differences, in each of the 3 classes, produce a series of different numbers. Moreover, in the two first classes these differences must be estimated both positively and negatively to modulus 7, or, which comes to the same thing since 7 is prime, the *square* of the differences may be taken. In the third class we must estimate the differences always in the same manner, always subtracting, suppose, the subscript of the q from the subscript of the p.

The duads of the first class all occur in the fourth triad of the primary arrangement. The squares of the differences of their subscripts are (to modulus 7) 4, 2, and 1, all different numbers.

The duads of the second class occur in the 2nd, 3rd, and 5th triad of the primary arrangement. The squares of the differences of their subscripts (to modulus 7) are 1, 4, and 2. All different numbers.

The duads of the third class occur in the 1st, 2nd, 3rd, and 5th triad of the primary arrangement. The differences of their subscripts estimated as defined above, are 0, 5, and 4, 1 and 3, 2 and 6. All different numbers.

There is still a fourth class of duads, where the constant term is associated with a term from one or other of the cycles. These manifestly cannot recur till the cycle is repeated. Thus the efficiency of the cycles is demonstrated. I may

[redacted]

k that these cycles will produce combinations of triads . Kirkman's form. (See *Journal* No. xxiv. p. 260, or the gement already given.) Indeed, it was from studying orm that I discovered them. But it is a mistake to se that all the combinations of triads which can be d belong to one form. There is another pair of cycles, ly efficient with that already given, which will give inations of triads of an entirely distinct form. The ary arrangement, expressed in terms of the successive bers of these cycles, will be

$$kp_0q_0 \mid p_2q_1q_3 \mid p_4q_2q_6 \mid p_1q_4q_5 \mid p_3p_5p_6$$

e first triad is as before. In the three next a duad of econd cycle is associated with a single term of the first. nd the duads proceed as follows: with q_1 join q_3, 3 being mitive root of 7, and from this as base form the other s by multiplying the subscripts by the square of 3 to dlus 7, that is by 2.

e subscript of the associated term of the first cycle is emisum (estimated to modulus 7) of the subscripts of luad. The three remaining members of the first cycle the last triad. The efficiency of the cycles may be ed by estimating the differences of the subscripts as e.

ow I say that the combination of triads effected by these s will be entirely distinct from the former. To show use for a moment the former notation. Write the primary gement (in a slightly different order), and over each triad its characteristic letter, thus:

$$\begin{array}{ccccc} a & b & c & d & e \\ \mid kp_0q_0 & \mid p_1q_4q_5 & \mid p_2q_1q_3 & \mid p_3p_5p_6 & \mid p_4q_2q_6 \mid \end{array}$$

we have (according to our former notation) the two s *abcdedd* and *acecbbe*; where any of the three subts 1, 2, 3 may be appended to any of the letters, provided lo not use the same subscript twice to the same letter. s we may take for cycles $\left.\begin{array}{l} a_1b_1c_1d_1e_1d_2d_3 \\ a_2c_2e_2c_3b_2b_3e_3 \end{array}\right\}$,

irst of which is the same as before, the other different. g these cycles to complete the system, we get

$a_1a_2a_3$	$b_1b_2b_3$	$c_1c_2c_3$	$d_1d_2d_3$	$e_1e_2e_3$
$b_1c_2a_3$	$c_1b_3e_3$	$d_1e_1b_3$	$e_1d_2a_1$	$d_2c_2a_2$
$c_1e_2a_3$	$d_1e_2a_2$	$e_1c_2b_2$	$d_2a_1b_1$	$d_3b_3c_2$
$d_1c_3a_3$	$e_1a_2c_3$	$d_2b_3c_2$	$d_3b_1c_1$	$a_1b_2e_2$
$e_1b_2a_3$	$d_2c_3e_2$	$d_3b_2a_2$	$a_1c_1d_1$	$b_1e_2c_3$
$d_2b_3a_3$	$d_3e_2a_3$	$a_1e_3c_3$	$b_1d_1e_1$	$c_2a_3b_3$
[illegible] a_3	$a_2c_3b_2$	$b_1a_2e_2$	$c_1e_1d_2$	$d_1c_3b_2$

a combination of triads entirely distinct from the o
form.

Now I say that whenever n is odd and $6n+1$
$12n+3$ symbols can be exhibited in $6n+1$ series of
each series containing all the symbols, and the who
hausting but not repeating all the duads, by help of
of cycles similar to the second of those I have explai
the case of 15.

Let k be the constant term,

$$\left.\begin{array}{l} P_0,\ P_1,\ P_2 \ldots\ldots P_{6n-2},\ P_{6n-1},\ P_{6n} \\ Q_0,\ Q_1,\ Q_2 \ldots\ldots Q_{6n-2},\ Q_{6n-1},\ Q_{6n} \end{array}\right\}$$

be the successive members of the two cycles. Let
primitive root of $6n+1$. Let ρ be any value of x
satisfies the equivalence

$$\frac{x^{3n}+1}{x+1} \equiv 0, \text{ (modulus } 6n+1).$$

That is to say, let

$$\rho \equiv r^{\mu}, \text{ (modulus } 6n+1),$$

where μ is any odd number not divisible by $3n$.

Then the primary arrangement, expressed in terms o
will consist,

(1). Of a triad in which the constant term is as
with one member from each of the cycles.

This will in every case be kP_0, Q_0.

(2). Of $3n$ triads, in each of which one term of th
cycle is associated with a duad of the second. These
will be of the form

$$\Sigma P_{r^{2i}\frac{\rho+1}{2}}\, Q_{r^{2i}}\, Q_{r^{2i}\rho},$$

where all integral values are to be substituted for i
to $3n-1$.

(3). Of n triads, each consisting of terms from th
cycle only. These triads will be of the form

$$\Sigma P_{r^{2i+1}\frac{(\rho+1)}{2}}\ P_{r^{2i+2n+1}\frac{(\rho+1)}{2}}\ P_{r^{2i+4n+1}\frac{(\rho+1)}{2}}$$

where all integral values are to be substituted for
0 to $n-1$. The subscripts are of course all estima
modulus $6n+1$.

For, if possible, let the cycles be inefficient.

First, if possible, let the difference of the subscripts of two duads of the second class be equivalent, estimated positively or negatively to modulus $6n+1$; that is, if possible, in the triads

$$Pr^{2i}_{\frac{\rho+1}{2}}\; Qr^{2i}\; Q\rho r^{2i} \text{ and } Pr^{2i_1}_{\frac{\rho+1}{2}}\; Qr^{2i_1}\; Q\rho r^{2i_1}.$$

Let $\rho r^{2i} - r^{2i} \equiv \pm(\rho r^{2i_1} - r^{2i_1})$, (modulus $6n+1$);

therefore, dividing by $\rho - 1$,

$$r^{2i} \equiv \pm r^{2i_1}, \text{ (modulus } 6n+1),$$

and i, i_1 being by hypothesis different, one of them must be the greater. Let $i > i_1$; therefore dividing by r^{2i_1}, we must have

$$r^{2(i-i_1)} \equiv \pm 1, \text{ (modulus } 6n+1).$$

To satisfy this equivalence, we must have, if we take the upper sign $2(i-i_1) = 6\lambda n$, λ being some integer, the value $\lambda = 0$ inadmissible because i and i_1 are different, and so is any other value, for $i > i_1$ and both are less than $3n$; therefore $i - i_1$ is positive and less than $3n$, and $2(i-i_1)$ less than $6n$.

If we take the lower sign, we must have

$$2(i - i_1) = 3n + 6\lambda n,$$

λ being some integer, that is to say, an even number equal to an odd one, which is absurd.

Next, if possible, let the difference of subscripts of the duads of the third class in these same triads be equivalent. But we have proved the difference of subscripts of the duads of the second class not to be equivalent. And the differences in question, being the halves of these former differences to modulus $6n+1$, must likewise not be equivalent.

Lastly, if possible, let the differences of the subscripts, estimated positively or negatively to modulus $6n+1$, of two duads of the first class be equivalent.

(1). Let the duads occur in the same triad,

$$Pr^{2i+1}_{\frac{\rho+1}{2}}\; Pr^{2i+2n+1}_{\frac{\rho+1}{2}}\; Pr^{2i+4n+1}_{\frac{\rho+1}{2}};$$

therefore, (expunging the common factor $r^{2i+1}_{\frac{\rho+1}{2}}$), we must have two of the three quantities

$$r^{2n} - 1, \quad r^{4n} - 1, \quad r^{4n} - r^{2n}$$

equivalent, estimated positively or negatively.

$$\left.\begin{array}{ll}\text{Let} & r^{4n} - 1 \equiv \pm (r^{2n} - 1); \\ \text{therefore either} & r^{4n} - 1 \equiv 0, \\ \text{or} & r^{2n} \equiv 0, \\ \text{or else} & r^{2n} \equiv - 2, \\ \text{and therefore} & 2^3 + 1 \equiv 0. \end{array}\right\} \text{(modulus } 6n + 1)$$

All impossible. And so of the others.

(2). Let the duads occur in two different triads

$$Pr^{2i+1}_{\frac{\rho+1}{2}},\ Pr^{2i+2n+1}_{\frac{\rho+1}{2}},\ Pr^{2i+4n+1}_{\frac{\rho+1}{2}} \text{ and } Pr^{2i_1+1}_{\frac{\rho+1}{2}},\ Pr^{2i_1+2n+1}_{\frac{\rho+1}{2}},\ Pr^{2i_1+4n+1}_{\frac{\rho+1}{2}}.$$

If we take corresponding duads from the two triads, the difference of subscripts are obviously not equivalent, estimated positively or negatively, from the same reasoning as before. Take then two duads which do not correspond, say the first and third term of the one triad and the first and second of the other; therefore we must have (expunging the common factor $\frac{r(\rho + 1)}{2}$),

$$\left.\begin{array}{ll} & r^{2i}(r^{4n} - 1) \equiv \pm r^{2i_1}(r^{2n} - 1); \\ \text{therefore} & r^{2(i-i_1)} \times (r^{2n} + 1) \equiv \pm 1, \\ \text{or, since} & 1 + r^{2n} + r^{4n} \equiv 0, \\ & r^{4n+2(i-i_1)} \equiv \mp 1. \end{array}\right\} \text{(modulus } 6n + 1)$$

To satisfy this equivalence, if we take the upper sign, we may reduce, as before, to the absurdity of an even number equalling an odd one. If we take the lower, we must have

$$\left.\begin{array}{ll} & 4n + 2(i - i_1) = 6\lambda n \\ \text{or} & 2n + i - i_1 = 3\lambda n \end{array}\right\} \lambda \text{ being some integer.}$$

But both i and i_1 lie between the limits (inclusive) 0 and $n - 1$, and therefore $i - i_1$ between the limits $\pm (n - 1)$, and $2n + i - i_1$ between the limits $n + 1$, $3n - 1$.

Therefore the above equation is impossible, and similarly of the other duads which do not correspond.

There remain to be considered the three duads which occur in the triad kP_0Q_0 of the primary arrangement.

Two of them are duads of the fourth class and cannot recur till the cycle is repeated. The other is of the third class, and the difference of its subscripts $= 0$: while the difference of subscripts of no other duad of the third class can be equivalent to 0. Thus the efficiency of the cycles not to reproduce duads is demonstrated. Moreover, the primary

arrangement (and therefore all the arrangements) will contain all the symbols. This follows from two well-known propositions in the theory of numbers.

PROP. I. If $2N + 1$ is a prime number, and r a primitive root thereto, not only will the different terms of the series

$$r^0\ r^1\ r^2 \ldots\ldots r^{2N-2}\ r^{2N-1},$$

estimated to modulus $2N + 1$, comprise all the natural numbers from 1 to $2N$; but these terms, all multiplied by any of the same number, will do so likewise, provided that number is itself prime to $2N + 1$. From this it follows that to the letter P all the subscripts are annexed.

PROP. II. The same things supposed, let ρ be any value of x which satisfied the equivalence

$$x^N + 1 \equiv 0, \text{ (modulus } 2N + 1).$$

Then the different terms of the two series

$$\left.\begin{matrix} r^0 & r^2 & r^4 \ldots\ldots & r^{2(n-2)} & r^{2(n-1)} \\ \rho r^0 & \rho r^2 & \rho r^4 \ldots\ldots & \rho r^{2(n-2)} & \rho r^{2(n-1)} \end{matrix}\right\} .$$

will comprise all the natural numbers from 1 to $2N$.

From hence it follows that to the letter Q all the subscripts are annexed.

To apply our formula, let $n = 3$. Then $6n + 1 = 19$, and is prime. A primitive root of 19 is 10. Taking then $r = \rho = 10$, we find at once, by help of Jacobi's *Canon. Arith.*, the primary arrangement of 39 symbols, as follows:

$$kp_0q_0$$
$$+ p_{16}q_1q_{10} + p_{15}q_5q_{13} + p_{14}q_6q_2 + p_{13}q_{11}q_{15} + p_8q_{17}q_{18} + p_3q_9q_{14}$$
$$+ p_{10}q_7q_{18} + p_{12}q_{16}q_6 + p_2q_4q_3$$
$$+ p_{17}p_{16}p_4 + p_9p_4p_5 + p_7p_1p_{11}.$$

And by successively increasing the subscripts by unit, all the nineteen arrangements may be completed.

We may also obtain combinations of triads analogous to Mr. Kirkman's form, by help of cycles similar to the first of those explained in the case of 15. Our symbols having the same signification as before, the general expression for the primary arrangement will be as follows. It will consist—

(1). Of a triad in which the constant term is associated with one number from each of the cycles. This will be, as before, kP_0Q_0.

(2). Of $3n$ triads, in each of which one term of the fi[rst] cycle is associated with a duad of the second. These tri[ads] will be of the form

$$\Sigma P_{\frac{\rho+1}{\rho-1} r^{2n+i}} Q_i Q_{6n+1-i},$$

where all integral values are to be substituted for i from to $3n$. Or, as it may be otherwise expressed,

$$\Sigma P_{\frac{\rho+1}{\rho-1} r^{2i}} Q r^{2i} Q r^{2i+3n},$$

where all integral values are to be substituted for i from to $3n-1$.

(3). Of n triads, each consisting of terms from the fi[rst] cycle only. These triads will be of the form

$$\Sigma P_{\frac{\rho+1}{\rho-1} r^{2i+1}} P_{\frac{\rho+1}{\rho-1} r^{2i+2n+1}} P_{\frac{\rho+1}{\rho-1} r^{2i+4n+1}},$$

where all integral values are to be substituted for i from to $n-1$. The subscripts are of course all estimated t[o] modulus $6n+1$.

The proof of this presents no difficulty and is left to th[e] reader.

To apply this formula take, as before, $n = 3$,

$$r = \rho = 10.$$

Then $$\frac{\rho+1}{\rho-1} = r^{14} = r^{-4}, \text{ (modulus 19)},$$

and we can easily construct the following expression for th[e] primary arrangement of 39 symbols:

$$kp_0q_0$$
$$+ p_{16}q_1q_{18} + p_4q_2q_{17} + p_9q_3q_{16} + p_7q_4q_{15} + p_6q_5q_{14} + p_1q_6q_{13}$$
$$+ p_{17}q_7q_{12} + p_5q_8q_{11} + p_{11}q_9q_{10}$$
$$+ p_8p_{12}p_{18} + p_3p_2p_{14} + p_{10}p_{15}p_{13}.$$

And by successively increasing the subscripts by uni[ty] throwing out nineteens, complete the whole 19 arrang[e]ments.

The efficiency of either of these cycles may be tested, the way already explained, by estimating the difference the subscripts.

When n is even, I can obtain no solution of the proble[m] in terms of either cycles.

POSTSCRIPT.

The number of distinct species of combinations of triads comprised in each of the general expressions given, will be $\frac{1}{2}(3n-1)$. For it is evident that if we multiply all the subscripts of a primary arrangement by any the same number, provided that number is itself prime to $6n+1$, we shall not affect the system resulting therefrom, but only the order in which the different arrangements composing that system are generated by the cycles. It is also evident that, in both the general expressions for the primary arrangement, such multiplication will either have no effect upon its form, or will change ρ into $\frac{1}{\rho}$ to modulus $6n+1$, according as the multiplier is equivalent to an even or an odd power of a primitive root. Consequently the different values of ρ, $3n-1$ in number, may be arranged in $\frac{1}{2}(3n-1)$ pairs, the constituents of any pair being inverse to each other to modulus $6n+1$, and either substituted in the primary arrangement producing the same combination. The number of distinct species therefore will be $\frac{1}{2}(3n-1)$ in each of the general expressions, or $3n-1$ altogether. Thus for fifteen symbols there will be two distinct species of combinations; for thirty-nine, eight; and so on.

February 7, 1852.

There is, I find, still another expression which gives efficient cycles. In the general expression, the proof of which I have given at length, we may take for the subscript of the associated member of the first cycle the *sum* of the subscripts of the duads of the second, instead of the semisum as there given; *i.e.* in all the subscripts of terms of the first cycle we may write $\rho+1$ in place of $\frac{1}{2}(\rho+1)$ where it occurs, and the proof will still hold good.

The two general expressions I originally gave generate systems mutually correspondent, depending on the twofold arrangement of duads which I first explained. For the duads of the second class, which are associated with P_0 in one of the systems when expanded, will be the duads of the second class which occur in the primary arrangement of the other system. In this new system, however, the correspondent system will be of the same form as the original one. I have applied it to 15 symbols, and find the form to be a distinct species, though closely resembling Mr. Kirkman's.

The number of distinct species of combinations of triads of $12n+3$ symbols (n being odd and $6n+1$ prime) will be then $\frac{3}{2}(3n-1)$. Or at least these are all which I can discover.

May 3, 1852.